SPDEs with α-stable Lévy noise

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Poisson random measure (PRM)

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$$\nu_\alpha(dz) = [p\alpha z^{-\alpha-1} 1_{(0,\infty)}(z) + q\alpha(-z)^{-\alpha-1} 1_{(-\infty,0)}(z)]dz$$

$p, q \geq 0$ with $p + q = 1$ and $0 < \alpha < 2$, $\alpha \neq 1$

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The building blocks

$$L_j(B) = \int_{B \times \{\varepsilon_j < |z| \leq \varepsilon_{j-1}\}} zN(dt, dx, dz)$$

$B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\varepsilon_j \downarrow 0$ with $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots$

Lemma 1 (Itô's construction of a Lévy process)

$$Y(B) = \sum_{j \geq 1} [L_j(B) - E(L_j(B))] + L_0(B)$$

is an "independently scattered random measure" with

$$Y(B) \sim S_\alpha(\sigma |B|^{1/\alpha}, \beta, \mu |B|)$$

$\beta = p - q$; $\sigma > 0, \mu \in \mathbb{R}$ are constants depending on α.

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Definition

$$Z = \{Z(B) = Y(B) - \mu|B|; B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\}$$

is an *α-stable Lévy noise* (or α-stable random measure).

The case $\alpha < 1$:

$$\mu|B| = -|B| \int_{|z|\leq 1} z\nu_\alpha(dz) = -\sum_{j\geq 1} E(L_j(B))$$

$$Z(B) = \sum_{j\geq 0} L_j(B) =: \int_{B\times(\mathbb{R}-\{0\})} zN(dt,dx,dz)$$

The case $\alpha > 1$:

$$\mu|B| = |B| \int_{|z|>1} z\nu_\alpha(dz) = E(L_0(B))$$

$$Z(B) = \sum_{j\geq 0}[L_j(B) - E(L_j(B))] =: \int_{B\times(\mathbb{R}-\{0\})} z\widehat{N}(dt,dx,dz)$$

Rajput and Rosinski (1989)

A function $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is Z-integrable iff $\varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)$.

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Samorodnitsky and Taqqu (1994)

The process

$$\{Z(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x) Z(dt,dx); \varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)\}$$

has jointly α-stable finite dimensional distributions. In particular,

$$Z(\varphi) \sim S_\alpha(\sigma \|\varphi\|_\alpha, \beta, 0).$$

The linear equation

$$Lu(t,x) = \dot{Z}(t,x), \quad t > 0, x \in \mathcal{O} \tag{1}$$

$\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$
$G(t,x,y)$ is the fundamental solution of $Lu = 0$ on $\mathbb{R}_+ \times \mathcal{O}$

Definition

$$u(t,x) = \int_0^t \int_{\mathcal{O}} G(t-s,x,y) Z(ds,dy)$$

is a *solution* of (1) with zero initial conditions, provided that the stochastic integral is well-defined, i.e. $\forall t > 0, \forall x \in \mathcal{O}$

$$\int_0^t \int_{\mathcal{O}} G(t-s,x,y)^{\alpha} dyds < \infty$$

Example 1: Heat equation with $d \geq 1$

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(u,x) + \dot{Z}(t,x), \quad t > 0, x \in \mathbb{R}^d$$

The solution exists iff $\alpha < 1 + 2/d$.

Example 2: Wave equation with $d = 1, 2$

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(u,x) + \dot{Z}(t,x), \quad t > 0, x \in \mathbb{R}^d$$

The solution exists for any $\alpha \in (0, 2)$.

The stochastic integral

Filtration

$\mathcal{F}_t^N$ is the σ-field generated by $N([0,s] \times A \times \Gamma)$, $s \leq t$, $\Gamma \subset \mathbb{R} - \{0\}$ bounded away from 0; $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{N}$

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Elementary processes

$$X(t,x) = 1_{(a,b]}(t) 1_A(x) Y$$

Y is $\mathcal{F}_a$-measurable and bounded; $A \in \mathcal{B}_b(\mathbb{R}^d)$

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Simple processes

$$X(t,x)=\sum_{i=0}^{n-1}1_{(t_{i-1},t_i]}(t)\sum_{j=1}^{m_i}1_{A_{ij}}(x)Y_{ij}$$

Integral of a simple process X

$$I(X)(t,B) = \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} Y_{ij} Z((t_i \wedge t, t_{i+1} \wedge t] \times (A_{ij} \cap B))$$

is càdlàg in t. (The α-stable Lévy process $\{Z(t,B) = Z([0,t] \times B); t \geq 0\}$ has a càdlàg modification.)

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Theorem 1 (Maximal Inequality, B. 2013)

For any bounded simple process X,

$$\sup_{\lambda>0} \lambda^{\alpha} P(\sup_{t \leq T} |I(X)(t,B)| > \lambda) \leq c_{\alpha} E \int_0^T \int_B |X(t,x)|^{\alpha} dx dt$$

The space of integrands

$\mathcal{L}_\alpha$ is the class of all *predictable* processes X such that

$$\|X\|_{\alpha,T,B}^{\alpha} := E\int_0^T \int_B |X(t,x)|^\alpha dxdt < \infty$$

for any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$.
Let $E_k \in \mathcal{B}_b(\mathbb{R}^d)$ with $E_k \uparrow \mathbb{R}^d$. Define

$$\|X\|_\alpha = \sum_{k\geq 1} \frac{1 \wedge \|X\|_{\alpha,k,E_k}}{2^k} \quad \text{if} \quad \alpha > 1$$

$$\|X\|_\alpha^\alpha = \sum_{k\geq 1} \frac{1 \wedge \|X\|_{\alpha,k,E_k}^\alpha}{2^k} \quad \text{if} \quad \alpha < 1$$

Extending the integral to $\mathcal{L}_\alpha$

Proposition 1.

The set of bounded simple processes is dense in $(\mathcal{L}_\alpha, \|\cdot\|_\alpha)$.

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Fix $X \in \mathcal{L}_\alpha$

By Proposition 1, $\exists\ (X_n)_n$ simple processes such that $\|X_n - X\|_\alpha \to 0$. By **Theorem 1** (maximal inequality),

$$\|I(X_n)(\cdot, B) - I(X_m)(\cdot, B)\|_{\alpha,T}^{\alpha} \leq c_\alpha \|X_n - X_m\|_{\alpha,T,B}^{\alpha} \to 0,$$

as $n, m \to \infty$, where

$$\|Y\|_{\alpha,T}^{\alpha} = \sup_{\lambda>0} \lambda^\alpha P(\sup_{t\leq T} |Y(t)| > \lambda)$$

$\Longrightarrow \exists$ a random element $I(X)(\cdot, B)$ in $D[0,T]$ such that

$$P(\sup_{t \leq T} |I(X_n)(t,B) - I(X)(t,B)| > \lambda) \to 0$$

For any $B \in \mathcal{B}_b(\mathbb{R}^d)$, we write

$$I(X)(t,B) = \int_0^t \int_B X(s,x) Z(ds,dx)$$

Lemma 2

Theorem 1 holds for any $X \in \mathcal{L}_\alpha$.

The case $B = \mathbb{R}^d$

Let X be a predictable such that

$$E \int_0^T \int_{\mathbb{R}^d} |X(t,x)|^\alpha dxdt < \infty \quad \forall T > 0$$

Then $I(X)(\cdot, \mathbb{R}^d)$ is defined as the limit in probability of $\{I(X)(\cdot, E_k)\}_{k \geq 1}$ in $D[0,T]$ equipped with the sup-norm, where $E_k \in \mathcal{B}_b(\mathbb{R}^d)$ with $E_k \uparrow \mathbb{R}^d$. We have:

$$\sup_{\lambda > 0} \lambda^\alpha P(\sup_{t \leq T} |I(X)(t, \mathbb{R}^d)| > \lambda) \leq c_\alpha E \int_0^T \int_{\mathbb{R}^d} |X(t,x)|^\alpha dxdt$$

The non-linear equation

σ is a Lipschitz function. We consider the SPDE:

$$Lu(t,x) = \sigma(u(t,x))\dot{Z}(t,x) \quad t > 0, x \in \mathcal{O} \tag{2}$$

Definition

A predictable process $u = \{u(t,x); t > 0, x \in \mathcal{O}\}$ satisfying:

$$u(t,x) = \int_0^t \int_{\mathcal{O}} G(t-s,x,y)\sigma(u(s,y))Z(ds,dy) \quad \text{a.s.}$$

for all $t > 0, x \in \mathcal{O}$ is called a *solution* of (2) with zero initial conditions and Dirichlet boundary conditions.

The main result

Theorem 2 (B. 2013)

Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain and $\forall T > 0$

$$\lim_{h \to 0} \int_0^T \int_{\mathcal{O}} |G(t,x,y) - G(t+h,x,y)|^p dydt = 0 \tag{3}$$

$$\lim_{h \to 0} \int_0^T \int_{\mathcal{O}} |G(t,x,y) - G(t,x+h,y)|^p dydt = 0 \tag{4}$$

$$\int_0^T \left(\sup_{x \in \mathcal{O}} \int_{\mathcal{O}} G(t,x,y)^p dy \right) dt < \infty \tag{5}$$

for some $p > \alpha$ with $p < 1$ if $\alpha < 1$, or $p \leq 2$ if $\alpha > 1$. Then equation (2) has a solution u. Moreover, $\exists$ stopping times $\tau_K \uparrow \infty$ a.s. such that for any $T > 0$ and $K \geq 1$

$$\sup_{t \leq T} \sup_{x \in \mathcal{O}} E(|u(t,x)|^p 1_{\{t \leq \tau_K\}}) < \infty.$$

Example: Heat equation ($\mathcal{O} \subset \mathbb{R}^d$ bounded)

$G(t,x,y) \leq \overline{G}(t,x-y)$ where

$$\overline{G}(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right)$$

$$\sup_{x\in\mathcal{O}} \int_{\mathcal{O}} G(t,x,y)^p dy \leq \int_{\mathbb{R}^d} \overline{G}(t,y)^p dy = ct^{(1-p)d/2}$$

Condition (5) holds if

$$p < 1 + \frac{2}{d}$$

Conditions (3) and (4) hold by the continuity of G

Steps of the proof of Theorem 2:

1. Define a truncated noise Z_K (remove from Z the jumps which exceed a value K) and a stochastic integral with respect to Z_K for which one has a p-moment inequality
2. Solve equation (2) with noise $\dot{Z}_K$ (instead of $\dot{Z}$) yielding a solution $u_K(t,x)$
3. Show that for any $t > 0, x \in \mathcal{O}, L > K$

$$u_K(t,x) = u_L(t,x) \quad \text{a.s. on } \{t \leq \tau_K\}$$

for some stopping times $\tau_K \uparrow \infty$ as $K \to \infty$.
Show that the process $u(t,x) = u_K(t,x)$ on $\{t \leq \tau_K\}$ is a solution of (2)

1. The truncated noise: the case $\alpha < 1$

$$Z_K(B) = \int_{B\times\{0<|z|\le K\}} zN(dt, dx, dz)$$

$$E(e^{iuZ_K(B)}) = \exp\left\{|B| \int_{|z|\le K} (e^{iuz} - 1)\nu_\alpha(dz)\right\}$$

Lemma 3 (tail behavior: Giné and Marcus, 1983)

$$\begin{aligned} P(|Z_K(B)| > \lambda) &\le c_\alpha |B| \lambda^{-\alpha} \quad \text{for all} \quad \lambda > 0 \\ P(|Z_K(B)| > \lambda) &\le c'_\alpha K^{1-\alpha} |B| \lambda^{-1} \quad \text{for all } \lambda > K \end{aligned}$$

Consequently, $E|Z_K(B)|^p \le C_{\alpha,p} K^{p-\alpha} |B|$ for all $p \in (\alpha, 1)$.

1. The truncated noise: the case $\alpha < 1$

Theorem 1 (maximal inequality) holds for

$$I_K(X)(t,B) = \int_0^t \int_B X(s,x) Z_K(ds,dx)$$

for X simple. For $X \in \mathcal{L}_\alpha$, $I_K(X)$ is defined similarly to $I(X)$.

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Theorem 3 (Moment Inequality, B. 2013)

For any $p \in (\alpha, 1)$ and for any $X \in \mathcal{L}_p$,

$$E|I_K(X)(t,B)|^p \leq C_{\alpha,p} K^{p-\alpha} E \int_0^t \int_B |X(s,x)|^p dx ds$$

1. The truncated noise: the case $\alpha > 1$

$$Z_K(B) = \int_{B \times \{0 < |z| \leq K\}} z \widehat{N}(dt, dx, dz)$$

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Saint Loubert Bié (1998): $p \in [1, 2]$ fixed

For any bounded simple process $\{Y(t, x, z)\}$

$$E \sup_{t \leq T} \left| \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}-\{0\}} Y(s, x, z)\widehat{N}(ds, dx, dz) \right|^p \leq$$

$$c_p \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} |Y(t, x, z)|^p \nu_\alpha(dz)dxdt := [Y]^p_{p,T}$$

1. The truncated noise: the case $\alpha > 1$

For $p \in [1, 2]$ *fixed*, we define the integral w.r.t. $\widehat{N}$ for any process Y with $[Y]_{p,T} < \infty$, as an element in $L^p(\Omega; D[0, T])$

1. The truncated noise: the case $\alpha > 1$

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Definition of $I_K(X)$

Let

$$Y(t, x, z) = X(t, x) z 1_{\{|z| \leq K\}}$$

If $p > \alpha$ and $X \in \mathcal{L}_p$, then $[Y]_{p,T} < \infty$ and we can define

$$I_K(X)(t, B) := \int_0^t \int_B \int_{|z| \leq K} X(s, x) z \widehat{N}(ds, dx, dz)$$

Maximal Inequality

$$E \sup_{t \leq T} |I_K(X)(t, B)|^p \leq C_{\alpha,p} K^{p-\alpha} E \int_0^T \int_B |X(s, x)|^p dx ds$$

2. The equation with truncated noise

$$Lu(t,x) = \sigma(u(t,x))\dot{Z}_K(t,x) \quad t>0, x \in \mathcal{O} \qquad (6)$$

$\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, or $\mathcal{O} = \mathbb{R}^d$

Theorem 4 (B. 2013)

Under (3), (4) and (5), equation (6) with zero initial conditions has a unique solution u_K. This solution is $L^p(\Omega)$-continuous and satisfies

$$\sup_{t \leq T} \sup_{x \in \mathcal{O}} E|u_K(t,x)|^p < \infty$$

Proof: As in Dalang (1999), using the previous p-moment inequalities for $I_K(X)$ (instead of Burkholder's inequality).

3. The stopping times

Peszat and Zabczyk (2006, 2007)

For any $B \in \mathcal{B}_b(\mathbb{R}^d)$, define

$$\tau_K(B) = \begin{cases} \inf\{t > 0; Z(t,B) - Z(t-,B) > K\} & \text{if } \alpha < 1 \\ \inf\{t > 0; L_0(t,B) - L_0(t-,B) > K\} & \text{if } \alpha > 1 \end{cases}$$

$$P(\tau_K(B) > T) = \exp(-T|B|K^{-\alpha})$$

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Remark

$\tau_K(\mathbb{R}^d)$ is *not* defined! For this reason, the study of equation (2) on $\mathcal{O} = \mathbb{R}^d$ is an open problem.

Thank you!