August 19, 2003

PRELIMINARY EXAMINATION: STT 871-872,
WEDNESDAY, AUGUST 20, 2003, 1 - 5:30 P.M.,
A506 WELLS HALL

NOTE: This examination contains 8 problems. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. In a problem consisting of multiple parts you may use one part to solve the other part even if you are unable to solve the part being used.

Please start solution of each problem on the page containing the statement of the problem. Be sure to put your assigned number on the right hand top corner of each page of your solutions.

GOOD LUCK!!!

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1. Consider the following population:

$$\begin{array}{c|ccc} x & 1 & 2 & 3 \\ \hline f_\theta(x) & \theta^2 & 2\theta(1-\theta) & (1-\theta)^2 \end{array}, \qquad 0<\theta<1.$$

In a random sample of size n obtained from this population with replacement, let $X_j =\#$ of times the outcome j is observed, $j = 1, 2, 3$.

(a) Assuming that $2X_1 + X_2$ and $X_2 + 2X_3$ are both positive, show that the MLE of θ is $(2X_1 + X_2)/2X_1$. **(3)**

(b) Does MLE always exist ? **(5)**

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2. (a) Based on one observation X from a $N(0, \sigma^2)$ population, give a uniformly minimum variance unbiased estimator of σ^7. **(3)**

(b) Let $\Theta := \{0, 1, 2, \cdots\}$. Suppose for each $\theta \in \Theta$, the random variable X is uniformly distributed on the set of two values θ and $\theta + 1$. Show that there does not exist an uniformly minimum variance unbiased estimator of θ based on X. **(7)**

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3. Let F be a known absolutely continuous d.f. with density f positive on $\mathbb{R}$. Define

$$f_\theta(x) := (\theta+1)F^\theta(x)f(x), \qquad x \in \mathbb{R},\ \theta \geq 0.$$

Let $X_1, \cdots, X_n$ be a random sample of size n from f_θ, for some $\theta \geq 0$, and let $0 < \alpha < 1$.

(a) Give uniformly most powerful size α test of $H_0 : \theta = 0$, against the alternatives $H_1 : \theta > 0$ based on the given sample. **(5)**

(b) Show that the power of the test developed in (a) is $P(\chi^2_{2n} > (\theta+1)c_{2n,\alpha})$, where $c_{2n,\alpha}$ is $100(1-\alpha)\%$ percentile of the χ^2_{2n} distribution. **(5)**

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4. Let

$$\begin{aligned} g_{\theta,a}(x) &:= \frac{a}{\pi[a^2+(x-\theta)^2]}, \qquad x,\theta \in \mathbb{R},\ a>0, \\ \mathcal{P}_0 &:= \{\mathcal{N}(\theta,1);\ \theta\in\mathbb{R}\}, \qquad \mathcal{P}_1 := \{g_{\theta,1};\ \theta\in\mathbb{R}\}. \end{aligned}$$

Let X_1 and X_2 denote the random sample of size 2 from a common density f. It is desired to test $H_0 : f \in \mathcal{P}_0$, versus the alternatives $H_1 : f \in \mathcal{P}_1$. Let $\delta(X_1, X_2)$ be an invariant test with respect to the group of translations for this problem.

(a) Show that $\delta(X_1, X_2)$ is a function of $X_1 - X_2$ only. **(3)**

(b) Show that under H_1, the density of $X_1 - X_2$ is $g_{0,2}$. **(3)**

(c) Show that the most powerful invariant tests rejects H_0 if $|X_1 - X_2| > k$, where k is a constant determined by the size requirement. **(4)**

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5. For any two densities f_0, f_1, define the Kullback-Leibler divergence

$$K(f_0, f_1) := \int f_0 \log f_0/f_1.$$

Recall that $K(f_0, f_1) \geq 0$ in general, with equality holding if and only if, $f_0 = f_1$.

Let k be a known positive integer, $\Theta := \{1, 2, , \ldots, k\}$ and $\{P_\theta,\, \theta \in \Theta\}$ denote a family of probability distributions on $\mathbb{R}$, dominated by the Lebesgue measure with a common support. Fix a $\theta_0 \in \Theta$ and let $f_\theta := dP_\theta/dP_{\theta_0}$. Let $\Lambda_n(\theta)$ denote the log-likelihood function of the random sample X_i, $1 \leq i \leq n$ from f_θ, and $\hat{\theta}_n$ denote the maximum likelihood estimator of θ.

(a) (i) Show that $\Lambda_n(\hat{\theta}_n) \geq 0$, and that for any $\theta \neq \theta_0$, $\lim_{n\to\infty} \Lambda_n(\theta) < 0$, a.s. P_{θ_0}. **(5)**

(ii) Show that $\hat{\theta}_n$ is a.s. consistent for θ, at all θ.

(b) If Π is a prior probability distribution on Θ with $\Pi\{\theta_0\} > 0$, then show that **(5)**

$$\Pi(\theta_0 | X_1, X_2, \ldots, X_n) \to 1, \quad \text{with } f_{\theta_0} \text{ probability } 1.$$

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6. Let Θ be a compact subset of $\mathbb{R}$, $\mathcal{X}$ be a finite set and let $\{P_\theta : \theta \in \Theta\}$ be a family of probability measures on $\mathcal{X}$ such that for each $x \in \mathcal{X}$, $P_\theta(x)$ is continuous in θ. Consider the usual estimation problem with squared error loss, i.e. the decision space is Θ, and the loss function is $L(\theta, a) = (\theta - a)^2$.

(a) Show that the set $\mathcal{D} := \{\delta; \delta \text{ a nonrandomized estimator}\}$ is compact under the convergence $\delta_n \to \delta$ iff $\delta_n(x) \to \delta(x)$ for all $x \in \mathcal{X}$. **(2)**

(b) Let Π be a prior on Θ. Show that the Bayes rule always exists. If in addition the prior Π has full support, that is every open set U of Θ has positive measure, then the corresponding Bayes rule is admissible. **(5)**

(c) Give an example to show that if the prior does not have full support (as defined in above) then the corresponding Bayes rule rule may not be admissible. **(3)**

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7. Let $\mu := (\mu_1, \mu_2)' \in \mathbb{R}^2$, and let

$$\Sigma := \begin{pmatrix} \sigma_1 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2 \end{pmatrix}, \qquad \sigma_j > 0,\ j = 1, 2;\ |\rho| < 1.$$

Let $\{(X_i, Y_i); 1 \le i \le n\}$ be a random sample from $\mathcal{N}_2(\mu, \Sigma)$ distribution and $T_n := \bar{Y}_n - \bar{X}_n$.

(a) Obtain the $1 - \alpha$ confidence level confidence interval for $\theta := \mu_2 - \mu_1$ based on T_n. **(5)**

(b) Let L_n denote the length of the confidence interval obtained in (a). Give the limiting distribution of $n^{1/2} L_n$. **(5)**

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8. Let $\mu := (\mu_1, \mu_2, \mu_3)' = (1, -5, 6)'$. Let $\bar{X}_n$ denote the sample mean vector of a random sample of size n from $\mathcal{N}_3(\mu, I)$ distribution, where I is the 3×3 identity matrix. Define the function

$$g(x, \theta) := \theta_1 x^2 + \theta_2 x + \theta_3, \qquad x \in \mathbb{R},\ \theta := (\theta_1, \theta_2, \theta_3)' \in \mathbb{R}^3.$$

Let θ_n, θ denote the largest root of the equations $g(x, \bar{X}_n) = 0$, $g(x, \mu) = 0$, respectively.

(a) Show that $P(\theta_n$ exists, as $n \to \infty) = 1$, and that $\theta_n \to \theta$, a.s. (4)

(b) Determine the limiting distribution of $n^{1/2}(\theta_n - \theta)$. (8)

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