The Packing Measure of the Trajectories of Multiparameter Fractional Brownian Motion *

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Abstract

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a multiparameter fractional Brownian motion of index α $(0 < \alpha < 1)$ in $\mathbb{R}^d$. We prove that if $N < \alpha d$, then there exist positive finite constants K_1 and K_2 such that with probability 1,

$$K_1 \le \ \varphi\text{-}p(X([0,1]^N)) \le \ \varphi\text{-}p(\mathrm{Gr}X([0,1]^N)) \le K_2$$

where $\varphi(s) = s^{N/\alpha}/(\log\log 1/s)^{N/(2\alpha)}$, φ-$p(E)$ is the φ-packing measure of E, $X([0,1]^N)$ is the image and $\mathrm{Gr}X([0,1]^N) = \{(t, X(t));\ t \in [0,1]^N\}$ is the graph of X, respectively. We also establish liminf and limsup type laws of the iterated logarithm for the sojourn measure of X.

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1 Introduction

For a fixed $\alpha \in (0,1)$, an N-parameter fractional Brownian motion of index α in $\mathbb{R}$ is a centered, real-valued Gaussian random field $Y = \{Y(t),\ t \in \mathbb{R}^N\}$ with $Y(0) = 0$ and covariance function

$$\mathbb{E}(Y(t)Y(s)) = \frac{1}{2}\Big(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\Big),$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$ which is endowed with the usual inner product $\langle t, x\rangle$. There is a very useful stochastic integral representation for Y. Such a representation is based upon the fact that for each $t \in \mathbb{R}^N$

$$|t|^{2\alpha} = c^2 \int_{\mathbb{R}^N} \Big(1 - \cos\langle t, x\rangle\Big) \frac{dx}{|x|^{2\alpha+N}},$$

where $c = c(\alpha, N) > 0$ is a normalizing constant depending on α and N only. Let m be a scattered Gaussian random measure on $\mathbb{R}^N$ with Lebesgue measure λ_N as its control measure, that is, $\{m(A),\ A \in \mathcal{E}\}$ is a centered Gaussian process on $\mathcal{E} = \{E \subset \mathbb{R}^N : \lambda_N(E) < \infty\}$ with covariance function

$$\mathbb{E}\Big(m(E)m(F)\Big) = \lambda_N(E \cap F).$$

Let m' be an independent copy of m. Then it is easy to verify that $Y = \{Y(t),\ t \in \mathbb{R}^N\}$ has the following stochastic integral representation

$$Y(t) = \frac{c}{\sqrt{2}} \int_{\mathbb{R}^N} (1 - \cos\langle t, x\rangle) \frac{dm(x)}{|x|^{\alpha+\frac{N}{2}}} + \frac{c}{\sqrt{2}} \int_{\mathbb{R}^N} \sin\langle t, x\rangle \frac{dm'(x)}{|x|^{\alpha+\frac{N}{2}}}, \quad t \in \mathbb{R}^N. \tag{1.1}$$

We refer to Samorodnitsky and Taqqu [17] for other representations and more properties of fractional Brownian motion.

Associated with the real-valued Gaussian field Y, we define a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ by

$$X(t) = (X_1(t), \ldots, X_d(t)),$$

where $X_1, \cdots, X_d$ are independent copies of Y. Using the terminology of Kahane ([4], chapter 18), we call X the (N, d, α) process or an N-parameter fractional Brownian motion of index α in $\mathbb{R}^d$. It is easy to see that X is self-similar with exponent α in the sense that for any $a > 0$,

$$X(a\ \cdot) \stackrel{\mathrm{d}}{=} a^\alpha X(\cdot), \tag{1.2}$$

and has stationary increments, that is, for every $b \in \mathbb{R}^N$

$$X(\cdot + b) - X(b) \stackrel{d}{=} X(\cdot) - X(0), \tag{1.3}$$

where $X \overset{\mathrm{d}}{=} Y$ means that the two processes X and Y have the same finite dimensional distributions.

For any Borel set $E \subseteq \mathbb{R}^N$, the image $X(E) = \{X(t);\ t \in E\}$ and graph set $\mathrm{Gr}X(E) = \{(t, X(t));\ t \in E\}$ of fractional Brownian motion are random fractals. It is well known that with probability one,

$$\dim_{\mathrm{P}} X([0,1]^N) = \min\{d;\ \frac{N}{\alpha}\},$$

$$\dim_{\mathrm{P}} \mathrm{Gr}X([0,1]^N) = \begin{cases} \min\left\{d;\ \frac{N}{\alpha}\right\}, & \text{if } \ N \le \alpha d, \\ N + (1-\alpha)d, & \text{if } \ N > \alpha d. \end{cases}$$

where $\dim_{\mathrm{P}} E$ is the packing dimension of E (See Section 2 for definition and basic properties).

There has been a lot of interest in studying the exact packing measure of the image and graph of Brownian motion. See Taylor and Tricot [22], LeGall and Taylor [7], Rezakhanlou and Taylor [15]. Many of these results have been extended to Lévy processes by Taylor [20], Fristedt and Taylor [2]. Their methods rely heavily on special properties of Lévy processes such as the independence of increments, hence can not be applied to calculate the packing measure of the sample paths of non-Markovian processes. Taylor ([21], p.392) raised the question of finding the exact packing measure for the sample paths of fractional Brownian motion. By applying general Gaussian methods and by direct conditioning, Xiao [23] solved the exact packing measure problem for the image of one-parameter transient fractional Brownian motion. However, some key arguments in Xiao [23] such as the proofs of Theorem 3.2, Lemma 4.1 and Theorem 4.1 depend on the fact that t is one-dimensional and they do not work in the multiparameter case. Hence the packing measure problems for the image and graph of multiparameter fractional Brownian motion had remained open. The main objective of this paper is to prove that, in the transient case (that is, $N < \alpha d$), there exist positive constants K_1 and K_2, such that

$$K_1 \le \varphi\text{-}p(X([0,1]^N)) \le \varphi\text{-}p(\mathrm{Gr}X([0,1]^N)) \le K_2, \quad \text{a.s.} \tag{1.4}$$

where $\varphi(s) = s^{N/\alpha}/(\log\log 1/s)^{N/(2\alpha)}$ and φ-p is the φ-packing measure. For this purpose, we develop a more general conditioning argument which may also be useful in other circumstances.

The following are some remarks about the other cases that are not addressed in this paper. If $N > \alpha d$, then X is recurrent and it has a continuous local time (see Pitt [12], or Kahane ([4], Ch.18, Theorem 2). This implies that $X([0,1]^N)$ a.s. contains interior points, hence $0 < s^d\text{-}p(X([0,1]^N)) < \infty$ a.s. However, for the graph set, we have $\dim_{\mathrm{P}} \mathrm{Gr}X([0,1]^N) = N + (1-\alpha)d$ a.s.. It would be interesting to determine the exact packing measure function for $\mathrm{Gr}X([0,1]^N)$. In the critical case of $N = \alpha d$, the problems of finding the packing measure of $X([0,1]^N)$ and

$\mathrm{Gr}X([0,1]^N)$ are both open in general. The only known result is due to LeGall and Taylor [7] who proved that if X is planar Brownian motion, then φ-$p(X([0,1]))$ is either zero or infinite.

The rest of the paper is organized as follows. In Section 2, we collect some definitions and lemmas which will be useful to our calculations. In Section 3, we prove liminf and limsup type laws of the iterated logarithm for the sojourn time of a transient fractional Brownian motion. Besides of their applications in determining the fractal measures of the sample paths of fractional Brownian motion, these results also suggest that the sojourn measure of fractional Brownian motion has a non-trivial logarithmic multifractal structure. It would be interesting to determine its multifractal spectrum. In Section 4, we consider the packing measure of the image and graph of fractional Brownian motion and prove (1.4).

We will use K to denote unspecified positive and finite constants whose value may be different in each occurrence. Constants that are referred to in the sequel will be denoted by $K_1, K_2, \ldots, K_{20}$.

2 Preliminaries

We start by recalling the definitions of packing measure and packing dimension which were introduced by Taylor and Tricot [22] as dual concepts to Hausdorff measure and Hausdorff dimension. See also Falconer [1] or Mattila [11] for more information. Let Φ be the class of functions $\varphi : (0, \delta) \to (0, 1)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K_3 > 0$ for which

$$\frac{\varphi(2s)}{\varphi(s)} \leq K_3 \qquad \text{for } 0 < s < \delta/2. \tag{2.1}$$

For $\varphi \in \Phi$, define the φ-packing premeasure φ-$P(E)$ on $\mathbb{R}^N$ by

$$\varphi\text{-}P(E) = \lim_{\epsilon \to 0} \sup\Big\{\sum_i \varphi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \epsilon\Big\}, \tag{2.2}$$

where $B(x,r)$ denotes the open ball of radius r centered at x and $\overline{B}(x,r)$ is its closure. A sequence of closed balls satisfying the conditions in the right hand side of (2.2) is called an ϵ-packing of E. The φ-packing measure, denoted by φ-p, on $\mathbb{R}^N$ is defined by

$$\varphi\text{-}p(E) = \inf\Big\{\sum_n \varphi\text{-}P(E_n) : \quad E \subseteq \bigcup_n E_n\Big\}. \tag{2.3}$$

It is known that φ-p is a metric outer measure and hence every Borel set in $\mathbb{R}^N$ is φ-p measurable. If $\varphi(s) = s^\alpha$, s^α-$p(E)$ is called the α-dimensional packing measure of E. The packing dimension of E is defined by

$$\dim_{\mathrm{P}} E = \inf\{\alpha > 0 : \quad s^{\alpha}\text{-}p(E) = 0\}.$$

By (2.3), we see that, for any $E \subset \mathbb{R}^N$,

$$\varphi\text{-}p(E) \le \varphi\text{-}P(E), \tag{2.4}$$

which gives a way to determine the upper bound for φ-$p(E)$. The following density theorem for packing measures (see Taylor and Tricot [22] and Saint Raymond and Tricot [16] for a proof) is very useful in determining the lower bound of φ-$p(E)$.

Lemma 2.1 *Let μ be a Borel measure on $\mathbb{R}^N$ and $\varphi \in \Phi$. Then for any Borel set $E \subseteq \mathbb{R}^N$,*

$$\varphi\text{-}p(E) \ge K_3^{-3} \mu(E) \inf_{x \in E} \{\underline{D}_{\mu}^{\varphi}(x)\}^{-1},$$

where K_3 is the constant in (2.1) and

$$\underline{D}_{\mu}^{\varphi}(x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{\varphi(2r)}$$

is the lower φ-density of μ at x.

Now we collect some general facts about Gaussian processes. Let $Y = \{Y(t),\ t \in S\}$ be a centered Gaussian process. We define a pseudo-metric d on S by

$$d(s,t) = \|Y(s) - Y(t)\|_2 := (E(Y(s) - Y(t))^2)^{1/2}.$$

Denote by $N_d(S, \epsilon)$ the smallest number of open d-balls of radius ϵ needed to cover S, and write $D = \sup\{d(s,t);\ s,\ t \in S\}$ for the d-diameter of S.

Lemma 2.2 below is well known. It is a consequence of the Gaussian isoperimetric inequality and Dudley's entropy bound (cf. Ledoux and Talagrand [6], or Talagrand [18]).

Lemma 2.2 *There exists an absolute constant $K > 0$ such that for any $u > 0$, we have*

$$\mathbb{P}\Big\{\sup_{s,t \in S} |Y(s) - Y(t)| \ge K\Big(u + \int_0^D \sqrt{\log N_d(S,\epsilon)}\, d\epsilon\Big)\Big\} \le \exp\Big(-\frac{u^2}{D^2}\Big).$$

The first part of the following lemma is a corollary of Lemma 2.2 and the second part was proved by Marcus [9] for $N = 1$. Extension to the case of $N > 1$ is immediate.

Lemma 2.3 *Let $Y = \{Y(t),\ t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $Y(0) = 0$. If there exist constants $0 < \alpha < 1$ and $\sigma > 0$ such that*

$$\mathbb{E}(Y(t) - Y(s))^2 \le \sigma^2 |t - s|^{2\alpha},$$

then there exist finite constants $K,\ K_4 > 0$ such that for any $r > 0$, any hypercube $I \subset \mathbb{R}^N$ with edge length r and any $u \ge K r^\alpha$, we have

$$\mathbb{P}\Big\{\sup_{s,t\in I} |Y(t) - Y(s)| \ge \sigma u\Big\} \le \exp\Big(-\frac{u^2}{K_4 r^{2\alpha}}\Big) \tag{2.5}$$

and

$$\limsup_{h\to 0} \sup_{\substack{t,t+s\in[0,1]^N\\ |s|\le h}} \frac{|Y(t+s) - Y(t)|}{\sigma h^\alpha (\log 1/h)^{1/2}} \le \sqrt{K_4}, \quad a.s.. \tag{2.6}$$

The following delayed hitting probability estimate for multiparameter fractional Brownian motion is a special case of Theorem 5.2 of Mason and Xiao [10], which extends the corresponding result of Xiao [23] in the one-parameter case. A similar lower bound follows from the arguments in Xiao [26].

Lemma 2.4 *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a d-dimensional fractional Brownian motion of index α $(0 < \alpha < 1)$ with $N < \alpha d$. Then for any $T > 0$ and any $0 < r < T^\alpha$, we have*

$$\mathbb{P}\Big\{\exists\ t \in \mathbb{R}^N \text{ such that } |t| > T \text{ and } |X(t)| < r\Big\} \le K\Big(\frac{r}{T^\alpha}\Big)^{d-N/\alpha},$$

where $K > 0$ is a constant depending on α, N and d only.

Remark 2.5 It is an open problem to estimate the hitting probability for fractional Brownian motion in the case of $N = \alpha d$. Classical results for planar Brownian motion can be found in Port and Stone [14].

We end this section with the following Borel-Cantelli lemma. Part (i) is well known and Part (ii) in this form is from Talagrand [19]. See also Marcus [9].

Lemma 2.6 *Let $\{A_k\}$ be a sequence of events in a probability space.*

(i) *if $\sum_{k=1}^\infty \mathbb{P}(A_k) < \infty$, then $\mathbb{P}(\limsup_{k\to\infty} A_k) = 0$.*

(ii) if there exist positive constants $K,\ \epsilon$ and positive integers $k_0,\ J$ such that for $k_0 \le k < J$,

$$\sum_{j=k+1}^{J} \mathbb{P}(A_k \cap A_j) \le \mathbb{P}(A_k)\Big(K + (1+\epsilon) \sum_{j=k+1}^{J} \mathbb{P}(A_j)\Big) \tag{2.7}$$

and

$$\sum_{k=k_0}^{J} \mathbb{P}(A_k) \ge \frac{1+2K}{\epsilon}, \tag{2.8}$$

then

$$\mathbb{P}\Big\{\bigcup_{k=k_0}^{J} A_k\Big\} \ge \frac{1}{1+2\epsilon}\ .$$

Remark 2.7 If $\sum_k \mathbb{P}(A_k) = \infty$, then for any $\epsilon > 0$ and any fixed integer $k_0 \ge 1$, we can take J large enough so that (2.8) holds. Hence only (2.7) needs to be verified.

3 Limit Theorems for the Sojourn Time

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a fractional Brownian motion in $\mathbb{R}^d$ with index $\alpha \in (0,1)$. We assume $N < \alpha d$, hence X is transient in the sense that $\lim_{t\to\infty} |X(t)| = \infty$ a.s. (cf. Kôno [5], Theorem 10). For any $r > 0$ and any $y \in \mathbb{R}^d$, let

$$T_y(r) = \int_{\mathbb{R}^N} \mathbb{1}_{B(y,r)}(X(t))dt$$

be the sojourn time of $X(t)$ ($t \in \mathbb{R}^N$) in $B(y,r)$. If $y = 0$, we denote $T_y(r)$ by $T(r)$. It follows from the self-similarity of $X(t)$ (cf. (1.2)) that $T(r)$ has the following scaling property: for any $a > 0$ and $r > 0$

$$T(ar) \stackrel{\mathrm{d}}{=} a^{N/\alpha} T(r). \tag{3.1}$$

We also need to make use of the "truncated" sojourn time

$$T(b,\tau) = \int_{|t|\le\tau} \mathbb{1}_{B(0,b)}(X(t))dt. \tag{3.2}$$

In this section, we prove liminf and limsup type laws of the iterated logarithm for the sojourn time $T(r)$. Combined with Lemma 2.1, the liminf theorem will be applied in Section 4 to prove the lower bound in (1.4). The limsup result is related to the Hausdorff measure of the sample paths of fractional Brownian motion. See Talagrand [18] or Xiao [24].

We will first prove some hitting probability estimates for Gaussian random fields under more general conditions than those in Xiao [23].

Let $Z_0 = \{Z_0(t),\ t \in \mathbb{R}^N\}$ be a real valued Gaussian random field with $Z_0(0) = 0$. Denote

$$\sigma^2(t,s) = \mathbb{E}(Z_0(t) - Z_0(s))^2\ , \quad \sigma^2(t) = \mathbb{E}(Z_0(t))^2.$$

We consider an $\mathbb{R}^d$-valued Gaussian field $Z = \{Z(t), t \in \mathbb{R}^N\}$ defined by

$$Z(t) = (Z_1(t), \cdots, Z_d(t)),$$

where $Z_1, \ldots, Z_d$ are independent copies of Z_0.

For any $a > 0$, let $S = \{t \in \mathbb{R}^N : a \leq |t| \leq 2a\}$. We assume that the random field Z_0 and a function $c(t) : \mathbb{R}^N \mapsto \mathbb{R}$ satisfy the following conditions: there exist positive and finite constants δ_0, η and K such that for all $t \in S$,

$$K^{-1}|t|^{2\alpha} \leq \sigma^2(t) \leq K|t|^{2\alpha}, \tag{3.3}$$

$$K^{-1} \leq |c(t)| \leq K \tag{3.4}$$

and for all $s,\ t \in S$ with $|s - t| \leq \delta_0$,

$$K^{-1}|t-s|^{2\alpha} \leq \sigma^2(t,s) \leq K|t-s|^{2\alpha}, \tag{3.5}$$

$$|c(t) - c(s)| \leq K|t-s|^{\alpha+\eta}. \tag{3.6}$$

Lemma 3.1 *Suppose that the Gaussian random field Z and the function $c(t)$ satisfy the conditions (3.3)–(3.6) on S. If $N < \alpha d$, then there exist positive constants K_5, K_6 and K_7, depending on $\alpha,\ d,\ N$ only, such that for all $r > 0$ and all $y \in \mathbb{R}^d$ with $|y| \geq K_5 r$, we have*

(i) if $a \geq r^{1/\alpha}$, then

$$\mathbb{P}\Big\{\exists t \in S \quad \text{such that} \quad |Z(t) - c(t)y| < r\Big\} \leq K_6 \exp\Big(-\frac{|y|^2}{K_7 a^{2\alpha}}\Big)\ \Big(\frac{r}{a^\alpha}\Big)^{d-N/\alpha}; \tag{3.7}$$

(ii) if $a \leq r^{1/\alpha}$, then

$$\mathbb{P}\Big\{\exists t \in S \quad \text{such that} \quad |Z(t) - c(t)y| < r\Big\} \leq K_6 \exp\Big(-\frac{|y|^2}{K_7 a^{2\alpha}}\Big). \tag{3.8}$$

Remark 3.2 We note that the constants K_5, K_6 and K_7 are independent of a. This is important when we apply Lemma 3.1 to finish the proof of Lemma 3.4 below.

Proof of Lemma 3.1 Even though the proof follows a similar line to that of Lemma 3.1 in Xiao [23], several technical modifications have to be made. For the convenience of the reader, we include it here.

We prove Part (i) first. Let a, $r > 0$ be fixed and $a \geq r^{1/\alpha}$. Denote by $N(S, r^{1/\alpha})$ the smallest number of open balls of radius $r^{1/\alpha}$ that are needed to cover S. Then

$$N(S, r^{1/\alpha}) \leq K\ a^N\ r^{-N/\alpha}. \tag{3.9}$$

Let $\{S_p,\ 1 \leq p \leq N(S, r^{1/\alpha})\}$ be a family of balls of radius $r^{1/\alpha}$ that cover S. We define the following events

$$A = \Big\{ \inf_{t \in S} |Z(t) - c(t)y| < r \Big\},$$

$$A_p = \Big\{ \inf_{t \in S_p} |Z(t) - c(t)y| < r \Big\}.$$

Then

$$A \subseteq \bigcup_{p=1}^{N(S, r^{1/\alpha})} A_p. \tag{3.10}$$

Let $b = \max\{2, a^\alpha, \log(r^{1/\alpha}/\delta_0)\}$ and, for every integer $n \geq 1$, let

$$\epsilon_n = r^{1/\alpha} \exp(-b^{n+1}).$$

Then $\epsilon_n \leq \delta_0$ for all $n \geq 0$. Set

$$r_n = \beta\, d\, \epsilon_n^\alpha\, b^{\frac{n+1}{2}},$$

where $\beta \geq K_4 + 1$ is a constant to be determined later (recall that K_4 is the constant in (2.5)). It is easy to verify that there is a constant K_8 such that

$$r + \sum_{k=n}^{\infty} r_k \leq K_8\, r.$$

Consequently, we can find a finite constant K_5 with the following property: if y, $u \in \mathbb{R}^d$, $|y| \geq K_5 r$ and $|u - c(t)y| \leq K_8 r$ for some $t \in S$, then $|u| \geq \frac{1}{2}|y|$. This fact will be used in (3.19) and (3.20) below.

Now, we fix $y \in \mathbb{R}^d$ with $|y| \geq K_5\, r$ and define

$$n_0 = \inf\{n \geq 0 :\ K\epsilon_n^\eta\ |y| \leq a^\alpha\}, \tag{3.11}$$

where K is the constant in (3.6). If no such n exists, we let $n_0 = 0$.

Let $1 \leq p \leq N(S, r^{1/\alpha})$ be fixed. For every integer $n \geq 1$, let $T_n = \{t_i^{(n)},\ 1 \leq i \leq N(S_p, \epsilon_n)\}$ be a set of the centers of open balls with radius ϵ_n that cover S_p. Let

$$A^{(n_0)} = \bigcup_{i=1}^{N(S_p,\epsilon_{n_0})} \Big\{ |Z(t_i^{(n_0)}) - c(t_i^{(n_0)})y| < r + \sum_{k=n_0}^{\infty} r_k \Big\}.$$

For every $n > n_0$ and $1 \leq i \leq N(S_p, \epsilon_n)$, we define the following events

$$A_i^{(n)} = \Big\{ |Z(t_i^{(n)}) - c(t_i^{(n)})y| < r + \sum_{k=n}^{\infty} r_k \Big\} \tag{3.12}$$

and

$$A^{(n)} = A^{(n-1)} \bigcup \bigcup_{i=1}^{N(S_p,\epsilon_n)} A_i^{(n)}. \tag{3.13}$$

Clearly, $\{A^{(n)}\}$ is a sequence of increasing events. We claim that

$$\mathbb{P}(A_p) \leq \lim_{n\to\infty} \mathbb{P}(A^{(n)}). \tag{3.14}$$

To see this, we assume that for some $s_0 \in S_p$, $|Z(s_0) - c(s_0)y| < r$. Then there is a sequence of points $\{s^{(n)}\}$ such that $s^{(n)} \in T_n$, $s^{(n)} \to s_0$ and $|s^{(n)} - s^{(n+1)}| \leq 2\epsilon_n$. The triangle inequality yields that for all $n > n_0$,

$$|Z(s^{(n)}) - c(s^{(n)})y| < r + |Z(s^{(n)}) - Z(s_0)| + |y| \cdot |c(s^{(n)}) - c(s_0)|.$$

Hence (3.14) follows from (2.5) in Lemma 2.3, (3.6), and the fact that $\epsilon_n^{\eta}\,|y| \to 0$ as $n \to \infty$.

It follows from (3.13) that

$$\mathbb{P}(A^{(n)}) \leq \mathbb{P}(A^{(n-1)}) + \mathbb{P}(A^{(n)} \backslash A^{(n-1)}) \tag{3.15}$$

and

$$\mathbb{P}(A^{(n)} \backslash A^{(n-1)}) \leq \sum_{i=1}^{N(S_p,\epsilon_n)} \mathbb{P}(A_i^{(n)} \backslash A_{i'}^{(n-1)}), \tag{3.16}$$

where i' is chosen so that $|t_i^{(n)} - t_{i'}^{(n-1)}| < \epsilon_{n-1}$. Note that

$$\begin{aligned}
\mathbb{P}(A_i^{(n)} \backslash A_{i'}^{(n-1)}) &= \mathbb{P}\Big\{ |Z(t_i^{(n)}) - c(t_i^{(n)})y| < r + \sum_{k=n}^{\infty} r_k\,, \\
&\qquad\qquad |Z(t_{i'}^{(n-1)}) - c(t_{i'}^{(n-1)})y| > r + \sum_{k=n-1}^{\infty} r_k \Big\} \\
&\leq \mathbb{P}\Big\{ |Z(t_i^{(n)}) - c(t_i^{(n)})y| < r + \sum_{k=n}^{\infty} r_k\,, \\
&\qquad\qquad |Z(t_i^{(n)}) - Z(t_{i'}^{(n-1)}) + (c(t_{i'}^{(n-1)}) - c(t_i^{(n)}))\,y| \geq r_{n-1} \Big\}.
\end{aligned}$$

By the elementary properties of Gaussian random variables, we can write

$$\frac{Z(t_i^{(n)}) - Z(t_{i'}^{(n-1)})}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)})} = \rho \ \frac{Z(t_i^{(n)})}{\sigma(t_i^{(n)})} + \sqrt{1-\rho^2}\ \Xi, \tag{3.17}$$

where

$$\rho = \frac{\mathbb{E}\Big[(Z(t_i^{(n)}) - Z(t_{i'}^{(n-1)}))Z(t_i^{(n)})\Big]}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)})\sigma(t_i^{(n)})}$$

and where Ξ is a standard Gaussian vector and is independent of $Z(t_i^{(n)})$.

It follows from (3.17) and the triangle inequality that $\mathbb{P}(A_i^{(n)} \backslash A_{i'}^{(n-1)})$ is at most

$$\begin{aligned}
&\mathbb{P}\Big\{|Z(t_i^{(n)}) - c(t_i^{(n)})y| < K_8\ r,\ |\Xi| \geq \frac{\beta d}{2}\ b^{\frac{n}{2}}\Big\} \\
&\quad + \mathbb{P}\Big\{|Z(t_i^{(n)}) - c(t_i^{(n)})y| < K_8\ r,\ \Big|\rho\ \frac{Z(t_i^{(n)})}{\sigma(t_i^{(n)})} - \frac{c(t_{i'}^{(n-1)}) - c(t_i^{(n)})}{\sigma(t_{i'}^{(n-1)}, t_i^{(n)})} \cdot y\Big| \geq \frac{\beta d}{2}\ b^{\frac{n}{2}}\Big\} \\
&\quad \hat{=} I_1 + I_2.
\end{aligned} \tag{3.18}$$

By the independence of Ξ and $Z(t_i^{(n)})$, we have

$$\begin{aligned}
I_1 &= \mathbb{P}\Big\{|Z(t_i^{(n)}) - c(t_i^{(n)})y| \leq K_8 r\ \Big\} \cdot \mathbb{P}\Big\{|\Xi| \geq \frac{\beta d}{2}\ b^{\frac{n}{2}}\Big\} \\
&= \int_{\{|u-c(t_i^{(n)})y| \leq K_8 r\}} \frac{1}{(2\pi)^{d/2}\sigma^d(t_i^{(n)})}\ \exp\Big(-\frac{|u|^2}{2\sigma^2(t_i^{(n)})}\Big) du \cdot \mathbb{P}\Big\{|\Xi| \geq \frac{\beta d}{2}\ b^{\frac{n}{2}}\Big\} \\
&\leq K\ \exp\Big(-\frac{|y|^2}{K\sigma^2(t_i^{(n)})}\Big)\Big(\frac{r}{\sigma(t_i^{(n)})}\Big)^d \cdot \mathbb{P}\Big\{|\Xi| \geq \frac{\beta d}{2}\ b^{\frac{n}{2}}\Big\} \quad (\text{ recall } |y| \geq K_5 r) \\
&\leq K\ \exp\Big(-\frac{|y|^2}{K(2a)^{2\alpha}}\Big)\Big(\frac{r}{a^\alpha}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{16} b^n\Big),
\end{aligned} \tag{3.19}$$

where the last inequality follows from (3.3) and the tail probability of the standard Gaussian vector. On the other hand, noting that (3.5), (3.6) and (3.11) imply for $n \geq n_0 + 1$,

$$\begin{aligned}
\frac{|c(t_{i'}^{(n-1)}) - c(t_i^{(n)})|}{\sigma(t_{i'}^{(n-1)}, t_i^{(n)})} \cdot |y| &\leq K\ \frac{|c(t_{i'}^{(n-1)}) - c(t_i^{(n)})|}{|t_{i'}^{(n-1)} - t_i^{(n)}|^\alpha} \cdot |y| \\
&\leq K\ \epsilon_{n-1}^\eta\, |y| \leq a^\alpha,
\end{aligned}$$

we have

$$I_2 \leq \mathbb{P}\Big\{|Z(t_i^{(n)}) - c(t_i^{(n)})y| < K_8\ r,\ \Big|\frac{Z(t_i^{(n)})}{\sigma(t_i^{(n)})}\Big| \geq \frac{\beta d}{2}\ b^{\frac{n}{2}}\Big\}$$

$$
\begin{aligned}
&= \int_{\{|u-c(t_i^{(n)})y|\le K_8 r,\ |u|\ge \frac{\beta d}{2} b^{n/2}\sigma(t_i^{(n)})\}} \Big(\frac{1}{2\pi}\Big)^{\frac{d}{2}} \frac{1}{\sigma^d(t_i^{(n)})} \exp\Big(-\frac{|u|^2}{2\sigma^2(t_i^{(n)})}\Big) du \\
&\le K \int_{\{|u-c(t_i^{(n)})y|\le K_8 r\}} \frac{1}{\sigma^d(t_i^{(n)})} \exp\Big(-\frac{|u|^2}{4\sigma^2(t_i^{(n)})}\Big) du \cdot \exp\Big(-\frac{(\beta d)^2}{16}\ b^n\Big) \\
&\le K\ \exp\Big(-\frac{|y|^2}{8\sigma^2(t_i^{(n)})}\Big)\Big(\frac{r}{\sigma(t_i^{(n)})}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{16}\ b^n\Big) \quad (\text{if } |y|\ge K_5 r) \\
&\le K\ \exp\Big(-\frac{|y|^2}{K(2a)^{2\alpha}}\Big)\ \Big(\frac{r}{a^\alpha}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{16}\ b^n\Big). \qquad (3.20)
\end{aligned}
$$

Combining (3.19) and (3.20), we obtain that for $|y| \ge K_5\, r$,

$$
\mathbb{P}(A_i^{(n)} \backslash A_{i'}^{(n-1)}) \le K_9 \exp\Big(-\frac{|y|^2}{K_{10}\ a^{2\alpha}}\Big)\ \Big(\frac{r}{a^\alpha}\Big)^d \cdot \exp\Big(-\frac{(\beta d)^2}{16}\ b^n\Big), \qquad (3.21)
$$

where K_9 and K_{10} are positive and finite constants.

We choose the constant $\beta \ge K_4+1$ such that $(\beta d)^2/16 > bN+1$. Inequalities (3.15), (3.16) and (3.21) imply

$$
\begin{aligned}
\mathbb{P}(A^{(n)}) &\le \mathbb{P}(A^{(n_0)}) + \sum_{k=n_0+1}^{n} \mathbb{P}\Big(A^{(k)} \backslash A^{(k-1)}\Big) \\
&\le K_9 \Big[N(S_p, \epsilon_{n_0}) + \sum_{k=n_0+1}^{\infty} N(S_p, \epsilon_k) \exp\Big(-\frac{(\beta d)^2}{16}\ b^k\Big)\Big] \cdot \exp\Big(-\frac{|y|^2}{K_{10} a^{2\alpha}}\Big)\Big(\frac{r}{a^\alpha}\Big)^d \\
&\le K_{11}\ \exp\Big(-\frac{|y|^2}{K_{12} a^{2\alpha}}\Big)\Big(\frac{r}{a^\alpha}\Big)^d. \qquad (3.22)
\end{aligned}
$$

In the above, we have used the fact that by (3.11),

$$
N(S_p, \epsilon_{n_0}) \le K\Big(\frac{|y|}{a^\alpha}\Big)^{2N/\eta}
$$

which can be absorbed by the exponential factor. Finally, (3.7) follows from (3.9), (3.10), (3.14) and (3.22).

The proof of Part (ii) is simpler. Since $a \le r^{1/\alpha}$, there is a constant K such that S can be covered by at most K balls of radius $r^{1/\alpha}$. The rest of the proof is almost the same as that of Part (i), we only need to note that in this case,

$$
\mathbb{P}\Big\{|Z(t_i^{(n)}) - c(t_i^{(n)})y| \le K_8 r\ \Big\} \le K\ \exp\Big(-\frac{|y|^2}{16\sigma^2(t_i^{(n)})}\Big).
$$

This finishes the proof of Lemma 3.1.

Lemma 3.3 *There exists a positive and finite constant* K_{13} *such that for any* $t \in \mathbb{R}^N \backslash \{0\}$ *and any* $a > 0$,

$$K_{13}^{-1} a^{2\alpha} \le \int_{|x|\ge a^{-1}} (1-\cos\langle t, x\rangle)\ \frac{dx}{|x|^{2\alpha+N}} \le K_{13} a^{2\alpha}. \tag{3.23}$$

Proof Since $1-\cos\langle t,x\rangle \le 2$, the right inequality in (3.23) is immediate. To prove the left inequality, we consider the case $N=1$ first. By a change of variables, we see that it is sufficient to show that there exists a constant $K>0$ such that for $0<a<1$,

$$\int_{|x|\ge a^{-1}} (1-\cos x)\ \frac{dx}{|x|^{2\alpha+1}} \ge K a^{2\alpha}. \tag{3.24}$$

Denote $n_1 = \min\{n \in \mathbb{N}_+ :\ 2n\pi + \pi/3 \ge a^{-1}\}$. Since on each interval $[2n\pi+\pi/3,\ 2(n+1)\pi - \pi/3]$, $1-\cos x \ge 1/2$, we see that the left-hand side of (3.24) is at least

$$\begin{aligned}\frac{1}{2}\sum_{n=n_1}^{\infty}\int_{2n\pi+\pi/3}^{2(n+1)\pi-\pi/3} \frac{dx}{|x|^{2\alpha+1}} \quad &\ge \quad K \sum_{n=n_1}^{\infty} \frac{1}{[2(n+1)\pi - \pi/3]^{2\alpha+1}} \\ &\ge \quad \sum_{n=n_1}^{\infty}\int_{2n\pi+5\pi/3}^{2(n+1)\pi+5\pi/3} \frac{dx}{x^{2\alpha+1}} \ \ge\ K a^{2\alpha}.\end{aligned}$$

This proves (3.24). Now consider the case that $N>1$. We write

$$\int_{|x|\ge a^{-1}} (1-\cos\langle t,x\rangle)\ \frac{dx}{|x|^{2\alpha+N}} = c_N \int_{S_{N-1}}\int_{a^{-1}}^{\infty} [1-\cos(\langle t,\theta\rangle\rho)]\ \frac{d\rho}{\rho^{2\alpha+1}}\,\mu(d\theta),$$

where μ is the normalized surface area on the unit sphere S_{N-1} in $\mathbb{R}^N$ and c_N is a positive finite constant depending on N only. The desired inequality follows from the inequality for $N=1$.

The following lemma will play an important role in the proof of Theorem 3.1.

Lemma 3.4 *Assume that* $N < \alpha d$. *Then there exists a positive and finite constant* K_{14}, *depending only on* α, N *and* d *such that for any* $0<u<1$,

$$\exp\Big(-\frac{K_{14}}{u^{2\alpha/N}}\Big) \le \mathbb{P}\{T(1)<u\} \le \exp\Big(-\frac{1}{K_{14}u^{2\alpha/N}}\Big). \tag{3.25}$$

Remark 3.5 When X is an ordinary Brownian motion in $\mathbb{R}^d$ with $d\ge 3$, Gruet and Shi [3] showed that

$$\mathbb{P}(T(1)<u) \sim \gamma(d) u^{\beta} \exp\Big(-\frac{2}{u}\Big) \quad \text{as} \ \ u \to 0, \tag{3.26}$$

where

$$\beta = \frac{7}{2} - d, \quad \gamma(d) = \frac{(8\pi)^{1/2}}{(\Gamma(d/2-1))^2}.$$

Their proof of (3.26) depends heavily on the relationship between the sojourn time of Brownian motion in $\mathbb{R}^d$ and a Bessel process of dimension d, hence can not be used to study the similar problem for fractional Brownian motion. Nevertheless, it is natural to conjecture that

$$\lim_{u\to 0} u^{2\alpha/N} \log \mathbb{P}\{T(1) < u\} \ \text{ exists.}$$

This problem is also about the small ball probability of the self-similar process $T = \{T(r), r \geq 0\}$. For an extensive survey of results and techniques for estimating small ball probability of Gaussian processes, we refer to Li and Shao [8].

Proof of Lemma 3.4 The right inequality in (3.25) is easy to prove. By the self-similarity of X, cf. (1.2), and Lemma 2.3 , we have

$$\begin{aligned}
\mathbb{P}\{T(1) < u\} &\leq \mathbb{P}\Big\{\max_{t\in[0,u^{1/N}]^N} |X(t)| \geq 1\Big\} \\
&= \mathbb{P}\Big\{\max_{t\in[0,1]^N} |X(t)| \geq u^{-\alpha/N}\Big\} \\
&\leq \exp\Big(-\frac{1}{Ku^{2\alpha/N}}\Big).
\end{aligned}$$

In order to prove the left inequality in (3.25), for any $0 < u < 1$, we consider the Gaussian random vector $\xi = (\xi_1, \cdots, \xi_d)$, where $\xi_1, \ldots, \xi_d$ are independent and each has the same distribution as

$$\xi_0 = \int_{|x|\geq 1} \frac{dm(x)}{|x|^{\alpha+N/2}}.$$

It is clear that ξ_0 is a mean zero Gaussian random variable with

$$\mathbb{E}(\xi_0^2) = \int_{|x|\geq 1} \frac{dx}{|x|^{2\alpha+N}}. \tag{3.27}$$

Using conditional expectation, we can write the $\mathbb{R}$-valued fractional Brownian motion Y as

$$Y(t) = Y^1(t) + c(t)\xi_0, \quad t \in \mathbb{R}^N \tag{3.28}$$

where the $\mathbb{R}$-valued Gaussian random field Y^1 is independent of ξ_0 and

$$c(t) = \frac{\mathbb{E}(Y(t)\xi_0)}{\mathbb{E}(\xi_0^2)}.$$

It follows from the integral representation (1.1) that

$$\mathbb{E}(Y(t)\xi_0) = \int_{|x|\geq 1} (1-\cos\langle t, x\rangle)\ \frac{dx}{|x|^{2\alpha+N}}.$$

Thus Lemma 3.3 and (3.27) imply that there exists a positive and finite constant K such that

$$K^{-1} \le c(t) \le 2 \quad \text{for all} \quad t \in \mathbb{R}^N \backslash \{0\}. \tag{3.29}$$

Furthermore, some elementary computations show that for some finite constant $K > 0$ and for all $s, t \in \mathbb{R}^N$,

$$|c(s) - c(t)| \le K \begin{cases} |s-t|^{2\alpha} & \text{if } \ 0 < \alpha < 1/2, \\ |s-t| \log |t-s|^{-1} & \text{if } \ \alpha = 1/2, \\ |s-t| & \text{if } \ 1/2 < \alpha < 1. \end{cases} \tag{3.30}$$

Hence the function $c(t)$ satisfies conditions (3.4) and (3.6) on $\mathbb{R}^N \backslash \{0\}$. On the other hand, we have

$$\mathbb{E}(Y^1(s) - Y^1(t))^2 = |s-t|^{2\alpha} - [c(s) - c(t)]^2 \mathbb{E}(\xi_0^2).$$

By (3.30), we see that there exists a constant $\delta_0 > 0$ such that for all $s, t \in \mathbb{R}^N$ with $|s-t| < \delta_0$,

$$\frac{1}{2}|s-t|^{2\alpha} \le \mathbb{E}\Big(Y^1(s) - Y^1(t)\Big)^2 \le |s-t|^{2\alpha}. \tag{3.31}$$

Also, it follows from (3.28) that there is a finite constant $K_{15} > 0$ such that for all $t \in \mathbb{R}^N$ with $|t| \ge K_{15}$,

$$\frac{1}{2}|t|^{2\alpha} \le \mathbb{E}[(Y^1(t))^2] \le |t|^{2\alpha}. \tag{3.32}$$

We note that

$$\begin{aligned} \mathbb{P}\{T(1) < u\} \ &\ge \ \mathbb{P}\Big\{ \text{ for any } \ t \in \mathbb{R}^N \backslash \Big(-\frac{u^{1/N}}{2}, \frac{u^{1/N}}{2} \Big)^N, \text{ we have } |X(t)| > 1 \Big\} \\ &= \ 1 - \mathbb{P}\Big\{ \exists \ t \in \mathbb{R}^N \backslash \Big(-\frac{u^{1/N}}{2}, \frac{u^{1/N}}{2} \Big)^N \ \text{ such that } |X(t)| \le 1 \Big\} \\ &= \ 1 - \mathbb{P}\Big\{ \exists \ t \in \mathbb{R}^N \backslash (-K_{15}, K_{15})^N \ \text{ such that } |X(t)| \le K_{16} u^{-\alpha/N} \Big\}, \end{aligned} \tag{3.33}$$

where $K_{16} = (2K_{15})^\alpha$ by the scaling property (1.2).

Let X^1 be the $\mathbb{R}^d$-valued Gaussian random field whose components are independent copies of Y^1. By (3.28), we can decompose X into

$$X(t) = X^1(t) + c(t)\xi, \tag{3.34}$$

where X^1 is independent of ξ.

Using conditioning and Eq. (3.34), the last probability in (3.33) can be written as

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathbb{P}\Big\{ \exists \ t \in \mathbb{R}^N \backslash (-K_{15}, K_{15})^N \ \text{ such that } |X(t)| \le K_{16} u^{-\alpha/N} \Big| \xi = y \Big\} \, p_\xi(y) dy \\ &= \int_{\mathbb{R}^d} \mathbb{P}\Big\{ \exists \ t \in \mathbb{R}^N \backslash (-K_{15}, K_{15})^N \ \text{ such that } |X^1(t) + c(t)y| \le K_{16} u^{-\alpha/N} \Big\} \, p_\xi(y) dy. \end{aligned} \tag{3.35}$$

In the above, $p_\xi(y)$ is the density function of the Gaussian random vector ξ. In order to estimate the probability in (3.35), we set

$$S_n = \Big\{t \in \mathbb{R}^N : \ 2^n K_{15} \le |t| \le 2^{n+1} K_{15}\Big\}.$$

For simplicity of notations, we do not distinguish balls and cubes and write

$$\mathbb{R}^N \backslash (-K_{15}, K_{15})^N = \bigcup_{n=0}^{\infty} S_n.$$

Let $n_2 = \max\{n : \ 2^n \le u^{-1/N}\}$. Then the probability in (3.35) is bounded by

$$\begin{aligned}
&\mathbb{P}\Big\{\exists\, t \in \bigcup_{n=0}^{n_2} S_n \ \text{ such that } |X^1(t) + c(t)y| \le K_{16} u^{-\alpha/N}\Big\} \\
&\qquad + \sum_{n=n_2+1}^{\infty} \mathbb{P}\Big\{\exists\, t \in S_n \ \text{ such that } |X^1(t) + c(t)y| \le K_{16} u^{-\alpha/N}\Big\} \\
&\qquad \hat{=}\ I_3 + I_4.
\end{aligned}$$

Now, we apply Lemma 3.1 to each S_n. Inequalities (3.29) – (3.32) show that the Gaussian random field $X^1(t)$ and the function $c(t)$ satisfy the conditions of Lemma 3.1 on each S_n. Hence it follows from Lemma 3.1 that for all $y \in \mathbb{R}^N$ with $|y| \ge K_{17} u^{-\alpha/N}$, we have

$$\begin{aligned}
I_4 \ &\le\ K \sum_{n=n_2+1}^{\infty} \exp\Big(-\frac{|y|^2}{K 2^{2n\alpha}}\Big) \Big(\frac{u^{-\alpha/N}}{2^{n\alpha}}\Big)^{d-N/\alpha} \\
&\le\ K u^{1-\alpha d/N} \int_0^{\infty} \exp\Big(-\frac{|y|^2}{K x^{2\alpha}}\Big) \frac{1}{x^{\alpha d - N+1}}\, dx \\
&\le\ K \Big(\frac{u^{-\alpha/N}}{|y|}\Big)^{d-N/\alpha}.
\end{aligned}$$

On the other hand, Part (ii) of Lemma 3.1 or its proof implies

$$I_3 \le K \exp\Big(-\frac{|y|^2}{K\, 2^{2n_2\alpha}}\Big) \le K \exp\Big(-\frac{|y|^2}{K\, u^{-2\alpha/N}}\Big).$$

Consequently, for all $y \in \mathbb{R}^N$ with $|y| \ge K_{17} u^{-\alpha/N}$,

$$\begin{aligned}
&\mathbb{P}\Big\{\exists\, t \in \mathbb{R}^N \backslash (-K_{15}, K_{15})^N \ \text{ such that } |X^1(t) + c(t)y| \le K_{16} u^{-\alpha/N}\Big\} \\
&\le K_{18} \Big(\frac{u^{-\alpha/N}}{|y|}\Big)^{d-N/\alpha},
\end{aligned} \tag{3.36}$$

where K_{17} and K_{18} are positive and finite constants depending on N, α and d only.

Putting things together we see that for all $0 < u < 1$,

$$\begin{aligned}\mathbb{P}\{T(1) < u\} &\geq \int_{|y| \geq K_{17} u^{-\alpha/N}} \Big[1 - K_{18}\Big(\frac{u^{-\alpha/N}}{|y|}\Big)^{d-N/\alpha}\Big]\, p_\xi(y)dy \\ &\geq \exp\Big(-\frac{K}{u^{2\alpha/N}}\Big).\end{aligned}$$

This completes the proof of (3.25).

We need one more technical lemma from Xiao [23]. It gives some information about the local density of an arbitrary probability measure.

Lemma 3.6 *Given any constant $\lambda_0 > 0$, there exists a positive and finite constant $K = K(\lambda_0)$ with the following property: for any Borel probability measure μ on $\mathbb{R}$, there exists $x \in [\lambda_0, 2\lambda_0]$ such that*

$$\mu((x-\delta, x+\delta)) \leq K\, \delta^{\frac{1}{2}} \quad \textit{for every } 0 < \delta < \frac{1}{8}.$$

We are ready to prove the following Chung-type law of the iterated logarithm for the sojourn measure $T(\cdot)$.

Theorem 3.1 *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a fractional Brownian motion of index α in $\mathbb{R}^d$ and $N < \alpha d$. Then with probability one,*

$$\liminf_{r \to 0} \frac{T(r)}{\varphi(r)} = \gamma, \tag{3.37}$$

where $\varphi(s) = s^{N/\alpha}/(\log\log 1/s)^{N/(2\alpha)}$ and $0 < \gamma < \infty$ is a constant depending on α, N and d only.

Proof We first prove that there exists a constant $\gamma_1 > 0$ such that

$$\liminf_{r \to 0} \frac{T(r)}{\varphi(r)} \geq \gamma_1 \quad \text{a.s.} \tag{3.38}$$

For $k = 1, 2, \cdots,$ let $a_k = \exp(-k/\log k)$ and consider the events

$$A_k = \{\omega : T(a_k) \leq \lambda\varphi(a_k)\},$$

where $\lambda > 0$ is a constant to be determined later. Then by the scaling property (3.1) of $T(r)$ and Lemma 3.4, we have, for k large enough,

$$\begin{aligned}\mathbb{P}(A_k) &= \mathbb{P}\Big\{T(1) \leq \frac{\lambda}{(\log\log 1/a_k)^{N/2\alpha}}\Big\} \\ &\leq \exp\Big(-\frac{1}{K\lambda^{2\alpha/N}} \log\log\frac{1}{a_k}\Big) \\ &\leq k^{-\frac{1}{K\lambda^{2\alpha/N}}}.\end{aligned}$$

Take $\lambda = \gamma_1 > 0$ such that $K\gamma_1^{2\alpha/N} < 1$. Then $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$. It follows from the Borel-Cantelli lemma that

$$\liminf_{k\to\infty} \frac{T(a_k)}{\varphi(a_k)} \geq \gamma_1 \quad \text{a.s.} \tag{3.39}$$

A standard monotonicity argument and (3.39) yield (3.38).

Next we prove that there exists a finite constant $\gamma_2 > 0$ such that

$$\liminf_{r\to 0} \frac{T(r)}{\varphi(r)} \leq \gamma_2 \quad \text{a.s.} \tag{3.40}$$

Let $b_k = \exp(-k^2)$, $\tau_k = k^\theta b_k^{1/\alpha}$, where $\theta = 3/(\alpha d - N)$ and, again, let $\lambda > 0$ be a constant to be determined later. Recall that

$$T(b_k, \tau_k) = \int_{|t|\leq \tau_k} \mathbb{1}_{B(0,b_k)}(X(t))dt.$$

Then by Lemma 2.4, we have

$$\begin{aligned} \mathbb{P}\Big\{T(b_k,\tau_k) \neq T(b_k)\Big\} &\leq \mathbb{P}\Big\{\exists\ t \text{ such that } \ |t| > \tau_k \text{ and } |X(t)| < b_k\Big\} \\ &\leq K\Big(\frac{b_k}{\tau_k^\alpha}\Big)^{d-N/\alpha} \\ &= Kk^{-3}. \end{aligned} \tag{3.41}$$

Hence by the Borel-Cantelli lemma, with probability 1, there exists $k_1 = k_1(\omega)$ such that

$$T(b_k,\tau_k) = T(b_k) \quad \text{for all } \ k \geq k_1. \tag{3.42}$$

For $k \geq 1$, consider the event

$$E_k = \Big\{T(b_k,\tau_k) < \lambda_k \varphi(b_k)\Big\},$$

where $\lambda \leq \lambda_k \leq 2\lambda$ will be chosen later. By (3.42), we see that, in order to prove (3.40), it is sufficient to show

$$\mathbb{P}\Big\{\limsup_{k\to\infty} E_k\Big\} = 1. \tag{3.43}$$

By (3.1) and Lemma 3.4, we have

$$\begin{aligned} \mathbb{P}(E_k) &\geq \mathbb{P}\Big\{T(b_k) < \lambda_k\varphi(b_k)\Big\} \\ &= \mathbb{P}\Big\{T(1) < \frac{\lambda_k}{(\log\log 1/b_k)^{\frac{N}{2\alpha}}}\Big\} \\ &\geq k^{-\frac{K}{\lambda^{2\alpha/N}}}. \end{aligned} \tag{3.44}$$

Thus, if we select $\lambda = \lambda_0$ such that $K/\lambda_0^{2\alpha/N} \le 1$, then

$$\sum_{k=1}^{\infty} \mathbb{P}(E_k) = \infty. \tag{3.45}$$

By the second Borel-Cantelli lemma (Part (ii) of Lemma 2.6), (3.43) will follow after we show that for any $\epsilon > 0$, (2.5) is satisfied.

For this purpose, we fix a positive integer k. For $j > k$, we need to estimate

$$\mathbb{P}(E_k \cap E_j) = \mathbb{P}\Big\{T(b_k, \tau_k) < \lambda_k \varphi(b_k),\ T(b_j, \tau_j) < \lambda_j \varphi(b_j)\Big\}.$$

We denote this probability by Q. In order to create independence, we make use of the stochastic integral representation (1.1) and decompose $X(t)$ in the following way. Set $v = (\sqrt{\tau_k \tau_j})^{-1}$ and consider the following two Gaussian random fields

$$X^1(t) = (Y_1^1(t), \cdots, Y_d^1(t)),$$

and

$$X^2(t) = (Y_1^2(t), \cdots, Y_d^2(t)),$$

where $Y_1^1, \dots, Y_d^1$ are independent copies of

$$Z^1(t) = \frac{c}{\sqrt{2}} \int_{|x| \le v} \Big(1 - \cos\langle t, x\rangle\Big) \frac{dm(x)}{|x|^{\alpha + N/2}} + \frac{c}{\sqrt{2}} \int_{|x| \le v} \sin\langle t, x\rangle \, \frac{dm'(x)}{|x|^{\alpha+N/2}},$$

and $Y_1^2, \dots, Y_d^2$ are independent copies of

$$Z^2(t) = \frac{c}{\sqrt{2}} \int_{|x| > v} \Big(1 - \cos\langle t, x\rangle\Big) \frac{dm(x)}{|x|^{\alpha + N/2}} + \frac{c}{\sqrt{2}} \int_{|x| > v} \sin\langle t, x\rangle \, \frac{dm'(x)}{|x|^{\alpha+N/2}}.$$

Note that the Gaussian random fields X^1 and X^2 are independent and, for every $t \in \mathbb{R}^N$, $X(t) = X^1(t) + X^2(t)$.

We will also need to use some estimates for

$$\mathbb{P}\Big\{\max_{|t| \le \tau_j} |X^1(t)| > \eta b_j\Big\} \ \text{ and } \ \mathbb{P}\Big\{\max_{|t| \le \tau_k} |X^2(t)| > \eta b_k\Big\}.$$

These estimates are obtained by applying Lemma 2.2.

Let $S_j = \{t \in \mathbb{R}^N : |t| \le \tau_j\}$ and $S_k = \{t \in \mathbb{R}^N : |t| \le \tau_k\}$. We note that the pseudo-metric d_1 on S_j and d_2 on S_k satisfy

$$d_1(s,t) := \|X^1(s) - X^1(s)\|_2 \le \sqrt{d}\ |s - t|^\alpha;$$

$$d_2(s,t) := \|X^2(s) - X^2(s)\|_2 \le \sqrt{d}\ |s-t|^{\alpha}.$$

By using the elementary inequality $1-\cos\langle t,x\rangle \le |t|^2|x|^2$, we have

$$\begin{aligned}\|Z^1(t)\|_2^2 &= c^2\int_{|x|\le v}\Big(1-\cos\langle t,x\rangle\Big)\frac{dx}{|x|^{2\alpha+N}}\\ &\le K|t|^2\int_{|x|\le v}\frac{dx}{|x|^{2\alpha+N-2}}\\ &= K|t|^2 v^{2(1-\alpha)}.\end{aligned}$$

Hence for any $|t|\le \tau_j$,

$$\|Z^1(t)\|_2 \le K\tau_j\cdot\Big(\frac{1}{\sqrt{\tau_k\tau_j}}\Big)^{1-\alpha} = K\sqrt{\frac{\tau_j}{\tau_k}}\ (\tau_k\tau_j)^{\alpha/2}. \tag{3.46}$$

On the other hand, since $1-\cos\langle t,x\rangle\le 2$, we obtain

$$\begin{aligned}\|Z^2(t)\|_2^2 &= c^2\int_{|x|>v}\Big(1-\cos\langle t,x\rangle\Big)\frac{dx}{|x|^{2\alpha+N}}\\ &\le Kv^{-2\alpha}.\end{aligned}$$

and hence for any $|t|\le\tau_k$,

$$\|Z^2(t)\|_2 \le K(\tau_k\tau_j)^{\alpha/2}. \tag{3.47}$$

It follows from (3.46) and (3.47) that the d_1-diameter D_j of S_j and the d_2-diameter D_k of S_k satisfy

$$D_j \le K\sqrt{\frac{\tau_j}{\tau_k}}\ (\tau_k\tau_j)^{\alpha/2}$$

and

$$D_k \le K(\tau_k\tau_j)^{\alpha/2}.$$

Straightforward computation shows that

$$\int_0^{D_j}\sqrt{\log N_d(S_j,\epsilon)}\ d\epsilon \le K\tau_j^{\alpha}\Big(\frac{\tau_j}{\tau_k}\Big)^{(1-\alpha)/2}\sqrt{\log\frac{\tau_k}{\tau_j}},$$

$$\int_0^{D_k}\sqrt{\log N_d(S_k,\epsilon)}\ d\epsilon \le K(\tau_k\tau_j)^{\alpha/2}\sqrt{\log\frac{\tau_k}{\tau_j}}.$$

Applying Lemma 2.2, we obtain

$$\mathbb{P}\Big\{\max_{|t|\le\tau_j}|X^1(t)|>\eta b_j\Big\} \le \exp\Big(-\frac{\eta^2}{K}\,j^{-2\theta\alpha}\Big(\frac{\tau_k}{\tau_j}\Big)^{1-\alpha}\Big) \tag{3.48}$$

and

$$\mathbb{P}\Big\{\max_{|t|\le\tau_k}|X^2(t)|>\eta b_k\Big\}\le\exp\Big(-\frac{\eta^2}{K}\,k^{-2\theta\alpha}\Big(\frac{\tau_k}{\tau_j}\Big)^{\alpha}\Big). \tag{3.49}$$

The rest of the proof is quite similar to that of Theorem 4.1 in Xiao [23] with appropriate modifications. For any $0<\delta<1$, let $\eta>0$ be determined by

$$\frac{1}{(1-2\eta)^{N/\alpha}}=1+\delta.$$

These numbers will be fixed for the moment. We note that $\eta\ge K\delta$ for some constant $K>0$.

Since the inequalities $|X^1(t)|\le\eta b_j$ and $|X^2(t)|\le(1-\eta)b_j$ together imply $|X(t)|\le b_j$, we see that, on the event $\{\max_{|t|\le\tau_j}|X^1(t)|\le\eta b_j\}$,

$$\int_{|t|\le\tau_j}\mathbb{1}_{B(0,(1-\eta)b_j)}(X^2(t))dt\le T(b_j,\tau_j).$$

Hence,

$$\begin{aligned}\{T(b_j,\tau_j)<\lambda_j\varphi(b_j)\}\ \ \subseteq\ \ &\Big\{\int_{|t|\le\tau_j}\mathbb{1}_{B(0,(1-\eta)b_j)}(X^2(t))dt<\lambda_j\varphi(b_j)\Big\}\\ &\qquad\bigcup\Big\{\max_{|t|\le\tau_j}|X^1(t)|>\eta b_j\Big\}.\end{aligned} \tag{3.50}$$

Similarly,

$$\begin{aligned}\{T(b_k,\tau_k)<\lambda_k\varphi(b_k)\}\ \ \subseteq\ \ &\Big\{\int_{|t|\le\tau_k}1_{B(0,(1-\eta)b_k)}(X^1(t))dt<\lambda_k\varphi(b_k)\Big\}\\ &\qquad\bigcup\Big\{\max_{|t|\le\tau_k}|X^2(t)|>\eta b_k\Big\}.\end{aligned} \tag{3.51}$$

It follows from (3.50) and (3.51) that Q is less than

$$\begin{aligned}&\mathbb{P}\Big\{\int_{|t|\le\tau_k}\mathbb{1}_{B(0,(1-\eta)b_k)}(X^1(t))dt<\lambda_k\varphi(b_k),\ \int_{|t|\le\tau_j}\mathbb{1}_{B(0,(1-\eta)b_j)}(X^2(t))dt<\lambda_j\varphi(b_j)\Big\}\\ &\qquad+\mathbb{P}\Big\{\max_{|t|\le\tau_j}|X^1(t)|>\eta b_j\Big\}+\mathbb{P}\Big\{\max_{|t|\le\tau_k}|X^2(t)|>\eta b_k\Big\}.\end{aligned}$$

By the independence of X^1 and X^2, we have

$$\begin{aligned}&\mathbb{P}\Big\{\int_{|t|\le\tau_k}\mathbb{1}_{B(0,(1-\eta)b_k)}(X^1(t))dt<\lambda_k\varphi(b_k),\ \int_{|t|\le\tau_j}\mathbb{1}_{B(0,(1-\eta)b_j)}(X^2(t))dt<\lambda_j\varphi(b_j)\Big\}\\ &=\mathbb{P}\Big\{\int_{|t|\le\tau_k}\mathbb{1}_{B(0,(1-\eta)b_k)}(X^1(t))dt<\lambda_k\varphi(b_k)\Big\}\\ &\qquad\cdot\mathbb{P}\Big\{\int_{|t|\le\tau_j}\mathbb{1}_{B(0,(1-\eta)b_j)}(X^2(t))dt<\lambda_j\varphi(b_j)\Big\}\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{P}\Big\{\int_{|t|\leq \tau_k} \mathbb{1}_{B(0,(1-2\eta)b_k)}(X(t))dt < \lambda_k\varphi(b_k)\Big\}\\
&\qquad\qquad\qquad \cdot \mathbb{P}\Big\{\int_{|t|\leq \tau_j} \mathbb{1}_{B(0,(1-2\eta)b_j)}(X(t))dt < \lambda_k\varphi(b_j)\Big\}\\
&\quad +\mathbb{P}\Big\{\max_{|t|\leq \tau_j} |X^1(t)| > \eta b_j\Big\} + \mathbb{P}\Big\{\max_{|t|\leq \tau_k} |X^2(t)| > \eta b_k\Big\}\\
&:= Q_1 + Q_2 + Q_3.
\end{aligned}
$$

Consequently, we have derived

$$
Q \leq Q_1 + 2Q_2 + 2Q_3. \tag{3.52}
$$

Since upper bounds for Q_2 and Q_3 have been obtained in (3.48) and (3.49), we only need to estimate Q_1. Note

$$
\begin{aligned}
&\mathbb{P}\Big\{\int_{|t|\leq \tau_j} \mathbb{1}_{B(0,(1-2\eta)b_j)}(X(t))dt < \lambda_j\varphi(b_j)\Big\}\\
&\leq \mathbb{P}\Big\{T((1-2\eta)b_j) < \lambda_j\varphi(b_j)\Big\} + \mathbb{P}\Big\{T((1-2\eta)b_j) \neq T((1-2\eta)b_j, \tau_j)\Big\}\\
&\leq \mathbb{P}\Big\{T(1) < \frac{1}{(1-2\eta)^{N/\alpha}} \cdot \frac{\lambda_j\varphi(b_j)}{b_j^{N/\alpha}}\Big\} + Kj^{-3},
\end{aligned} \tag{3.53}
$$

where the last inequality follows from scaling property (3.1) and (3.41). By applying Lemma 3.6 to the distribution of $T(1)(\log\log 1/b_j)^{N/(2\alpha)}$, we can choose $\lambda_j \in [\lambda_0, 2\lambda_0]$ such that

$$
\mathbb{P}\Big\{\lambda_j \leq T(1)(\log\log \frac{1}{b_j})^{N/(2\alpha)} < (1+\delta)\lambda_j\Big\} \leq K\delta^{1/2} \quad \text{for all } \ 0 < \delta < 1/8. \tag{3.54}
$$

Hence, by Lemma 3.4 and (3.54), the probability in (3.53) is less than

$$
\begin{aligned}
&\mathbb{P}\Big\{T(1) < \frac{\lambda_j\varphi(b_j)}{b_j^{N/\alpha}}\Big\} \cdot \Big(1 + \frac{\mathbb{P}\{\lambda_j b_j^{-N/\alpha}\varphi(b_j) \leq T(1) < (1+\delta)\lambda_j b_j^{-N/\alpha}\varphi(b_j)\}}{\mathbb{P}\{T(1) < \lambda_j b_j^{-N/\alpha}\varphi(b_j)\}}\Big)\\
&\leq \mathbb{P}\{T(b_j) < \lambda_j\varphi(b_j)\} \cdot (1 + Kj\ \delta^{1/2}).
\end{aligned} \tag{3.55}
$$

Note that, in the above, we have also used (3.44). Combining (3.53) and (3.55), we have

$$
\begin{aligned}
&\mathbb{P}\Big\{\int_{|t|\leq \tau_j} \mathbb{1}_{B(0,(1-2\eta)b_j)}(X(t))dt < \lambda_j\varphi(b_j)\Big\}\\
&\leq \mathbb{P}\{T(b_j) < \lambda_j\varphi(b_j)\} \cdot (1 + Kj\ \delta^{1/2}) + Kj^{-3}\\
&\leq \mathbb{P}\Big\{T(b_j, \tau_j) < \lambda_j\varphi(b_j)\Big\}\Big(1 + Kj\ \delta^{1/2} + Kj^{-2}\Big).
\end{aligned} \tag{3.56}
$$

Similarly, we have

$$
\begin{aligned}
&\mathbb{P}\Big\{\int_{|t|\leq \tau_k} \mathbb{1}_{B(0,(1-2\eta)b_k)}(X(t))dt < \lambda_k\varphi(b_k)\Big\}\\
&\leq \mathbb{P}\Big\{T(b_k, \tau_k) < \lambda_k\varphi(b_k)\Big\}\Big(1 + Kk\ \delta^{1/2} + Kk^{-2}\Big).
\end{aligned} \tag{3.57}
$$

It follows from (3.56) and (3.57) that

$$\begin{aligned} Q_1 \quad &\le \quad \mathbb{P}\Big\{T(b_j,\tau_j) < \lambda_j\varphi(b_j)\Big\}\mathbb{P}\Big\{T(b_k,\tau_k) < \lambda_k\varphi(b_k)\Big\} \\ &\qquad \cdot\Big(1 + Kj\ \delta^{1/2} + Kj^{-2}\Big)\Big(1 + Kk\ \delta^{1/2} + Kk^{-2}\Big). \end{aligned} \tag{3.58}$$

Putting (3.52), (3.58), (3.48), (3.44) and (3.49) together, we have

$$\begin{aligned} \mathbb{P}(E_k \cap E_j) \quad &\le \quad \mathbb{P}(E_k)\mathbb{P}(E_j)\Big(1 + Kj\ \delta^{1/2} + Kj^{-2}\Big)\Big(1 + Kk\ \delta^{1/2} + Kk^{-2}\Big) \\ &+ \quad 2\mathbb{P}(E_k)\Big\{k\exp\Big(-K\,\delta^2 j^{-2\theta\alpha}\Big(\frac{\tau_k}{\tau_j}\Big)^{1-\alpha}\Big) \\ &\qquad + k\exp\Big(-K\,\delta^2 k^{-2\theta\alpha}\Big(\frac{\tau_k}{\tau_j}\Big)^{\alpha}\Big)\Big\}. \end{aligned} \tag{3.59}$$

We now select

$$0 < \beta < \min\Big\{\frac{\alpha}{3},\ \frac{1-\alpha}{3}\Big\} \quad \text{and} \quad \delta = \Big(\frac{\tau_j}{\tau_k}\Big)^{\beta}.$$

Note that

$$\frac{\tau_k}{\tau_j} = \Big(\frac{k}{j}\Big)^{\theta}\exp\Big(\frac{1}{\alpha}(j^2 - k^2)\Big).$$

It follows from the above and (3.59) that, for any $\epsilon > 0$, there exist a constant $K > 0$ and a positive integer k_0 such that for any $k \ge k_0$ and any $J > k$, we have

$$\sum_{j=k+1}^{J}\mathbb{P}(E_k \cap E_j) \le \mathbb{P}(E_k)\Big(K + (1+\epsilon)\sum_{j=k+1}^{J}\mathbb{P}(E_j)\Big). \tag{3.60}$$

Consequently, by (3.45), (3.60) and Lemma 2.6 we have

$$\mathbb{P}\Big\{\limsup_{k\to\infty} E_k\Big\} \ge \frac{1}{1+2\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, we conclude

$$\mathbb{P}\Big\{T(b_k) < \lambda_k\varphi(b_k)\ \text{ infinitely often }\Big\} = 1.$$

Therefore, (3.40) holds almost surely with $\gamma_2 = 2\lambda_0$. Combining (3.38), (3.40) and the zero-one law of Pitt and Tran ([13], Theorem 2.1), we obtain (3.37). This completes the proof of Theorem 3.1.

Since for any $t_0 \in \mathbb{R}^N$, the random field $\{X(t+t_0) - X(t_0),\ t \in \mathbb{R}^N\}$ is also a fractional Brownian motion in $\mathbb{R}^d$ of index α, we have the following corollary.

Corollary 3.1 *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a fractional Brownian motion of index α in $\mathbb{R}^d$ and $N < \alpha d$. Then for any $t_0 \in \mathbb{R}^N$, with probability one,*

$$\liminf_{r \to 0} \frac{T_{X(t_0)}(r)}{\varphi(r)} = \gamma.$$

The following is a limsup theorem for the sojourn measure of fractional Brownian motion.

Theorem 3.2 *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a fractional Brownian motion of index α in $\mathbb{R}^d$ and $N < \alpha d$. Then for any $t_0 \in \mathbb{R}^N$, with probability one,*

$$\limsup_{r \to 0} \frac{T_{X(t_0)}(r)}{\psi(r)} = \gamma', \tag{3.61}$$

where $\psi(r) = r^{N/\alpha} \log\log 1/r$ and γ' is a positive and finite constant.

Proof It is sufficient to prove (3.61) for $t_0 = 0$. It has been proved in Talagrand [18] (see also Xiao [24]) that there exists a finite constant $K > 0$ such that

$$\limsup_{r \to 0} \frac{T(r)}{\psi(r)} \le K, \text{ a.s.} \tag{3.62}$$

Hence (3.61) will follow from (3.62) and the zero-one law of Pitt and Tran ([13], Theorem 2.1) after we prove that there exists a constant $K > 0$ such that

$$\limsup_{r \to 0} \frac{T(r)}{\psi(r)} \ge K, \text{ a.s.} \tag{3.63}$$

By the following Chung-type law of iterated logarithm for multiparameter fractional Brownian motion, see Xiao ([25], p.147), or Li and Shao ([8], Theorem 7.2),

$$\liminf_{h \to 0} \sup_{s \in B(0,h)} \frac{|X(s)|}{h^\alpha / (\log\log 1/h)^{\alpha/N}} = K, \quad \text{a.s.}$$

we see that there exists a sequence decreasing $\{h_n\}$ of positive numbers such that $h_n \downarrow 0$ and

$$\sup_{s \in B(0,h_n)} |X(s)| \le K\psi_1(h_n),$$

where $\psi_1(h) = h^\alpha/(\log\log 1/h)^{\alpha/N}$. Hence we have

$$T(r_n) \ge K h_n^N, \quad \text{where } r_n = K\psi_1(h_n). \tag{3.64}$$

Observing that $\psi_2(r) = r^{1/\alpha}(\log\log 1/r)^{1/N}$ is an asymptotic inverse function of $\psi_1(h)$, we see that (3.63) follows from (3.64). This completes the proof.

Remark 3.7 Using the moment estimates (see Xiao [24]) and the result on small ball probabilities for fractional Brownian motion (Talagrand, [18]), it is easy to derive the following estimates on the tail probability of $T(1)$: if $N < \alpha d$, then for some constant $K \geq 1$,

$$\exp\big(-Ku\big) \leq \mathbb{P}\{T(1) > u\} \leq \exp\big(-K^{-1}u\big).$$

In order to study the multifractal structure of the sojourn measure, one needs to have more precise tail asymptotics for the distribution of $T(1)$. We have the following conjecture

If $N < \alpha d$, then the limit $\lim_{u\to\infty} u^{-1} \log \mathbb{P}\{T(1) > u\}$ exists.

When $N \geq \alpha d$, both small and large tail asymptotics for the truncated sojourn measure $T(1,r)$, say, are not known. It seems that even the correct rate functions are non-trivial to obtain.

For application in the next section, we state another result from Xiao ([25], Theorem 3.1). Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a fractional Brownian motion of index α in $\mathbb{R}^d$. Then for any rectangle $T \subset \mathbb{R}^N$,

$$\liminf_{h\to 0} \inf_{t\in T} \sup_{s\in B(t,h)} \frac{|X(s)-X(t)|}{h^{\alpha}/(\log 1/h)^{\alpha/N}} \geq K, \quad \text{a.s.} \tag{3.65}$$

where $K > 0$ is a constant depending on $N,\ d$ and α only.

4 Packing Measure of Fractional Brownian Motion

In this section, we consider the packing measure of the image and graph of a transient fractional Brownian motion in $\mathbb{R}^d$ and prove (1.4).

Theorem 4.1 *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a fractional Brownian motion in $\mathbb{R}^d$ of index α. If $N < \alpha d$, then there exist positive constants K_1 and K_2 such that, with probability 1,*

$$K_1 \leq \varphi\text{-}p(X([0,1]^N)) \leq \varphi\text{-}p(\mathrm{Gr}X([0,1]^N)) \leq K_2, \tag{4.1}$$

where $\varphi(s) = s^{N/\alpha}/(\log\log 1/s)^{N/(2\alpha)}$.

Proof To prove the first inequality in (4.1), we define a random Borel measure μ on $X([0,1]^N)$ as follows. For any Borel set $B \subseteq \mathbb{R}^d$, let

$$\mu(B) = \lambda_N\{t \in [0,1]^N,\ X(t) \in B\}.$$

Recall λ_N is Lebesgue's measure in $\mathbb{R}^N$. Then $\mu(\mathbb{R}^d) = \mu(X([0,1]^N)) = 1$. By Corollary 3.1, for each fixed $t_0 \in [0,1]^N$,

$$\liminf_{r\to 0} \frac{\mu\Big(B(X(t_0), r)\Big)}{\varphi(r)} \le \liminf_{r\to 0} \frac{T_{X(t_0)}(r)}{\varphi(r)} = \gamma, \quad \text{a.s.} \tag{4.2}$$

Let $E(\omega) = \{X(t_0) :\ t_0 \in [0,1]^N$ and (4.2) holds $\}$. Then $E(\omega) \subseteq X([0,1]^N)$. A Fubini argument shows $\mu(E(\omega)) = 1$, a.s. Hence by Lemma 2.1, we have

$$\varphi\text{-}p(E(\omega)) \ge 2^{-3N/\alpha}\gamma^{-1}\ .$$

This proves the left hand inequality of (4.1) with $K_1 = 2^{-3N/\alpha}\gamma^{-1}$. The second inequality in (4.1), i.e.

$$\varphi\text{-}p(X([0,1]^N)) \le \varphi\text{-}p(\mathrm{Gr}X([0,1]^N)),$$

follows from the definition of packing measure easily.

To prove the right hand inequality of (4.1), we only need to show $\varphi\text{-}P\ (\mathrm{Gr}X([0,1]^N)) \le K$ a.s., thanks to (2.4). Let $\mathcal{J}_k$ be the family of dyadic cubes of length 2^{-k} in $[0,1]^N$. For each $\tau \in [0,1]^N$, let $I_k(\tau)$ denote the dyadic cube in $\mathcal{J}_k$ which contains τ and let

$$a_k(\tau) = \sup_{s,\ t\in I_k(\tau)} |X(t) - X(s)|.$$

For any $k \ge 1$ and any $I \in \mathcal{J}_k$, by Lemma 2.3, we have for any $u \ge K$,

$$\mathbb{P}\Big\{\sup_{s,\ t\in I} |X(t) - X(s)| \ge u 2^{-k\alpha}\Big\} \le \exp\Big(-\frac{u^2}{K_4}\Big).$$

Take $u = \sqrt{K_4 \lambda \log k}$, where $\lambda > 6 + N/(2\alpha)$. The above inequality yields

$$\mathbb{P}\Big\{\sup_{s,\ t\in I} |X(t) - X(s)| \ge \sqrt{K_4 \lambda \log k}\ 2^{-k\alpha}\Big\} \le \frac{1}{k^\lambda}.$$

Denote

$$M_k = \#\Big\{I \in \mathcal{J}_k,\quad \sup_{s,\ t\in I} |X(t) - X(s)| \ge \sqrt{K_4 \lambda \log k}\ 2^{-k\alpha}\Big\},$$

then

$$\mathbb{P}\Big\{M_k \ge 2^{kN} \cdot \frac{1}{k^{\lambda-5}}\Big\} \le \frac{1}{k^5}.$$

The Borel-Cantelli lemma implies that, with probability one,

$$M_k < \frac{2^{kN}}{k^{\lambda-5}} \quad \text{for all } k \text{ large enough.} \tag{4.3}$$

Let Ω_0 be the event that (4.3) holds, and let Ω_1 and Ω_2 be the events that (2.6) and (3.65) hold respectively. Then $P(\Omega_0 \cap \Omega_1 \cap \Omega_2) = 1$.

Fix an $\omega \in \Omega_0 \cap \Omega_1 \cap \Omega_2$, let $k_0 = k_0(\omega)$ be a positive integer, such that $k \geq k_0$ implies (4.3) and the following two inequalities

$$\sup_{t\in[0,1]^N} a_k(t) \leq K\sqrt{k}\, 2^{-\alpha k} \tag{4.4}$$

and

$$\inf_{t\in[0,1]^N} a_k(t) \geq K 2^{-\alpha k} k^{-\alpha/N} > \sqrt{N}\, 2^{-k}. \tag{4.5}$$

For any $0 < \epsilon < 2^{-k_0}$ and any ϵ-packing $\{\overline{B}((t_i, X_\omega(t_i)), r_i)\}$ of $\mathrm{Gr}X_\omega([0,1]^N)$, we will show that for some absolute constant $K_2 > 0$,

$$\sum_i \varphi(2r_i) \leq K_2. \tag{4.6}$$

From now on, we will suppress the subindex ω. For each i, let

$$k_i = \inf\{k : \; a_k(t_i) \leq r_i/2\}.$$

Then $a_{k_i-1}(t_i) > r_i/2$. We claim that the dyadic cubes $\{I_{k_i}(t_i)\}$ are disjoint. In fact, if

$$t_0 \in I_{k_i}(t_i) \cap I_{k_j}(t_j) \quad \text{ for some } \; i \neq j,$$

then the triangle inequality implies

$$\begin{aligned}
|(t_i, X(t_i)) - (t_j, X(t_j))| &\leq |(t_i, X(t_i)) - (t_0, X(t_0))| + |(t_0, X(t_0)) - (t_j, X(t_j))| \\
&\leq \sqrt{N} 2^{-k_i} + \sqrt{N}\, 2^{-k_j} + (r_i + r_j)/2 \\
&< r_i + r_j,
\end{aligned} \tag{4.7}$$

where in deriving the last inequality we have used the fact that $\sqrt{N} 2^{-k_i} < a_{k_i}(t_i) \leq r_i/2$, which follows from (4.5). The inequality (4.7) contradicts the fact that $\overline{B}((t_i, X(t_i)), r_i) \cap \overline{B}((t_j, X(t_j)), r_j) = \emptyset$. Summing up the volumes of $I_{k_i}(t_i)$, we have

$$\sum_i 2^{-k_i N} \leq 1. \tag{4.8}$$

We divide the points $\{(t_i, X(t_i))\}$ into two types: "good" points or "bad" points. $(t_i, X(t_i))$ is called a good point if

$$a_{k_i-1}(t_i) \leq \sqrt{K_4 \lambda \log(k_i - 1)}\, 2^{-(k_i-1)\alpha};$$

otherwise, it is called a bad point. Let G denote the set of the subscripts i of good points.

By (4.8), we obtain

$$\begin{aligned}\sum_{i\in G}\varphi(2r_i) &\le \sum_{i\in G}\varphi(4a_{k_i-1}(t_i)) \\ &\le K\sum_i 2^{-k_iN} \le K_{19}. \end{aligned} \tag{4.9}$$

In order to estimate $\sum_{i\notin G}\varphi(2r_i)$, we note that there are not many "bad" points and the oscillation of X at such points are controlled by the modulus of continuity. It follows from (4.3) and (4.4) that

$$\begin{aligned}\sum_{i\notin G}\varphi(2r_i) &\le \sum_{i\notin G}\varphi(4a_{k_i-1}(t_i)) \\ &\le \sum_{i\notin G}\frac{2^{(k_i-1)N}}{(k_i-1)^{\lambda-5}}\cdot\varphi\Big(K\sqrt{k_i-1}\,2^{-\alpha(k_i-1)}\Big) &(4.10)\\ &\le K\sum_{k=1}^{\infty}\frac{1}{k^{\lambda-5-N/(2\alpha)}} = K_{20}. &(4.11)\end{aligned}$$

Recall $\lambda > 6 + N/(2\alpha)$ in deriving the last equality.

Combining (4.9) and (4.10), we obtain (4.6). Thus, by (2.2), we see that

$$\varphi\text{-}P(\mathrm{Gr}X([0,1]^N)) \le K_2,$$

where $K_2 = K_{19} + K_{20}$. This finished the proof of the upper bound in (4.1).

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