Answers of Master's Exam, Spring, 2007

1) a) A subset of a sample space, or a set of outcomes of an experiment.

   b) $P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$ for all (Borel) subsets of the real-line. Equivalently, $P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y)$ for all pairs of real numbers (x, y).

2) $F(x) \lim_{x \to -\infty} = 0, \quad \lim_{x \to +\infty} F(x) = 1$, F is monotone non-decreasing, and F is continuous on the right.

3) $\sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} (X_i - a)^2 - (\bar{X} - a)^2 n$ for any constant a. Thus, $E(\sum_{i=1}^{n} (X_i - \bar{X})^2)$

$= n \sigma^2 - n \operatorname{Var}(\bar{X}) = n \sigma^2 - n(\sigma^2 /n) = (n - 1) \sigma^2$, and $E(S^2) = (n - 1) \sigma^2 /(n-1) = \sigma^2$.
Students could be asked to prove the first identity..

4) Use the normal approximation of the binomial, with the ½ correction.
Answer: 0.237

5) 0.926/0.94 = 0.9851

6) a) Var(Y) does not exist since the integral of $x^2 f(x)$ from 0 to 1 does not exist. We could say that Var(Y) is $+\infty$.
 b) $F_Y(y) = 1/(2 y^2)$ for $1/2 < y \leq 1$, $1/(4 y^3)$ for $½ < y < \infty$, 0 otherwise

7) a) $1 – 4 e^{-3}$ b) 82

8) a) 0.53056 b) 0.0643

9) a) $F_W(w) = w - w \log(w)$ for $0 < w \leq 1$, 0 for $w < 0$, 1 for $w > 1$.
b) $\Phi((0.2 – n/4)/ [7/(144n)]^{1/2})$, since E(W) = 14 and Var(W) = 7/144.
c) 0.65465

10) 3383

11) a) Sampling is without replacement. Sampling is done in such a way that all samples of 80 employees are equally likely.

   b) $\bar{X} = 1$, $S^2 = 0.65823$

c) 0.0049615

d) 200 ± 27.612

e) The sampling and the interval were determined in such a way that 95% of all possible samples produce intervals which contain the population total T.

12) a) Reject H_0 for $\bar{X} \geq 43.29$

b) 0.8037

c) [37.966, 48.034]

13) a) $\hat{\theta} = \bar{X}/(1 + \bar{X})$.

b) $\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| < \varepsilon \mid \theta) = 1$ for every $\varepsilon > 0$, for each θ in the parameter space.

c) By the weak law of large numbers $\bar{X}_n$ converges in probability to the mean μ. Since $\theta = \mu/(1 + \mu)$ is a continuous function of μ, $\hat{\theta}_n$ converges in probability to θ.

d) Let $Y_i = \log(X_i)$. The MLE is $n/\Sigma Y_i = 1/\bar{Y}$.

14) a) The first column: 0.36, 9, 6/7 Second column: 9, 9/4, 3/14
Third column: 6/7, 3/14, 1/49
b) Reject for (x_1, x_2) in {{1,1}, (1,2), (2,1), (2,2)}
c) Power o.81

15) a) $\hat{\beta}' = (3, 2, 1)$
b) 4
c) 5/2
d) 2 ± 6.80

16) Suppose that D_i is (A value – B value for plot i), with the D_i independent, each with the $N(\mu_D, \sigma_D^2)$ distribution, with μ_D, σ_D^2 unknown. We wish to test H_0: $\mu_D \leq 0$, vs H_a: $\mu_D > 0$.

We observe $D_1 = 2, D_2 = 3, D_3 = 2, D_4 = 5, D_5 = 3$, $\bar{D} = 3$, $S_D^2 = 6/4 = 1.5$
$T = (3 - 0)/[1.5/5]^{1/2} = 5.477$. Reject for $T > 2.132$, so we reject H_0.

) a) $\dfrac{\binom{32}{16}\binom{32}{16}}{\binom{64}{32}}$ $\lim_{n\to\infty} F_n(x)$

3) a) $M_n(t) = e^{t/n}\,(n+1)/(5n) + e^{(1-1/n)t}\,(4n+1)/(5n)$

b) $\lim_{n\to\infty} M_n(t) = 1/5 + (4/5)\,e^t$, for all t. This is the mgf of the Bernoulli distribution with parameter p = 4/5 = 0.8. By the continuity theorem for mgf's, Z_n converges in distribution to this Bernoulli distribution.

4) a) $E(Z_1) = p_1 + p_3$., $Cov(Z_1, Z_2) = Var(Y_3) = p_3\,(1 - p_3)$

b) (Z_1, Z_2) takes the following values with probabilities as given:

(0,0)	$q_1\,q_2\,q_3$	(0, 1)	$q_1\,p_2\,q_3$	(0, 2)	0
(1,0)	$p_1\,q_2\,p_3$	(1,1)	$q_1\,q_2\,p_3 + p_1\,p_2\,q_3$	(1,2)	$q_1\,p_2\,p_3$
(2,0)	0	(2,1)	$p_1\,q_2\,p_3$	(2,2)	$p_1\,p_2\,p_3$

5) a) $X_3 = F_3^{-1}(F_1(X_1)) = X_1^{-1/3}$

b) $f_M(m) = 6\,m^6$ for $0 \le m \le 1$, since each of $Y_1 = X_1$, $Y_2 = X_2^2$, $Y_3 = X_3^3$ has cdf $G(y) = y^2$ on [0,1]. .

c) Find $H(y) = P(Y_1 - Y_2 . > y)$ for each y, then integrate H(y) g(y) from 0 to one, where g is the derivative of G. Answer: 1/6

6) a) Let $\lambda(x_1, \dots, x_n)$ be the likelihood ratio with $f(x_1, \dots, x_n; \theta = \eta)$ in the numerator,

$f(x_1, \dots, x_n; \theta = 1)$ in the denominator. Then $\lambda(x_1, \dots, x_n) = \eta^n\,(\Pi\ x_i)^{\eta-1}$. We should reject for large $\log(\lambda) = n\log(\eta) + (\eta - 1)\log(\Pi\ x_i)$. For $\eta > 1$, this is an increasing function of the product W. Thus, we should reject for W > c for some constant c. We want $P(W > c \mid \theta = 1) = 1 - c$ for n = 1. Thus, c should be $1-\alpha$.

Comment: The power function of this test for any θ is $P(W > c \mid \theta) = 1 - c^{\theta}$. The general case for any n can be solved using the transformation $Y_i = -\log(X_i)$ and taking advantage of the fact that the sum of independent exponential rv's has the gamma distribution with shape parameter n. \

7) Could use the sign test. p-value = 58/128 Two-sided test.

8) a) $\hat{\theta} = (1 - \bar{X})/\bar{X}$

b) $\hat{\theta}_{MLE} = \bar{Y}$, where $Y_i = -\log(X_i)$ for each i.

c) Each Y_i has the exponential distribution with mean θ. Thus $E(\bar{Y}) = \theta$. Answer: Yes

9) a) Let $Q(\beta_1, \beta_2) = \Sigma (Y_{1j} - \beta_1)^2 + \Sigma (Y_{2j} - \beta_2)^2$. Differentiating wrt to β_i we get $\hat{\beta}_i$ for i = 1, 2.

b) $E(\hat{\beta}_1 - \hat{\beta}_2) = \beta_1 - \beta_2$, $Var(\hat{\beta}_1 - \hat{\beta}_2) = \sigma^2(1/n_1 + 1/n_2)$.

c) For $\sigma^2 = 10$, the Z statistic is $-2.6/[10(1/5 + 1/5)] = -1.3$, so the p-value for a one sided test is $P(Z \leq -1.3) \doteq 0.10$. Do not reject H_0.