

Space-time duality for fractional diffusion

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Abstract

Zolotarev proved a duality result that relates stable densities with different indices. In this paper, we show how Zolotarev duality leads to some interesting results on fractional diffusion. Fractional diffusion equations employ fractional derivatives in place of the usual integer order derivatives. They govern scaling limits of random walk models, with power law jumps leading to fractional derivatives in space, and power law waiting times between the jumps leading to fractional derivatives in time. The limit process is a stable Lévy motion that models the jumps, subordinated to an inverse stable process that models the waiting times. Using duality, we relate the density of a spectrally negative stable process with index $1 < \alpha < 2$ to the density of the hitting time of a stable subordinator with index $1/\alpha$, and thereby unify some recent results in the literature. These results provide a concrete interpretation of Zolotarev duality in terms of the fractional diffusion model. They also illuminate a current controversy in hydrology, regarding the appropriate use of space and time fractional derivatives to model contaminant transport in river flows.

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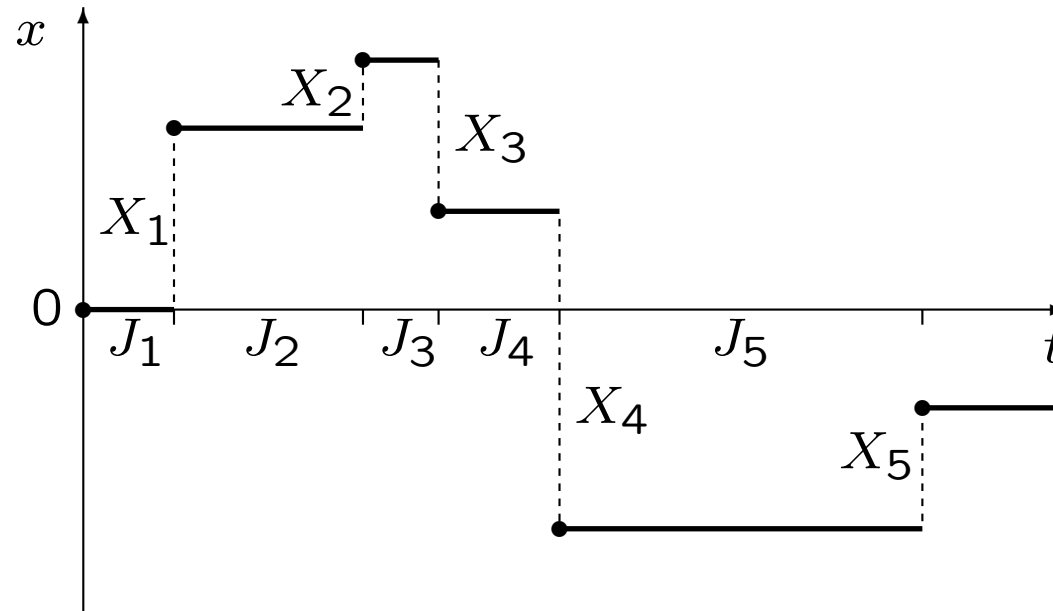
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Continuous time random walks



The CTRW has iid particle jumps X_n and iid waiting times J_n .

This talk assumes X_n independent of J_n (uncoupled CTRW).

CTRW notation

$T_n = J_1 + \cdots + J_n$ is the time of the n th particle jump

$S(n) = X_1 + \cdots + X_n$ is the particle position at time n

$N_t = \max\{n \geq 0 : T_n \leq t\}$ is the number of jumps by time $t > 0$

The CTRW $S(N_t)$ is the particle position at time $t > 0$.

Fractional derivatives and transforms

If the Laplace transform (LT) of $f(t)$ is defined by

$$\tilde{f}(s) = \int_0^{\infty} e^{-st} f(t) dt$$

then $\frac{d^\beta}{dt^\beta} f(t)$ has LT $s^\beta \tilde{f}(s) - s^{\beta-1} f(0)$ for $0 < \beta \leq 1$ (Caputo).

If the Fourier transform (FT) of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

then $\frac{d^\alpha}{dx^\alpha} f(x)$ has FT $(ik)^\alpha \hat{f}(k)$ (Riemann-Liouville).

Likewise $\frac{d^\alpha}{d(-x)^\alpha} f(x)$ has FT $(-ik)^\alpha \hat{f}(k)$ (negative R-L).

Random walk limits and governing equations

If $P(|X_n| > x) \approx x^{-\alpha}$ with $0 < \alpha < 2$ and

$$\frac{P(X_n < -x)}{P(|X_n| > x)} \rightarrow q \in [0, 1] \quad \text{as } x \rightarrow \infty$$

then $c^{-1/\alpha}S([ct]) \Rightarrow A(t)$, an α -stable Lévy process whose density $p(x, t)$ has FT

$$\hat{p}(k, t) = e^{tb[q(-ik)^\alpha + (1-q)(ik)^\alpha]}$$

where $b < 0$ for $0 < \alpha < 1$, and $b > 0$ for $1 < \alpha < 2$. Then

$$\frac{d}{dt}\hat{p}(k, t) = b[q(-ik)^\alpha + (1-q)(ik)^\alpha]\hat{p}(k, t).$$

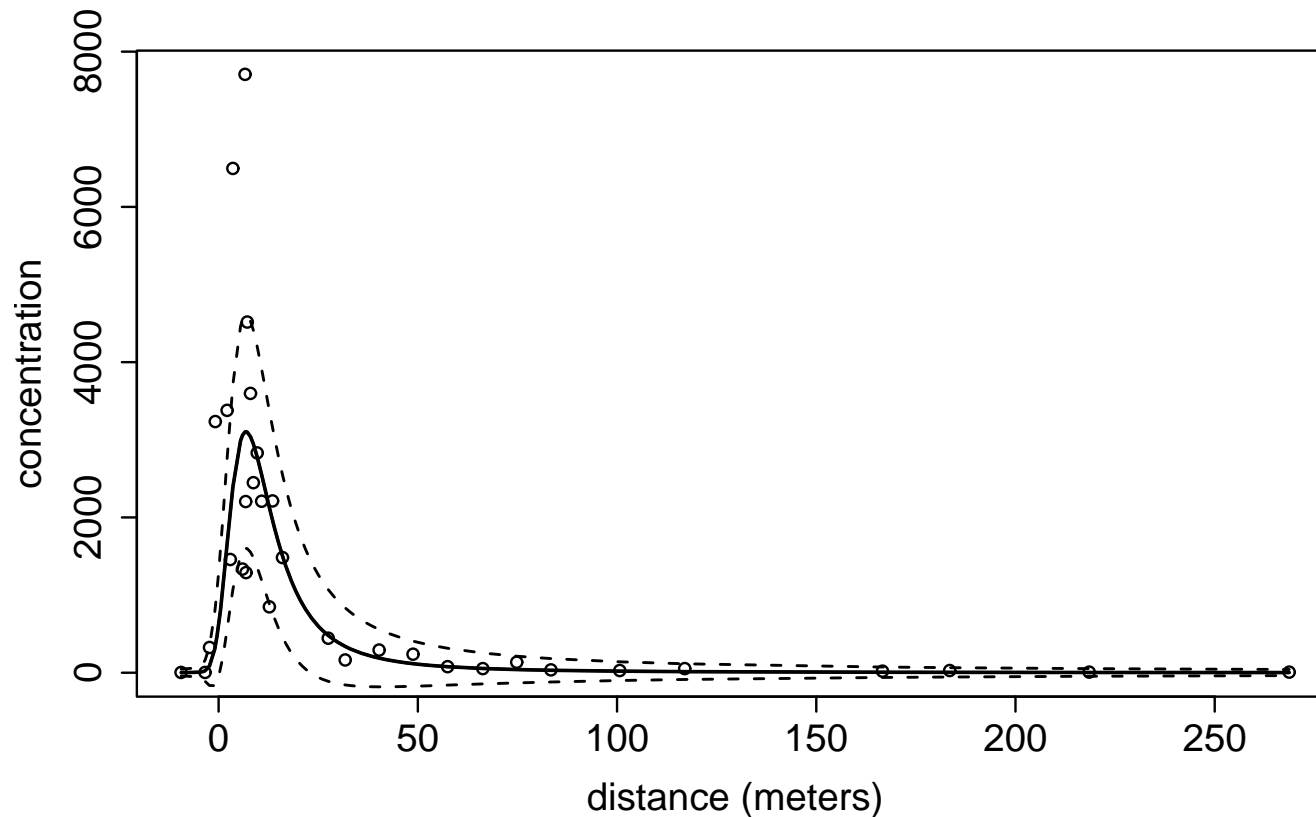
Invert the FT to get

$$\frac{\partial p(x, t)}{\partial t} = qb \frac{\partial^\alpha p(x, t)}{\partial (-x)^\alpha} + (1-q)b \frac{\partial^\alpha p(x, t)}{\partial x^\alpha}$$

The case $\alpha = 2$ is the diffusion equation for Brownian motion.

Tracer test in an underground aquifer

Positively skewed stable density $p(x, t)$ with $q = 0$ and $\alpha = 1.1$ gives a good fit. The positive skewness reflects downstream jumps attributed to high velocity channels.



The inverse subordinator

If $P(J > t) \approx t^{-\beta}$ with $0 < \beta < 1$ then

$$c^{-1/\beta} T_{[ct]} \Rightarrow D_t$$

where D_t is a β -stable subordinator.

Since $\{T_n \leq t\} = \{N_t \geq n\}$ (inverse processes) we also get $c^{-\beta} N_{[ct]} \Rightarrow E_t$ where

$$E_t = \inf\{x > 0 : D_x > t\}$$

is the right-continuous inverse of D_t . A continuous mapping argument yields

$$c^{-\beta/\alpha} S(N_{[ct]}) = (c^\beta)^{-1/\alpha} S(c^\beta \cdot c^{-\beta} N_{[ct]}) \Rightarrow A(E_t)$$

Hitting time density

The subordinator is self-similar: $D_t \stackrel{d}{=} t^{1/\beta} D_1$. Then

$$\begin{aligned} P(E_t \leq x) &= P(D_x \geq t) \\ &= P(x^{1/\beta} D_1 \geq t) \\ &= P(D_1 \geq tx^{-1/\beta}) \end{aligned}$$

Take derivatives to see that

$$h(x, t) = \frac{t}{\beta} x^{-1-1/\beta} g(tx^{-1/\beta}, 1)$$

where $g(x, t)$ is the density of D_t . A LT argument shows

$$\frac{\partial g(x, t)}{\partial t} = -\frac{\partial^\beta g(x, t)}{\partial x^\beta} \quad \text{and} \quad \frac{\partial h(x, t)}{\partial x} = -\frac{\partial^\beta h(x, t)}{\partial t^\beta}$$

CTRW governing equations

A simple conditioning argument shows that the CTRW limit $A(E_t)$ has a density

$$m(x, t) = \int p(x, u)h(u, t) du$$

where $p(x, u)$ is the density of $A(u)$ and $h(u, t)$ is the density of E_t . A Fourier-Laplace transform argument shows that

$$\frac{\partial^\beta m(x, t)}{\partial t^\beta} = qb \frac{\partial^\alpha m(x, t)}{\partial (-x)^\alpha} + (1 - q)b \frac{\partial^\alpha m(x, t)}{\partial x^\alpha}.$$

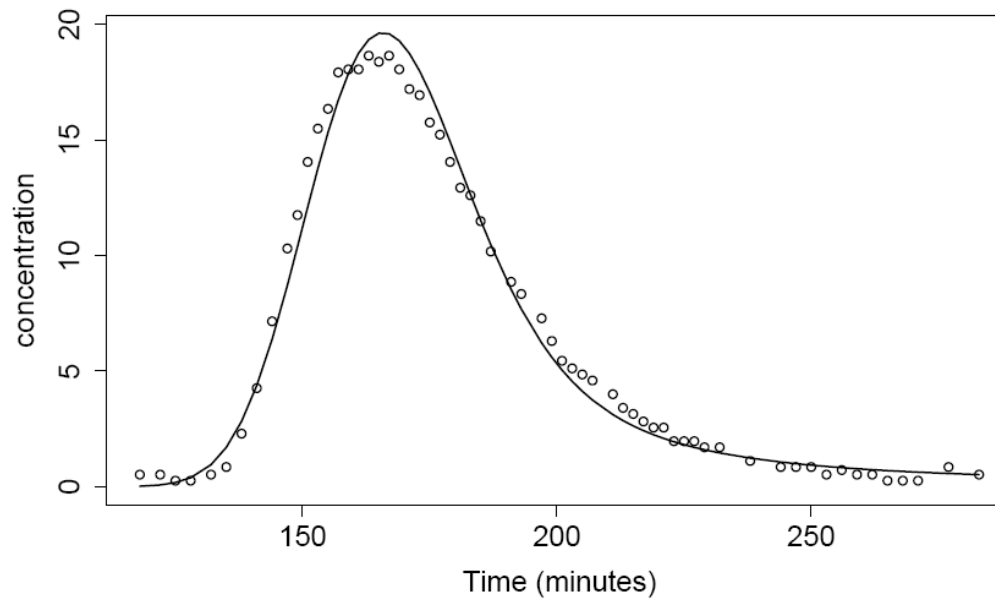
Power law jumps \Rightarrow fractional derivatives in space.

Power law waiting times \Rightarrow fractional derivatives in time.

Tracer test in the Red Cedar River

Here $\frac{\partial^\beta m(x, t)}{\partial t^\beta} = -v \frac{\partial m(x, t)}{\partial x} + b \frac{\partial^2 m(x, t)}{\partial x^2}$ with $\beta = 0.978$.

Fitted Values



Zolotarev duality

A spectrally negative α -stable $A(t)$ with $q = 1$ and $1 < \alpha \leq 2$ has series representation

$$p_\alpha(x, t) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(1 + k/\alpha)}{k!} t^{-k/\alpha} x^{k-1} \sin\left(\frac{\pi k}{\alpha}\right).$$

Substituting the hitting time density formula into the stable series representation for $\beta = (1/\alpha) < 1$ leads to

$$h(x, t) = \alpha \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(1 + k/\alpha)}{k!} t^{-k/\alpha} x^{k-1} \sin\left(\frac{\pi k}{\alpha}\right).$$

Hence $h(x, t) = \alpha p_\alpha(x, t)$ for $x > 0$, or $E_t \stackrel{d}{=} [A(t) | A(t) > 0]$.

Kotulski (1995) mentions this fact without proof.

Consequences of Zolotarev duality

The roles of space and time can be reversed.

Long negative jumps can lead to a time-fractional equation:

$$\frac{\partial h(x, t)}{\partial x} = -\frac{\partial^\beta h(x, t)}{\partial t^\beta} \quad \Rightarrow \quad \frac{\partial p_\alpha(x, t)}{\partial x} = -\frac{\partial^\beta p_\alpha(x, t)}{\partial t^\beta}$$

Long waiting times can lead to a space-fractional equation:

$$\frac{\partial p_\alpha(x, t)}{\partial t} = \frac{\partial^\alpha p_\alpha(x, t)}{\partial (-x)^\alpha} \quad \Rightarrow \quad \frac{\partial h(x, t)}{\partial t} = \frac{\partial^\alpha h(x, t)}{\partial (-x)^\alpha}$$

A controversy in hydrology

Negative skewness comes from negative jumps in the CTRW.

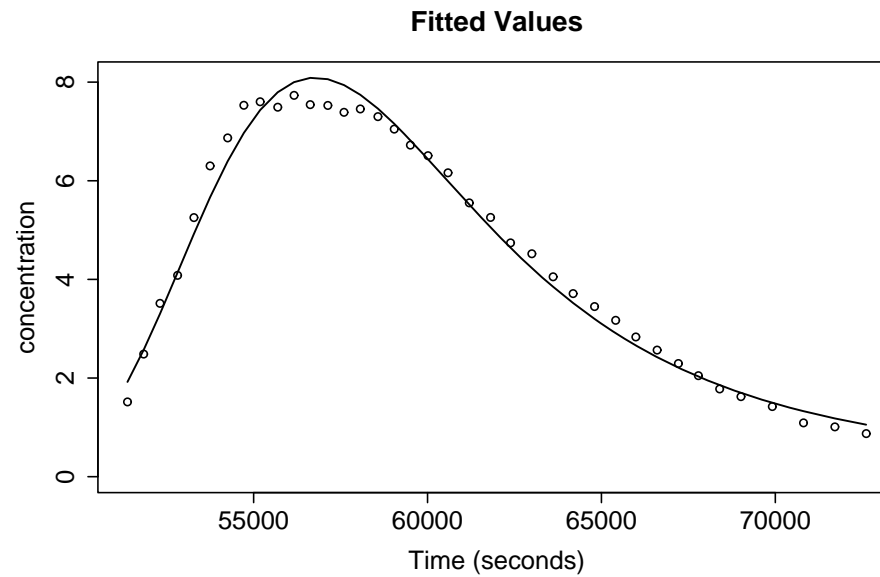
Particles cannot jump far upstream!

Duality equates negative skewness to power law waiting times.

Time-fractional model equivalent to negative space-fractional.

Tracer test in the Grand River

Negatively skewed stable fit $t \mapsto p_\alpha(x, t)$ with $\alpha = 1.38$ and $q = 1$.



Spectrally negative Lévy processes

Bertoin (1996) Theorem VII.1 p.189 implies that

$$D(x) = \inf\{t > 0 : A(t) > x\}$$

is a stable subordinator with index $\beta = 1/\alpha \in [1/2, 1)$ and

$$E_t = \inf\{x > 0 : D(x) > t\} = \sup\{A(u) : 0 \leq u \leq t\}$$

is the local time at zero of the reflected process $E_t - A(t)$.

Bertoin (1996) Corollary VII.3 p.190 implies that

$$tg(t, x) dt dx = xp_\alpha(x, t) dx dt \quad \text{for } (x, t) \in [0, \infty) \times [0, \infty).$$

Since $g(t, x) = x^{-1/\beta}g(tx^{-1/\beta}, 1)$ this also implies

$$p_\alpha(x, t) = tx^{-1-1/\beta}g(tx^{-1/\beta}, 1) = \frac{1}{\alpha}h(x, t).$$

Iterated Brownian motion

If $B(t)$ is a Brownian motion on \mathbb{R}^d independent of $A(t)$ then

$$B(E_t) \stackrel{d}{=} B(|A(t)|)$$

and the time-fractional governing equation for $\alpha = 2$

$$\frac{\partial^{1/2} m(x, t)}{\partial t^{1/2}} = \Delta m(x, t)$$

is equivalent to that of Allouba and Zheng (2001)

$$\frac{\partial}{\partial t} m(x, t) = \frac{\Delta m(x, 0)}{\sqrt{\pi t}} + \Delta^2 m(x, t)$$

where $\Delta = \partial^2 / \partial x_1^2 + \dots + \partial^2 / \partial x_d^2$.

Fractional telegraph equations

Orsingher and Beghin (2009) show that

$$\frac{\partial^\beta m(x, t)}{\partial t^\beta} = b \frac{\partial^2 m(x, t)}{\partial x^2}$$

has solution

$$m(x, t) = \frac{1}{\beta} \int_0^\infty p_2(x, u) p_{1/\beta}(u, t) du$$

which is equivalent to the CTRW solution

$$m(x, t) = \int_0^\infty p_2(x, u) h(u, t) du$$

using the duality relation $h(u, t) = \alpha p_\alpha(u, t)$ with $\alpha = 1/\beta$.

Fractional boundary value problem

The pdf $h(x, t)$ of E_t solves

$$\frac{\partial h(x, t)}{\partial t} = \frac{\partial^\alpha h(x, t)}{\partial (-x)^\alpha}$$

with boundary condition

$$\frac{\partial^{\alpha-1}}{\partial (-x)^{\alpha-1}} h(0, t) = 0.$$

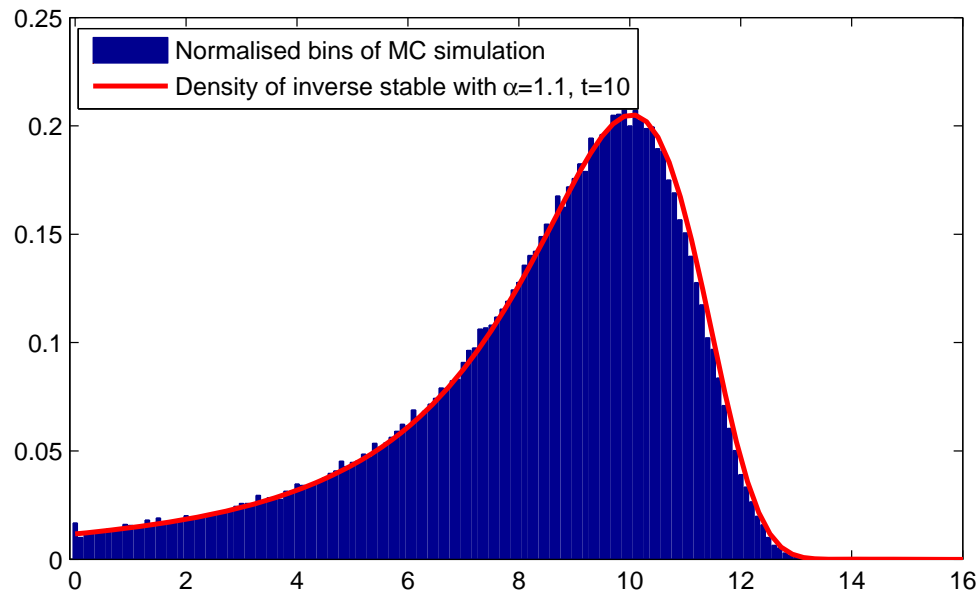
Proof: Since $\int_0^\infty p_\alpha(x, t) dx = 1/\alpha$ we must have

$$0 = \frac{\partial}{\partial t} \int_0^\infty p_\alpha(x, t) dx = \int_0^\infty \frac{\partial^\alpha}{\partial (-x)^\alpha} p_\alpha(x, t) dx = \frac{\partial^{\alpha-1}}{\partial (-x)^{\alpha-1}} p_\alpha(0, t)$$

and $h(x, t) = \alpha p_\alpha(x, t)$.

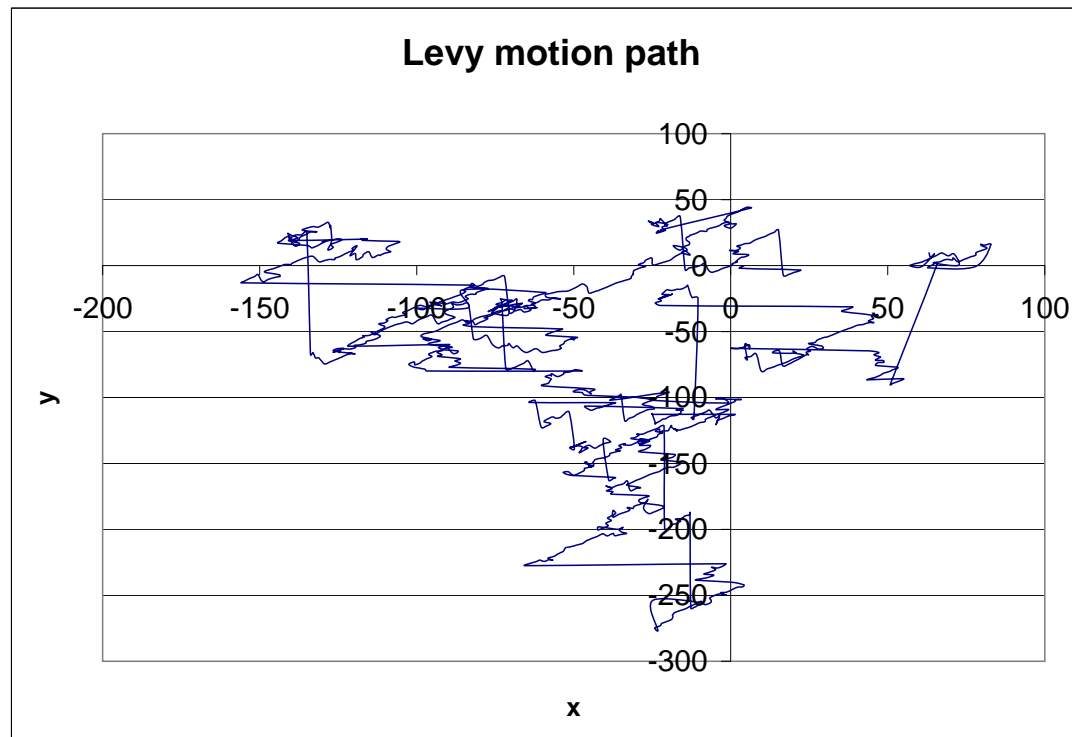
Open problem: Markovian subordinator

Is there a stable-like Markov process Z_t with the same probability density functions as $E_t = \sup\{A(u) : 0 \leq u \leq t\}$?



Open problem: Fractal properties

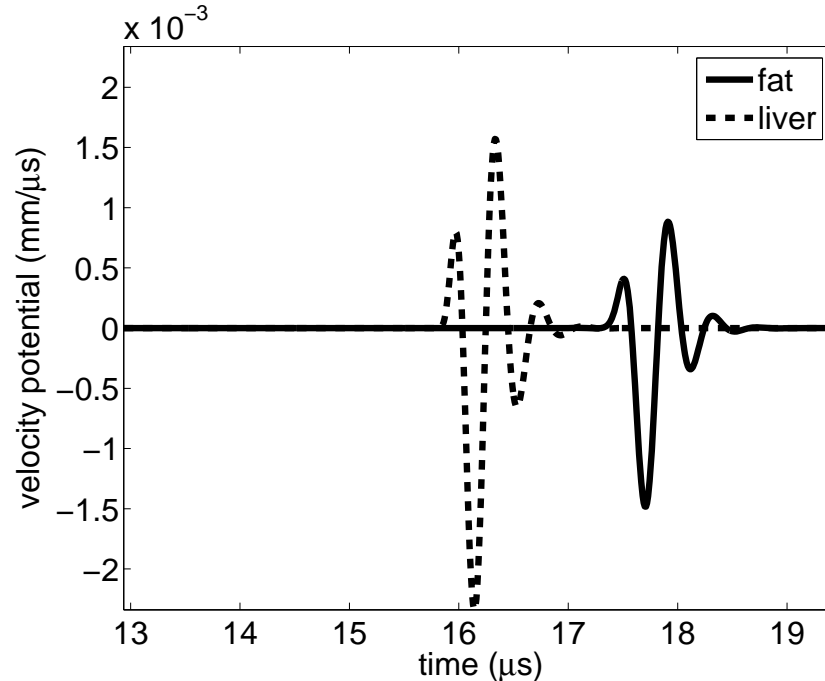
Stable Lévy sample paths are random fractals with dimension α .
What is the dimension of the graph of $A(E_t)$?



Open problem: Stochastic model for ultrasound?

Kelly et al. (2009) uses $\alpha = 1.5$ for human fat tissue and $\alpha = 1.1$ for liver tissue. Radial solution: $m(x, t) = p_\alpha(\|x\|, t)/(4\pi\|x\|)$.

$$\frac{\partial^2}{\partial t^2}m(x, t) + a\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}m(x, t) + c\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}m(x, t) = b\Delta m(x, t)$$



Open problem: Extended duality

A triangular array of jumps and waiting times leads to a CTRW limit with

$$E(e^{-ik \cdot A(t)}) = e^{-t\psi_A(k)} \quad \text{and} \quad E(e^{-sD_t}) = e^{-t\psi_D(s)}$$

where the density $m(x, t)$ of $A(E_t)$ solves

$$\psi_D(\partial_t)m(x, t) = -\psi_A(-iD_x)m(x, t) + \delta(x)\phi_D(t, \infty)$$

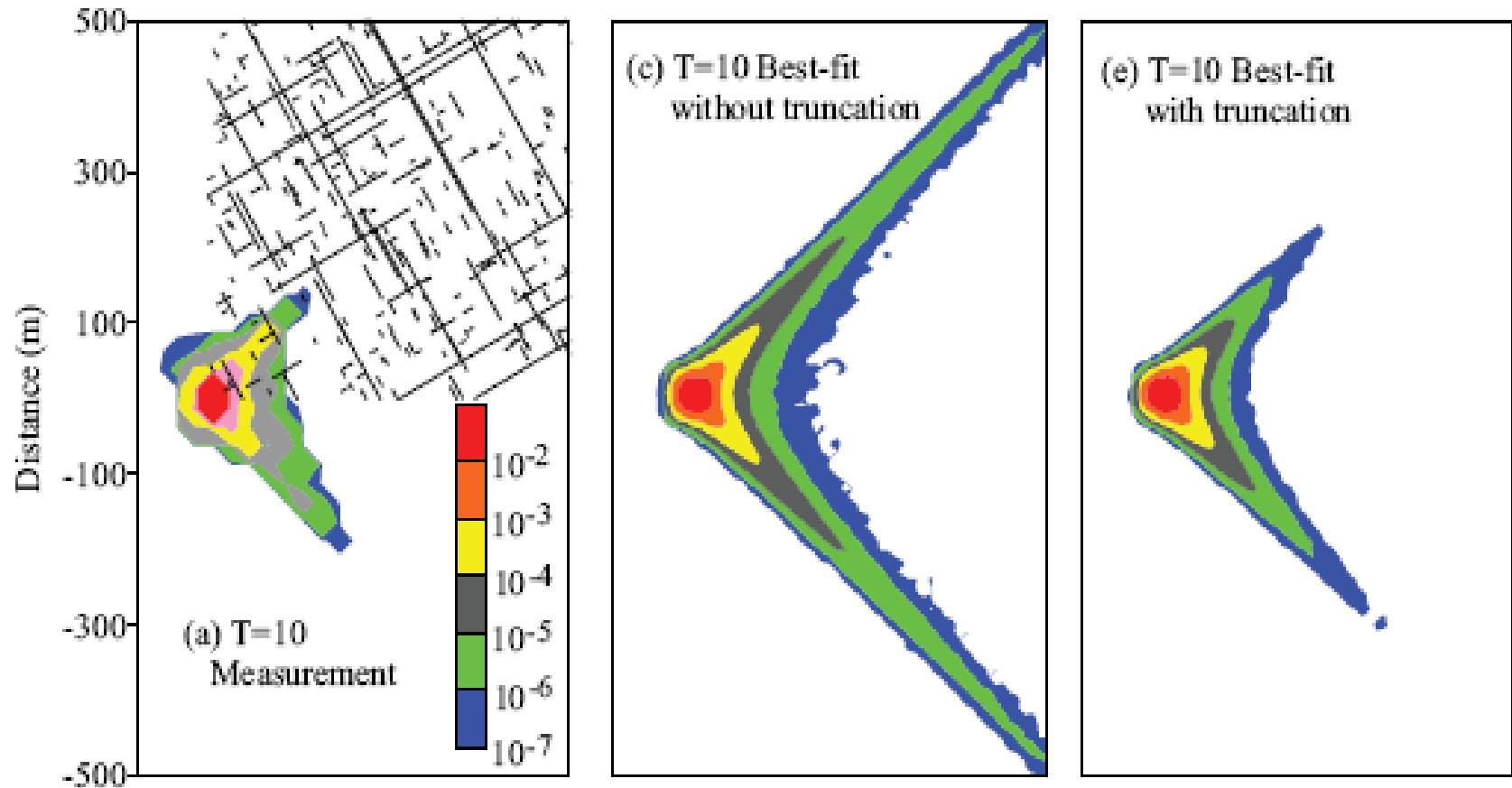
and the density $h(x, t)$ of E_t solves the governing equation

$$\partial_x h(x, t) = -\psi_D(\partial_t)h(x, t) + \delta(x)\phi_D(t, \infty).$$

Here ϕ_D is the Lévy measure of D_t .

Is there a duality result?

Tempered operator stable



Tempered operator stable model for flow in fractured rock (nuclear waste).

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Fractional derivative definitions

The Caputo fractional derivative of order $\beta \in (n, n + 1)$ is

$$\frac{d^\beta}{dt^\beta}[g(t)] = \frac{1}{\Gamma(n - \beta)} \int_0^t \frac{d^n g(u)}{du^n} \frac{du}{(t - u)^{\beta+1-n}}.$$

The Riemann-Liouville derivatives of order $\alpha \in (n, n + 1)$ are

$$\frac{d^\alpha}{dx^\alpha}[f(x)] = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(y) dy}{(x - y)^{\alpha+1-n}},$$

$$\frac{d^\alpha}{d(-x)^\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_x^\infty \frac{f(y) dy}{(y - x)^{\alpha+1-n}}.$$

Fractional derivatives of power laws

If $p > 0$ then the Laplace transform

$$\begin{aligned} \text{LT} \{t^p\} &= \int_0^{\infty} e^{-st} t^p dt && \boxed{\text{substitute } y = st} \\ &= \int_0^{\infty} e^{-y} (y/s)^p dy/s = s^{-p-1} \Gamma(p+1). \end{aligned}$$

Then

$$\begin{aligned} \text{LT} \left\{ \frac{d^\alpha}{dt^\alpha} [t^p] \right\} &= s^\alpha s^{-p-1} \Gamma(p+1) \\ &= s^{-(p-\alpha)-1} \Gamma(p-\alpha+1) \cdot \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} \\ &= \text{LT} \left\{ \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \right\} \end{aligned}$$

and the uniqueness of the LT yields

$$\frac{d^\alpha}{dt^\alpha} [t^p] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}.$$

Difference quotients

The derivative $\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} h^{-1} \Delta f(x)$ where

$$\Delta f(x) = f(x) - f(x - h).$$

For positive integers α , $\frac{d^\alpha}{dx^\alpha}f(x) = \lim_{h \rightarrow 0} h^{-\alpha} \Delta^\alpha f(x)$ where

$$\begin{aligned} \Delta^2 f(x) &= (f(x) - f(x - h)) - (f(x - h) - f(x - 2h)) \\ &= f(x) - 2f(x - h) + f(x - 2h), \end{aligned}$$

$$\Delta^3 f(x) = f(x) - 3f(x - h) + 3f(x - 2h) - f(x - 3h)$$

⋮

$$\Delta^\alpha f(x) = \sum_{m=0}^{\alpha} \binom{\alpha}{m} (-1)^m f(x - mh). \quad \text{Here } \binom{\alpha}{m} = \frac{\alpha!}{m!(\alpha - m)!}$$

Fractional difference quotients

For $\alpha > 0$ define $\frac{d^\alpha}{dx^\alpha} f(x) = \lim_{h \rightarrow 0} h^{-\alpha} \Delta^\alpha f(x)$ where

$$\Delta^\alpha f(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} (-1)^m f(x - mh), \quad \binom{\alpha}{m} = \frac{\Gamma(\alpha + 1)}{m! \Gamma(\alpha - m + 1)}$$

Since $f(x - h)$ has FT $e^{-ikh} \hat{f}(k)$, and using the Binomial formula

$$(1 + z)^\alpha = \sum_{m=0}^{\infty} \binom{\alpha}{m} z^m \quad \text{for any complex } |z| \leq 1$$

we see that $\Delta^\alpha f(x)$ has FT

$$\sum_{m=0}^{\infty} \binom{\alpha}{m} (-1)^m e^{-ikmh} \hat{f}(k) = (1 - e^{-ikh})^\alpha \hat{f}(k)$$

and then the FT of $h^{-\alpha} \Delta^\alpha f(x)$ is

$$h^{-\alpha} (ikh)^\alpha \left(\frac{1 - e^{-ikh}}{ikh} \right)^\alpha \hat{f}(k) \rightarrow (ik)^\alpha \hat{f}(k) \quad \text{as } h \rightarrow 0.$$