Preliminary Exam: Probability
9:00am – 2:00pm, August 26, 2005

Question 1. (10 points) Let $c > 0$ and $f(x) = (x + c^{-1})^2$. In what follows X is a random variable with $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) = 1$.

(i) Find $\min_{x \geq c} f(x)$ and $\mathbb{E}\big(f(X)\big)$.

(ii) Show that

$$\mathbb{P}\{X > c\} \leq \frac{1}{1 + c^2}.$$

Question 2. (15 points) Let $X_1, X_2, \ldots$ be a sequence of independent Cauchy random variables with parameter $a > 0$. That is, X_1 has density function

$$f(x) = \frac{a}{\pi(a^2 + x^2)}, \qquad x \in \mathbb{R}.$$

Denote by $F_n(x)$ the distribution function of $\frac{1}{n}\big(\sup_{1 \leq i \leq n} X_i\big)$.

(i) Find $F_n(x)$.

(ii) For every $x \in \mathbb{R}$, find $\lim_{n \to \infty} F_n(x)$.

(ii) Prove that for some exponential random variable T, we have

$$\frac{1}{n}\Big(\sup_{1 \leq i \leq n} X_i\Big) \Rightarrow \frac{1}{T} \qquad \text{as } n \to \infty.$$

Identify the parameter of T.

Question 3. (10 points) Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $\mathbb{E}(|X|^p) < \infty$ for all $p \in (0, 1)$. For a constant $\alpha > 1$, we consider the weighted partial sums

$$S_n = \sum_{k=1}^{n} k^{-\alpha} X_k, \quad n \geq 1.$$

(i) Prove that the series

$$\sum_{k=1}^{n} k^{-\alpha}\, \mathbb{E}\big(X_k\, \mathbb{1}_{\{|X_k| \leq k^\alpha\}}\big)$$

is convergent.

(ii) Prove that, as $n \to \infty$, S_n converges almost surely.

Question 4. (10 points) Let X and Y be real random variables such that (a) $X-Y$ and X are independent and (b) $X-Y$ and Y are independent. This problem is about showing that $X-Y$ is almost surely a constant.

(i) Let $\varphi(\xi)$ and $H(\xi)$ be the characterization functions of X and $X-Y$, respectively. Prove the following identity:

$$\varphi(\xi)\big(1-\big|H(\xi)\big|^2\big)=0, \qquad \forall \xi \in \mathbb{R}.$$

Prove that there exists an $\epsilon>0$ such that $|H(\xi)|=1$ for all $\xi \in \mathbb{R}$ with $|\xi| \leq \epsilon$.

(ii) Show that $X-Y$ is almost surely a constant.

Question 5. (20 points) Let $c>0$, $\frac{c^2}{1+c^2} \leq p < 1$ and define

$$f(p,c) = \sup_{X \in A(p,c)} \mathbb{E}\big(X; X > -c\big),$$

where $A(p,c)$ is the class of random variables satisfying $\mathbb{E}(X)=0$, $\mathbb{E}(X^2)=1$ and $\mathbb{P}(X > -c)=p$. The goal is to find $f(p,c)$ by using the following steps.

(i) Find a lower bound for $f(p,c)$ by finding a random variable in $A(p,c)$ that takes exactly 2 values.

(ii) Let $h(x) = x\mathbb{1}_{\{x>-c\}} - px - \sqrt{p(1-p)/4}\, x^2$. Find x where $\max_{x>-c} h(x)$ is achieved. Do the same for $\max_{x \leq -c} h(x)$.

(iii) Prove that if $X, Y \in A(p,c)$ then

$$\mathbb{E}(h(X)) - \mathbb{E}(h(Y)) = \mathbb{E}\big(X; X > -c\big) - \mathbb{E}\big(Y; Y > -c\big).$$

(iv) Find an upper bound for $f(p,c)$ by using the identity in (iii), together with parts (i) and (ii).

Question 6. (10 points) Let U be a uniform random variable on $[0,1]$ and let ε be a Bernoulli r.v., i.e., $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$, that is independent of U. Define the r.v. $X = \varepsilon/\sqrt{U}$.

(i) Compute the distribution of X.

(ii) Let $X_1, \dots, X_n, \dots$ be a sequence of independent random variables which are distributed as X. Let $S_n = \sum_{k=1}^n X_k$. Prove that

$$\frac{S_n}{\sqrt{n \log n}} \Rightarrow N \qquad \text{as } n \to \infty,$$

where N is a standard normal random variable.

Question 7. (10 points) Let $W = \{W(t), t \geq 0\}$ be a standard Brownian motion in $\mathbb{R}$. For any $t > 0$ and integer $n \geq 1$, let

$$Q_n(t) = \sum_{k=1}^{2^n} \left[W(kt\,2^{-n}) - W((k-1)t\,2^{-n})\right]^2.$$

(i) Compute $\mathbb{E}\big(Q_n(t)\big)$ and $\mathrm{Var}\big(Q_n(t)\big)$.

(ii) Show that for every $t > 0$, $Q_n(t) \to t$ almost surely as $n \to \infty$.

Question 8. (15 points) Let $X_1, \dots, X_n, \dots$ be i.i.d. r.v.'s with

$$\mathbb{P}(X_1 = 1) = p, \quad \mathbb{P}(X_1 = -1) = q \quad \text{and} \quad \mathbb{P}(X_1 = 0) = r,$$

where p, q, $r \in (0,1)$ and $p + q + r = 1$. Consider the random walk $\{S_n, n \geq 0\}$ on $\mathbb{Z}$ defined by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Given integers $a < 0 < b$, let $T = \inf\{n > 0 : S_n \notin (a,b)\}$. Prove the following statements:

(i) $\mathbb{E}(T) < \infty$.

(ii) For $\lambda \in \mathbb{R}$ and $\phi(\lambda) = pe^{\lambda} + qe^{-\lambda} + r$, define $Y_n = e^{\lambda S_n}\,\phi(\lambda)^{-n}$. Prove that $\{Y_n, n \geq 0\}$ is a martingale. Moreover, if λ satisfies $\phi(\lambda) \geq 1$, then $\mathbb{E}\big(e^{\lambda S_T}\,\phi(\lambda)^{-T}\big) = 1$.

(iii) Assume now $r = 0$ and $p > 1/2$. Compute the probabilities $\mathbb{P}(S_T = a)$ and $\mathbb{P}(S_T = b)$.

Question 9. (Optional) Let $W_i = \{W_i(t), t \geq 0\}$ $(i = 1, 2, \ldots, d)$ be d independent standard Brownian motions in $\mathbb{R}$. For $t \geq 0$, let $W(t) = (W_1(t), \ldots, W_d(t))$. Then $W = \{W(t), t \geq 0\}$ is called a Brownian motion in $\mathbb{R}^d$. Denote the Lebesgue measure in $\mathbb{R}$ by λ_1 and define the occupation measure μ_W of W by

$$\mu_W(A) = \lambda_1\{t \in [0,1] : W(t) \in A\}, \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

(i) Show that $\mathbb{E}(\|W(1)\|^{-\gamma}) < \infty$ if and only if $\gamma < d$.

(ii) Prove that for every $\gamma < \min\{d, 2\}$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{d\mu_W(x)d\mu_W(y)}{\|x-y\|^\gamma} < \infty \quad a.s.$$

[Hint: Use the change of variables formula.]