Solutions
Master's Exam - Fall 2010
October 28, 2010
1:00 pm - 5:00 pm

NUMBER:________________

A. The number of points for each problem is given.

B. There are problems with varying numbers of parts.
Problems 1 - 4 Probability (60 points)
Problems 5 - 8 Statistics (60 points)

C. Write your answers on the exam paper itself. If you need more room you may use the extra sheets provided. Answer as many questions as you can on each part. For Problems 2, 3 and 7 you can answer Part (b) even if you cannot answer Part (a). Tables are provided.

Good Luck!

Problem 1. (10 pts.) A class consists of 60% men and 40% women. Of the men, 25% are blond, while 45% of the women are blond. If a student is chosen at random and is found to be blond, what is the probability that student is a man?

Man: 0.6 — Blond | Man — 0.25 $\Rightarrow P(\text{Blond} \cap \text{Man}) = 0.15$

Woman: 0.4 — Blond | Woman — 0.45 $\Rightarrow P(\text{Blond} \cap \text{Woman}) = 0.18$

Law of Total Prob $\Rightarrow P(\text{Blond}) = 0.33$

$$P(\text{Man} \mid \text{Blond}) = \frac{P(\text{Blond} \cap \text{Man})}{P(\text{Blond})}$$

$$= \frac{0.15}{0.33} = \frac{15}{33} = 0.4545 \quad (\text{Ans.})$$

Problem 2. A population is made-up of items of three types: Type 1, Type 2 and Type 3 in proportions $\pi_1 > 0$, $\pi_2 > 0$ and $\pi_3 > 0$ where $\pi_1 + \pi_2 + \pi_3 = 1$. An item is drawn at random and replaced. The process is repeated until for the first time all types have been observed. Let N denote the number of selections until all types have been observed for the first time. (For example, $N = 7$ for the outcome (Type 1, Type 2, Type 2, Type 1, Type 1, Type 1, Type 3) and $N = 4$ for the outcome (Type 3, Type 2, Type 2, Type 1).)

(a) (10 pts.) Show that $P(N > n) = \sum_{i=1}^{3}(1-\pi_i)^n - \sum_{i=1}^{3}\pi_i^{\,n}$, $n = 3, 4, 5, \ldots$.

Hint: $(N > n) = (X_1 = 0) \cup (X_2 = 0) \cup (X_3 = 0)$ where X_i = number of Type i observed in the first n selections, $i = 1, 2, 3$.

Define A_i = No Type i is observed in the first n selections, $i = 1, 2, 3$.

Then $\forall\, n = 3, 4, \ldots$, $P(N > n) = P[\text{Not all the types are observed in first } n \text{ selections}]$

$$= P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3) \ldots (1)$$

Clearly, $P(A_1 \cap A_2 \cap A_3) = P(\text{Neither of the types are observed}) = 0 \ldots (2)$

$P(A_1 \cap A_2) = P(\text{In first } n \text{ selections, only Type 3 is observed}) = \pi_3^{\,n}$

Similarly $P(A_2 \cap A_3) = \pi_1^{\,n}$ and $P(A_3 \cap A_1) = \pi_2^{\,n}$. $\ldots (3)$

Also, $P(A_i) = P(\text{In first } n \text{ selections, no Type } i \text{ is observed}) = (1-\pi_i)^n$, $i = 1, 2, 3$. $\ldots (4)$

Combining (1) – (4), we have

$$P(N > n) = \sum_{i=1}^{3}(1-\pi_i)^n - \sum_{i=1}^{3}\pi_i^{\,n}, \quad n = 3, 4, 5, \ldots$$

(b) (10 pts.) Use (a) and the formula $E(N) = \sum_{n=0}^{\infty} P(N > n)$ to compute $E(N)$ when $\pi_1 = 0.5$, $\pi_2 = 0.3$ and $\pi_3 = 0.2$.

$$E(N) = \sum_{n=0}^{\infty} P(N>n) = \sum_{n=3}^{\infty} \left(\sum_{i=1}^{3}(1-\pi_i)^n - \sum_{i=1}^{3}\pi_i^n \right) + P(N>0) + P(N>1) + P(N>2)$$

$$= 3 + \sum_{i=1}^{3}\sum_{n=3}^{\infty}(1-\pi_i)^n - \sum_{i=1}^{3}\sum_{n=3}^{\infty}\pi_i^n \qquad \left[\text{Since } P(N \geq 3) = 1 \Rightarrow P(N>0) = P(N>1) = P(N>2) = 1 \right]$$

$$= 3 + \sum_{i=1}^{3}\left\{(1-\pi_i)^3 \sum_{n=3}^{\infty}(1-\pi_i)^{n-3}\right\} - \sum_{i=1}^{3}\left\{\pi_i^3 \sum_{n=3}^{\infty}\pi_i^{n-3}\right\}$$

Putting $m = n-3$

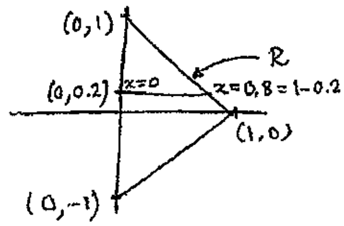
$$= 3 + \sum_{i=1}^{3}\left\{(1-\pi_i)^3 \sum_{m=0}^{\infty}(1-\pi_i)^m\right\} - \sum_{i=1}^{3}\left\{\pi_i^3 \sum_{m=0}^{\infty}\pi_i^m\right\}$$

$$= 3 + \sum_{i=1}^{3}\frac{(1-\pi_i)^3}{\pi_i} - \sum_{i=1}^{3}\frac{\pi_i^3}{1-\pi_i} = 6.65476 \quad (\text{Ans})$$

Problem 3. Suppose (X, Y) follows a uniform distribution on $R := \{(x, y): x > 0, x + |y| < 1\}$. This means that the joint probability density function of (X, Y) is constant on R and zero outside R.

(a) (10 pts.) Calculate $E(X \mid Y = 0.2)$.

(b) (10 pts.) Compute $P(Y > 3X - 1)$.



(a) Area $(R) = \frac{1}{2} \times 2 \times 1 = 1$

$\Rightarrow$ Joint pdf of (X,Y), ~~$f_{X,Y}(x,y)$~~

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R, \\ 0 & \text{o.w.,} \end{cases}$$

~~Range (Y)~~ . Let f_X and f_Y denote the marginal pdfs of X and Y, respectively.

Then $f_Y(0.2) = \int_{-\infty}^{\infty} f_{X,Y}(x, 0.2)\,dx = \int_0^{0.8} dx = 0.8.$

Conditional range of X given $Y = 0.2$ is $(0, 0.8)$.

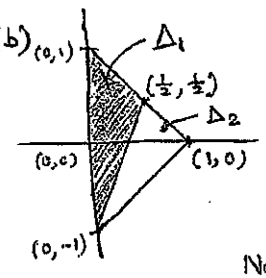
For any $x \in (0, 0.8)$, conditional pdf of X given $Y = 0.2$,

$$f_{X|Y}(x|0.2) = \frac{f_{X,Y}(x, 0.2)}{f_Y(0.2)} = \frac{1}{0.8} = 1.25, \quad 0 < x < 0.8.$$

$\Rightarrow$ Conditionally on $Y = 0.2$, $X \sim \text{Unif}(0, 0.8)$

$\Rightarrow \quad E(X|Y=0.2) = \frac{0+0.8}{2} = 0.4 \quad (\text{Ans}).$

(b)



Let Δ_1 be the shaded triangle and $\Delta_2 = R \setminus \Delta_1$.

Clearly $\Delta_1 = \{(x,y): x>0,\ x+|y|<1,\ y>3x-1\}$.

$= \{(x,y): x>0,\ x+y<1,\ y>3x-1\}$.

Note that $(\frac{1}{2}, \frac{1}{2})$ is the midpoint of the line-segment joining $(1,0)$ and $(0,1)$. Therefore

$$\text{Area}(\Delta_1) = \text{Area}(\Delta_2) = \frac{1}{2}\text{Area}(R) = \frac{1}{2}.$$

$$P(Y > 3X-1) = P[(X,Y) \in \Delta_1]$$

$$= \iint_{\Delta_1} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{\Delta_1} dx\,dy = \text{Area}(\Delta_1) = \frac{1}{2} \quad \text{(Ans).}$$

Problem 4. (10 pts.) Let X_n be a random variable that follows uniform distribution on $(-1/n, 1/n)$. Does $\{X_n\}$ converge in probability? If yes, what does it converge to? Please justify your answer.

Yes, $X_n \xrightarrow{P} 0$.

<u>Justification:</u> Fix $\varepsilon > 0$. We have to show $P[|X_n| > \varepsilon] \to 0$ as $n \to \infty$.

Choose N large enough so that $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$, $0 \leq P[|X_n| > \varepsilon] \leq P[|X_n| > \frac{1}{N}] = 0 \quad \cdots (*)$

Since $X_n \sim \text{Unif}(-\frac{1}{n}, \frac{1}{n}) \Rightarrow P[|X_n| \leq \frac{1}{n}] = 1$

$\Rightarrow P[|X_n| \leq \frac{1}{N}] = 1. \quad [n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N}]$

$\Rightarrow$ The last equality in $(*)$ holds.

$(*) \Rightarrow P[|X_n| > \varepsilon] = 0 \quad \forall\ n \geq N$

$\Rightarrow X_n \xrightarrow{P} 0.$

Problem 5. Let $X_1, X_2, \ldots, X_n$ be *iid* uniform on the interval $(0, \theta)$ where $\theta > 0$.

(a) (7 pts.) Determine the distribution of $Y = \max\{X_1, X_2, \ldots, X_n\}$.

Since $P_\theta(Y \leq y) = P_\theta(X_1 \leq y)P_\theta(X_2 \leq y) \cdots P_\theta(X_n \leq y)$, the *cdf* of Y is

$$F_\theta(y) = 0, \quad y \leq 0$$

$$F_\theta(y) = \left(\frac{y}{\theta}\right)^n, \quad 0 < y < \theta$$

$$F_\theta(y) = 1, \quad y \geq \theta$$

(b) (4 pts.) Consider a test of size 0.05 for testing H_0: $\theta = 1$ versus H_1: $\theta > 1$ based on Y and with rejection region $Y > c$. Determine the value of c.

Solve $F_1(y) = 0.95$ for c. Solution is $c = 0.95^{1/n}$.

(c) (4 pts.) Determine the power of the test as a function of θ, $\theta > 1$.

$$\text{Power} = P_\theta\left(Y > 0.95^{1/n}\right) = 1 - P_\theta\left(Y \leq 0.95^{\frac{1}{n}}\right) = 1 - \frac{0.95}{\theta^n}, \theta > 1.$$

Problem 6. (15 pts.) Let $X_1, X_2, \ldots, X_n$ be independent random variables with common probability density function

$$f_\theta(x) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, \quad x \geq \mu$$

$$= 0, \quad x < \mu.$$

where $n \geq 2$ and the parameter is $\theta = (\mu, \sigma)$ with $-\infty < \mu < \infty$, $\sigma > 0$.

Find the maximum likelihood estimator of θ.

Solution. The *ln likelihood* is

$$ln(f_\theta(x_1, x_2, \ldots, x_n) = -nln(\sigma) - \sum_{i=1}^{n} \frac{(x_i - \mu)}{\sigma}, \quad \sigma > 0, \ \mu \leq x_{(1)} := min\{x_1, x_2, \ldots, x_n\}$$

$$= -\infty, \quad \text{otherwise}$$

The term $-\sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma}$ is maximized by the choice $\mu = \hat{\mu} := x_{(1)}$ for each fixed $\sigma > 0$.

Differentiation with respect to σ and examination of the second derivative shows that

$\sigma = \hat{\sigma} := \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}) = \bar{x} - x_{(1)}$ maximizes the *ln likelihood* with respect to σ.

Problem 7. In a large population of trees, the trees have characteristics X and Y where X is at two levels and Y is at 3 levels. A random sample of $n = 100$ trees is cross-classified into the levels of X and Y with this resulting 2 x 3 contingency table.

	Characteristic Y			
Characteristic X	**1**	**2**	**3**	**Total**
1	10	20	30	60
2	10	20	10	40
Total	20	40	40	100

(a) (7 pts.) Compute an approximate 95% confidence interval for the population proportion of trees with characteristic Y at level 3.

Solution. Estimate $\hat{\pi} = \frac{X}{n} = \frac{40}{100} = 0.40$ has standard error $SE(\hat{\pi}) = \sqrt{\frac{(0.40)(0.60)}{100}} = 0.049$. The large sample 95% confidence interval estimate of π is

$$0.40 \pm (1.96)(0.049) = 0.40 \pm 0.096.$$

(b) (8 pts.) Carry-out a chi-square test of independence for the characteristics X and Y. Is the hypothesis of independence rejected at level 0.05? Why or why not?

Solution. The table of expected values estimated under independence is

Expected Values

	Characteristic Y			
Characteristic X	**1**	**2**	**3**	**Total**
1	12	24	24	60
2	8	16	16	40
Total	20	40	40	100

so the value of the chi-square statistic is $\frac{(10-12)^2}{12} + \frac{(20-24)^2}{24} + \cdots + \frac{(10-16)^2}{16} = 6.25$. The level 0.05 critical value is 5.99 (2 *df*). Since 6.25 > 5.99, independence is rejected.

Problem 8. Consider the model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i,\ i = 1, 2, \ldots, n$ where the x_i are constants and the ϵ_i are uncorrelated random variables with common expectation 0 and common variance σ^2. Suppose that one is supplied with the additional information that $\beta_0 = 2 + \beta_1$.

(a) (7 pts.) What is the least squares estimate of $\hat{\beta}_1$ of β_1 under this constraint?

Solution. The least squares estimate of β_1 is the minimizer of

$$\sum_{i=1}^{n} (y_i - (2 + \beta_1) - \beta_1 x_i)^2.$$

The first and second derivatives with respect to β_1 are

$$-2\sum_{i=1}^{n} (y_i - 2 - \beta_1 - \beta_1 x_i)(1 + x_i)$$

and

$$2\sum_{i=1}^{n} (1 + x_i)^2 > 0.$$

Setting the first derivative equal to 0 and solving for β_1 shows that the minimizer is

$$\beta_1 = \hat{\beta}_1 := \frac{\sum_{i=1}^{n} (y_i - 2)(1 + x_i)}{\sum_{i=1}^{n} (1 + x_i)^2} \qquad (1)$$

(b) (8 pts.) What is its expectation $E(\hat{\beta}_1)$ and what is its variance $V(\hat{\beta}_1)$?

Solution. Since $E(y_i) = 2 + \beta_1 + \beta_1 x_i$, (1) shows that $E(\hat{\beta}_1) = \frac{\sum_{i=1}^{n} \beta_1 (1 + x_i)(1 + x_i)}{\sum_{i=1}^{n} (1 + x_i)^2} = \beta_1$. Also from (1) we see that

$$V(\hat{\beta}_1) = \frac{\sum_{i=1}^{n} (1 + x_i)^2 \sigma^2}{\left[\sum_{i=1}^{n} (1 + x_i)^2\right]^2} = \frac{\sigma^2}{\sum_{i=1}^{n} (1 + x_i)^2}.$$