Preliminary Exam: Probability

9:00am – 2:00pm, August 26, 2011

The exam lasts from 9:00am until 2:00pm, with a walking break every hour.

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.

The exam consists of six main problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step. On you work, label the steps this way: (i), (ii),...

On each page you turn in, write your assigned code number instead of your name. Separate and staple each main part and return each in its designated folder.

Question 1. Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two sequences of r.v.s which take values in $\{0, 1\}$. Assume that all the r.v.s $\{X_n, Y_m;\, n, m \geq 1\}$ are independent and that, for any $n \geq 1$, one has:

$$\mathbb{P}(X_n = 1) = p \quad \text{and} \quad \mathbb{P}(Y_n = 1) = q,$$

where $0 < p,\, q < 1$ are constants.

(i). (5 points) Show that the r.v.s $Z_n = X_n Y_n$, $n \geq 1$, are independent and identically distributed. Compute their common distribution.

(ii). (4 points) Define $S_n = \sum_{m=1}^{n} X_m$, and $T_n = \sum_{m=1}^{n} Z_m$. What are the distributions of S_n and T_n?

(iii). (5 points) Define $\tau(\omega) = \inf\{n \geq 1 : T_n(\omega) = 1\}$, with the convention $\inf(\emptyset) = \infty$. Write S_τ for $S_{\tau(\omega)}(\omega)$. Show that τ and S_τ are random variables. What is the distribution of τ?

(iv). (6 points) Show that, for $n \geq 2$ and $1 \leq k < n$,

$$\mathbb{P}(X_k = 1 | \tau = n) = \mathbb{P}(X_k = 1 | Z_k = 0) = \frac{p(1-q)}{1-pq}.$$

Question 2. Let $\{\xi_n, n \geq 1\}$ be a sequence of i.i.d. $N(0, 1)$ random variables and let $X_n = \mu_n + \sigma_n \xi_n$.

(i). (5 points) Prove that $\sum_{n=1}^{\infty} X_n^2$ converges in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ if and only if

$$\sum_{n=1}^{\infty} (\mu_n^2 + \sigma_n^2) < \infty.$$

(ii). (5 points) For $p \in [1, \infty)$ and integers $n > m$, estimate $\| \sum_{k=m+1}^{n} X_k^2 \|_p$ in terms of $\{\mu_k\}$ and $\{\sigma_k\}$, where $\|X\|_p$ is the L^p-norm of X.

(iii). (5 points) Prove that, if the condition (i) is satisfied, then $\sum_{n=1}^{\infty} X_n^2$ converges in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for every $p \in [1, \infty)$.

(iv). (5 points) Assume $\mu_n = 0$ for every n, $\sigma_n \to 0$ as $n \to \infty$, and $\sum_{n=1}^{\infty} \sigma_n^2 = \infty$. Prove

$$\mathbb{P}\left(\sum_{n=1}^{\infty} X_n^2 = \infty \right) = 1.$$

[Hint: you may use the necessity part of Kolmogorov's Three Series Theorem.]

Question 3. Let $c > 0$, and assume that $\{N_t^{(c)},\ t \geq 0\}$ is a Poisson process with rate c.

(i). (3 points) Let $T_1 = \inf\{t > 0 :\ N_t^{(c)} = 1\}$. Find the distribution of T_1.

(ii). (5 points) Prove that $\lim_{n\to\infty} \frac{N_n^{(c)}}{n} = c$ a.s.

(iii). (7 points) Define

$$X_t^{(c)} = N_t^{(c)} - ct.$$

Show that, for any fixed $t > 0$, the random variables $\frac{1}{\sqrt{c}} X_t^{(c)}$ converge in distribution to $\beta(t)$, as $c \to \infty$. What is the distribution of $\beta(t)$?

Question 4. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with density

$$f(x) = \begin{cases} |x|^{-3}, & \text{if } \ |x| \geq 1, \\ 0, & \text{oterwise.} \end{cases}$$

Prove the following statements:

(i) (7 points) The characteristic function $\varphi(t)$ of X_1 satisfies

$$\lim_{t\to 0} \frac{\varphi(t) - 1}{t^2 \log |t|} = 1.$$

[Hint: You may use l'Hôpital's rule.]

(ii) (5 points) (i) shows that $\varphi(t) = 1 - \big(1 + o(1)\big) t^2 \log \frac{1}{|t|}$. Use this to show that, as $n \to \infty$,

$$\frac{S_n}{\sqrt{n \log n}} \Rightarrow N(0, 1).$$

Question 5. Let $\{X_n, n \geq 1\}$ be a sequence of random variables such that $\mathbb{P}(|X_i| \leq 1) = 1$ for all $i \geq 1$ and $\mathbb{E}(X_i | \mathcal{F}_{i-1}) = 0$ for all $i \geq 2$, where $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$. Let $S_n = \sum_{i=1}^n X_i$ and let τ be a stopping time with respect to $\{\mathcal{F}_i, i \geq 1\}$ and $\mathbb{E}(\tau) < \infty$.

(i). (5 points) Prove that $\mathbb{E}(|S_\tau|) < \infty$.

(ii). (5 points) Find $\mathbb{E}(S_\tau)$.

(iii). (5 points) Let $\{T_i, i \geq 1\}$ be a martingale in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_i, i \geq 1\}$. Show that for all $n \geq 1$,

$$\mathbb{E}(S_n T_n) = \sum_{i=1}^{n} \mathbb{E}(S_i - S_{i-1})(T_i - T_{i-1}),$$

where $S_0 = T_0 = 0$.

Question 6. Let $\{B(t), t \geq 0\}$ be a real-valued standard Brownian motion.

(i). (6 points) Prove that $\{e^{B(t)-\frac{t}{2}}, t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$, where $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$.

(ii). (6 points) Find $\lim_{t\to\infty} e^{B(t)-\frac{t}{2}}$ in the a.s. sense.

(iii). (6 points) Is $\{e^{B(t)-\frac{t}{2}}, t \geq 0\}$ uniformly integrable? You must give a proof of your statement.