Applications of Fractional Calculus to Stochastic Models for Finance

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Outline

- Empirical features of financial data and geometric Brownian motion (GBM) model.
- Use of random time change by a suitable stochastic process to obtain the fractal activity time geometric Brownian motion (FATGBM) model.
- Discuss several time change processes, an inverse of the standard stable subordinator and a fractional tempered stable Lévy motion.
- Models for high-frequency data.
- Use fractional Skellam process in these models to reflect the empirical features.

Geometric Brownian motion model

Geometric Brownian motion (GBM) or Black-Scholes model for risky asset:

$$P_t = P_0 e^{\{\mu t + \sigma B(t)\}}, \quad t > 0$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and B is Brownian motion.
Log returns: $X_t = \log P_t - \log P_{t-1}$, and in GBM

$$X_t = \mu + \sigma(B(t) - B(t-1)), \quad t \geq 1.$$

According to this model, the log returns X_t, $t = 1, 2, 3, \ldots$ are i.i.d. Gaussian

Stylized features

Features of log returns observed in practice (Granger 2005):

- Log returns are reasonably approximated by uncorrelated identically distributed random variables (independent in the Gaussian case)
- Squared and absolute log returns are dependent through time, with autocorrelation functions decreasing very slowly, remaining substantial after 50 to 100 lags
- Log returns have distributions that are heavier-tailed and higher-peaked than Gaussian distributions
- Variance is dependent upon time.

Example: USD:EUR exchange rate, log returns

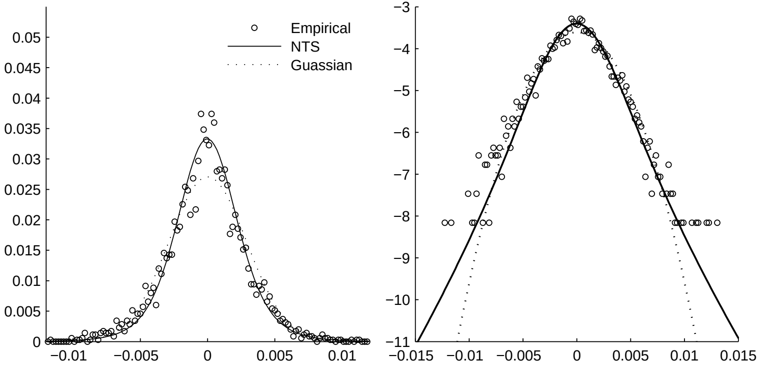


Figure: Empirical density and log density for daily return spot data

Correlation function

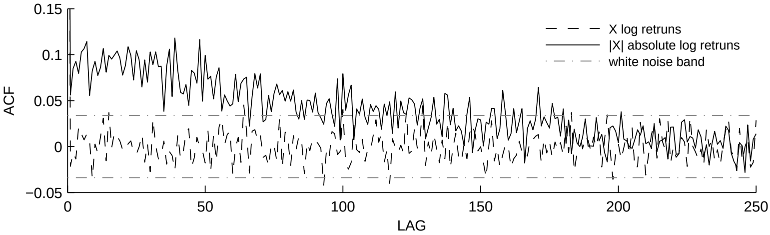


Figure: Autocorrelation plot

Approaches to incorporating empirical features

- Exponential Lévy models $P(t) = e^{-\chi(t)}$ for a Lévy process χ, capture non-Gaussian distributions of log returns but not other features.
- Time-changed models, where calendar time is replaced by a suitable stochastic process.
- Time change can be done in the GBM or in exponential Lévy models.

FATGBM

Fractal activity time GBM (FATGBM, Heyde (1999)):

$$\log P_t = \log P_0 + \mu t + \theta T_t + \sigma B(T_t),$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and $\theta \in R$.

The process $\{T_t\}$ is independent of the Brownian motion, positive, nondecreasing, with stationary or non-stationary (but not independent) increments $\tau_t = T_t - T_{t-1}$, and $T_0 = 0$.

The process $\{T_t\}$ has an attractive interpretation of information flow or trading volume (Howison and Lamper (2001)).

The process $\{\tau_t\}$ replaces one unit of calendar time.

If $T_t = t$, then FATGBM becomes the classical Black-Scholes model.

FATGBM model properties

Marginal distributions of log returns:

$$X_t = \log P_t - \log P_{t-1} \stackrel{d}{=} \mu + \theta\tau_t + \sigma\sqrt{\tau_t}B(1),$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Covariance of log returns (zero in the symmetric case $\theta = 0$):

$$cov(X_t, X_{t+k}) = \theta^2 cov(\tau_t, \tau_{t+k}),$$

Covariance of squared returns :

$$cov(X_t^2, X_{t+k}^2) = (\sigma^4 + 4\theta^2\mu^2 + 4\theta\mu\sigma^2) cov(\tau_t, \tau_{t+k}) + \theta^4 cov(\tau_t^2, \tau_{t+k}^2) +$$

$$(\theta^2\sigma^2 + 2\theta^3\mu)(cov(\tau_t^2, \tau_{t+k}) + cov(\tau_t, \tau_{t+k}^2)).$$

For $\mu = \theta = 0$ we also have

$$cov(|X_t|, |X_{t+k}|) = \frac{2}{\pi}\sigma^2 cov(\sqrt{\tau}_t, \sqrt{\tau_{t+k}}).$$

Inverse stable subordinator as an activity time

A standard α-stable subordinator $D(t)$ is a Lévy subordinator (positive strictly increasing Lévy process) with the Laplace transform $\mathbb{E}[e^{-\theta D(u)}] = e^{-u\theta^\alpha}$, $\theta > 0$, $t \geq 0$ with $\alpha \in (0,1)$.

The inverse α-stable subordinator $E(t)$ is defined as the inverse or first passage time of a stable subordinator $D(t)$, that is

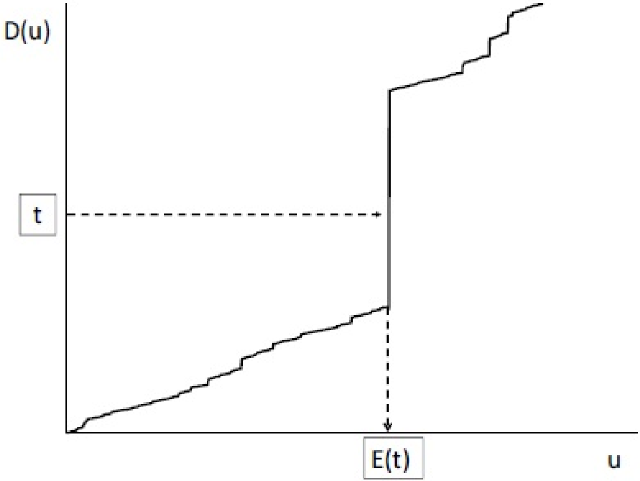
$$E(t) = \inf\{u \geq 0 : D(u) > t\}, \quad t \geq 0,$$

see, for example, Meerschaert and Sikorskii (2012).

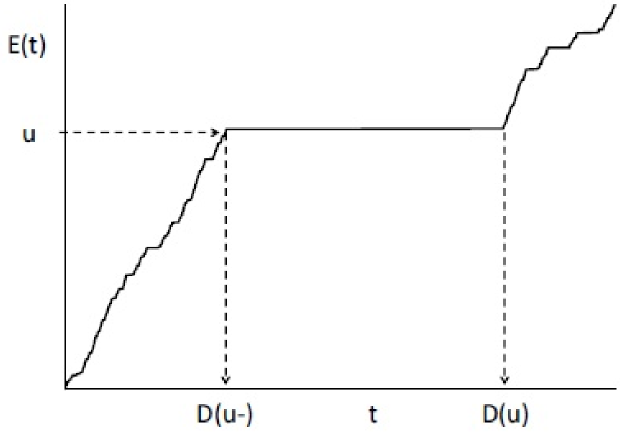
Note that $E(t)$, $t \geq 0$ is non-Markovian with non-stationary and non-independent increments. The moments are

$$\mathbb{E}[E^k(t)] = \frac{t^{\alpha k} k!}{\Gamma(\alpha k + 1)}.$$

Simulated stable subordinator



Simulated inverse stable subordinator



Properties of the inverse stable subordinator

The Laplace transform of the inverse stable subordinator is

$$\mathbb{E}[e^{-\theta E(t)}] = \mathcal{E}_\alpha(-\theta t^\alpha), \quad \theta > 0,\ t \geq 0,\ \alpha \in (0,1),$$

where

$$\mathcal{E}_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)} \qquad z \in \mathbb{C},\ \alpha \in (0,1)$$

is one-parameter Mittag-Leffler function, see for example Mainardi and Gorenflo (2000).

Time-changed Brownian motion

The time change process $B(E(t))$ is a limit of a CTRW (Meerschaert and Scheffler (2004)).

The outer process $B(t)$ models the random walk, and the inner process (time change) $E(t)$ accounts for waiting times.

The time-changed process is governed by

$$D_t^\alpha p = \frac{1}{2}\frac{\partial^2 p}{\partial x^2},$$

with the Caputo fractional derivative of order $0 < \alpha < 1$

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(\tau)(t-\tau)^{-\alpha}\, d\tau.$$

The Laplace transform of this derivative is $s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0)$, where $\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$.

Time change to the inverse stable subordinator in the GBM

- Considered in Janczura and Wyłomańska (2009), using the result of Magdziarz who showed that $B(E(t))$ is a martingale with respect to a suitably defined filtration.
- Magdziarz and Schilling (2015) prove the LLN for time changes of the Brownian motion by inverse subordinators.
- Using martingale property, Janczura and Wyłomańska computed that for $0 < s < t$:

$$\mathrm{corr}(B(E(t)), B(E(s))) = \left(\frac{s}{t}\right)^{\alpha/2}.$$

- When s is fixed and $t \to \infty$, correlation decays slowly.

More general result

From Leonenko, Meerschaert, Schilling and Sikorskii (2014):
Theorem. Suppose that $\chi(t)$, $t \geq 0$ is a homogeneous Lévy process with $X(0) = 0$, and $Y(t)$ is a non-decreasing process independent of χ. If $\psi(\xi) = -\log \mathbb{E} e^{i\xi\chi(1)}$ is the characteristic exponent of the Lévy process, then the characteristic function of the process $Z(t) = \chi(Y(t))$ is given by

$$\mathbb{E} e^{i\xi Z(t)} = \mathbb{E} e^{-\psi(\xi) Y(t)}.$$

Moreover, if $\mathbb{E}\chi(1)$ and $U(t) = \mathbb{E} Y(t)$ exist, then $\mathbb{E} Z(t)$ exists and

$$\mathbb{E}[Z(t)] = U(t)\mathbb{E}[\chi(1)];$$

if χ and Y have finite second moments, so does Z and

$$\mathrm{Var}[Z(t)] = [\mathbb{E}\chi(1)]^2 \,\mathrm{Var}[Y(t)] + U(t)\,\mathrm{Var}[\chi(1)],$$

and the covariance function is given by

$$\mathrm{Cov}[Z(t), Z(s)] = \mathrm{Var}[\chi(1)] U(\min(t,s)) + [\mathbb{E}\chi(1)]^2 \,\mathrm{Cov}[Y(t), Y(s)].$$

Second order properties

The covariance function of $Z(t) = \chi(E(t))$ is computed in Leonenko, Meerschaert, Schilling and Sikorskii (2014) (see also Veilette and Taqqu (2010) for the covariance of E_t):

$$\begin{aligned}\mathrm{Cov}[Z(t), Z(s)] = &\frac{s^\alpha \,\mathrm{Var}[\chi(1)]}{\Gamma(1+\alpha)} \\ &+ \frac{[E\chi(1)]^2}{\Gamma(1+\alpha)^2}\left[\alpha s^{2\alpha} B(\alpha, \alpha+1) + F(\alpha; s, t)\right]\end{aligned}$$

where

$$B(a, b; x) := \int_0^x u^{a-1}(1-u)^{b-1} du$$

is the incomplete beta function, $B(a, b; 1) = B(a, b)$, and

$$\begin{aligned}F(\alpha; s, t) &:= \alpha t^{2\alpha} B(\alpha, \alpha+1; s/t) - (ts)^\alpha \\ &= -\alpha \frac{(s/t)^{\alpha+1}}{\alpha+1} + O((s/t)^{\alpha+2}), \text{ for fixed } s > 0 \text{ and } t \to \infty.\end{aligned}$$

The correlation function asymptotics

Note that $F(\alpha; s, t) \to 0$ as $t \to \infty$, hence

$$\mathrm{Cov}[Z(t), Z(s)] \to \frac{s^{\alpha}\,\mathrm{Var}[X(1)]}{\Gamma(1+\alpha)} + \frac{s^{2\alpha}[\mathbb{E}\chi(1)]^2}{\Gamma(1+2\alpha)} \quad \text{as } t \to \infty.$$

If $\mathbb{E}[\chi(1)] = 0$, then the correlation function is

$$\mathrm{corr}[Z(t), Z(s)] = \left(\frac{s}{t}\right)^{\alpha/2}.$$

The correlation function of $Z(t)$ falls off like a power law $t^{-\alpha}$ when $\mathbb{E}[\chi(1)] \neq 0$, and even more slowly, like the power law $t^{-\alpha/2}$ when $\mathbb{E}[\chi(1)] = 0$.

In either case, the non-stationary time-changed process $Z(t)$ exhibits long range dependence (defined as a slow decay of the correlation function).

Applicability to financial data

If $\mathbb{E}[\chi(1)] = 0$, the time-changed process $Z(t) = \chi(E(t))$ also has uncorrelated increments. Since $\text{Cov}[Z(t), Z(s)]$ does not depend on t in this case, we have $\text{Var}[Z(s)] = \text{Cov}[Z(s), Z(s)] = \text{Cov}[Z(s), Z(t)]$ and hence, since the covariance is additive, $\text{Cov}[Z(s), Z(t) - Z(s)] = 0$ for $0 < s < t$ and t large.

Uncorrelated increments together with long range dependence is a hallmark of financial data, and hence the time-changes process $\chi(E(t))$ can be useful in modeling such data.

Since the outer process $\chi(t)$ can be any Lévy process with a finite second moment, the distribution of the time-changed process $Z(t) = \chi(E(t))$ can take many forms.

Distributions in FATGBM

Since $\log P_t \stackrel{d}{=} \log P_0 + \mu t + \theta T_t + \sigma\sqrt{T_t}B(1)$, the conditional distribution of X_t given $T = V$ is normal with mean $\mu + \theta V$ and variance $\sigma^2 V$, normal mixed or generalized hyperbolic distributions (Barndorff-Nielsen, Kent, and Sørensen (1982)).

If T_t is has a tempered stable distribution, then $\log P_t$ has a normal tempered stable distribution.

Take a stable random variable X distributed as $\sim s(\kappa, c, 1, 0)$, $\kappa(0, 2]$, $c > 0$ with the Laplace transform

$$\mathbb{E}[e^{-uX}] = exp\left\{-\frac{c^\kappa}{cos\frac{\pi\kappa}{2}}u^\kappa\right\}$$

Here κ is the index of stability, c is the scale parameter; other fixed parameters $\beta = 1$ (positively skewed), and shift $\mu = 0$.

Tempered stable

Re-parametrize a positively skewed κ-stable density function

$$s_{\kappa,\delta}(x) = s(x; \kappa, (\delta 2^{\kappa} cos(\pi\kappa/2))^{1/\kappa}, 1, 0).$$

Exponential tilting (or tempering) creates a new random variable Y by tempering an existing random variable X. We use the tempering function $e^{\lambda x}$ with $\lambda = -\frac{1}{2}\gamma^{1/\kappa}$ to arrive at the exponentially tilted version of the positively skewed κ-stable law $s(\kappa, \delta)$ with pdf

$$f_{TS}(x) = f_{TS}(x; \kappa, \delta, \gamma) = e^{\delta\gamma} s_{\kappa,\delta}(x) e^{-\frac{x}{2}\gamma^{1/\kappa}},\ x > 0,$$

and parameters $\kappa \in (0, 1)$, $\delta > 0$, $\gamma > 0$. The characteristic function is

$$\phi_{TS}(u) = \mathbb{E}e^{iuY} = e^{\delta\gamma - \delta(\gamma^{\frac{1}{\kappa}} - 2iu)^{\kappa}} = e^{\delta\gamma(1-(1-2iu\gamma^{-1/\kappa})^{\kappa})},$$

Construction

Define

$$T(t) = \int_0^t (1 - s/t)_+^{H-1/2} L(ds) = \int_0^t k(t,s) L(ds),$$

where $x_+ = \max(x, 0)$, $1/2 < H < 1$, L is a Lévy subordinator with $L(0) = 0$. Convolutions of this type were studied in Bender and Marquardt (2009) and James and Zhang (2010).

The kernel $k(t,s) = (1 - s/t)_+^{H-1/2}$ in is called the Holmgren-Liouville kernel.

We choose the process L in such a way that $T(t)$ has exact tempered stable marginal distributions (more generally: a self-decomposable distribution). Another possible choice of the kernel $k(t,s) = (t-s)_+^{H-1/2}$ yields the fractal activity time with similar properties.

Connection to fractional calculus

The process $T(t)$ can be viewed as fractional Lévy motion of type II, using the terminology applied to type II Brownian motion by Marinucci and Robinson (1999).

Recall that type II fractional Brownian motion is defined via Riemann-Liouville fractional integral

$$W_H(t) = (2H)^{1/2} \int_0^t (t-s)^{H-1/2} dB(s), \quad t > 0$$

$$W_H(t) = (2H)^{-1/2} \int_t^0 (s-t)^{H-1/2} dB(s), \quad t < 0$$

in contrast to type I fractional Brownian motion

$$B_H(t) = \frac{1}{A(H)} \int_{\mathbb{R}} [(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}] dB(s), \quad s \in \mathbb{R},$$

$$A(H) = \left(\frac{1}{2H} + D(H) \right)^{1/2}, \quad D(H) = \int_0^\infty [(1+s)^{H-1/2} - s^{H+1/2}]^2 ds.$$

Fractional motions

(Type I) fractional stable motion discussed in Samorodnitsky and Taqqu (1994); motions driven by Lévy processes with finite second moment and no Brownian component studied in Marquardt (2006); fractional tempered stable motion introduced in Meerschaert and Sabzikar (2016); other kernels similar to type I considered by Klüppelberg and Matsui (2014), others.

Houdré and Kawai (2006) define fractional tempered stable motion as an integral with respect to a tempered stable Lévy process with a very different kernel; parameters are not tractable.

Construction of the fractal activity time

Consider the Lévy process L defined as the sum of independent components:

$$L(t) = \zeta(t) + (H - 1/2)Z(t),$$

where $\zeta(t)$ is a tempered stable Lévy process with $\zeta(1)$ distributed as $TS(\kappa, \delta, \gamma)$, and $Z(t)$ is the background driving Lévy process (BDLP) of $TS(\kappa, \delta, \gamma)$ Ornstein-Uhlenbeck (OU) type process $Y(t)$, that is, a tempered stable stationary process $Y(t)$ is the unique strong solution of the stochastic differential equation

$$dY(t) = -\lambda Y(t)dt + dZ(\lambda t), \; t > 0, \quad Y(0) = Y_0.$$

The activity time T_t

$$T(t) = \int_0^t (1 - s/t)_+^{H-1/2} L(ds)$$

has the same exact marginal distribution as $\zeta(t)$ (Theorem 4.1, Kerss, Leonenko, Sikorskii (2014)).

Other properties of the activity time

The expectation and variance are

$$\mathbb{E}[T(t)] = 2\delta\kappa\gamma^{(\kappa-1)/\kappa}t, \ \mathbb{V}\text{ar}[T(t)] = 4\delta\kappa(1-\kappa)\gamma^{(\kappa-2)/\kappa}t,$$

and the covariance

$$\mathbb{C}\text{ov}[T(t), T(u)] = 8H\delta\kappa(1-\kappa)\gamma^{(\kappa-2)/\kappa}\times \int_0^{t\wedge u}(1-s/t)_+^{H-1/2}(1-s/u)_+^{H-1/2}\,ds.$$

For a fixed $t > 0$ and $u \to \infty$

$$\mathbb{C}\text{orr}[T(t), T(t+u)] \sim \frac{2H}{H+1/2}\sqrt{\frac{t}{t+u}}.$$

Asymptotic homogeneity

The mean and the variance of the increments of the process T are asymptotically homogeneous:

$$\mathbb{E}[(T(t+u) - T(t)] = EL(1)\frac{u}{H+1/2},$$

and when t is fixed and $u \to \infty$

$$\mathbb{V}\text{ar}[T(t+u) - T(t)] \sim \mathbb{V}\text{ar}[L(1)]\frac{u}{2H}.$$

The slow decay of correlations can be interpreted as a long-range dependence for non-stationary process $T(t)$. Since with this construction of the activity time,

$$\mathbb{C}\text{ov}[\log P_t, \log P_u] = \theta^2 \mathbb{C}\text{ov}[T(t), T(u)]$$

this long-range dependence is also present in the logarithm of the price.

High frequency data

Integer-valued processes are used in models for trade by trade data.

Difference of two Poisson processes (or Skellam process) was considered in Barndorff-Nielsen, Pollard and Shephard (2011) and Carr (2011):

$$N^{(1)}(t) - N^{(2)}(t), \qquad t \geq 0,$$

A drawback of this model, which may be at odds with empirical facts, is exponential inter-arrival time, or time between trades.

Skellam distribution and process

The Skellam distribution has been introduced in Irwin (1937) and Skellam (1946). The probability mass function of $S(t)$ is of the form

$$s_k(t) = \mathbb{P}(S(t) = k) = e^{-t(\lambda^{(1)}+\lambda^{(2)})} \left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)^{k/2} I_{|k|}\left(2t\sqrt{\lambda^{(1)}\lambda^{(2)}}\right),$$

$$k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\},$$

where I_k is the modified Bessel function of the first kind

$$I_k(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+k}}{n!(n+k)!}.$$

The Skellam Lévy process is a stochastic solution of the following system of differential equations:

$$\frac{d}{dt}s_k(t) = \lambda^{(1)}(s_{k-1}(t) - s_k(t)) - \lambda^{(2)}(s_k(t) - s_{k+1}(t)), \quad k \in \mathbb{Z}$$

with the initial conditions $s_0(0) = 1$ and $s_k(0) = 0$ for $k \neq 0$.

Fractional Poisson process

The fractional Poisson process (fPP) can be obtained as a renewal process with Mittag-Leffler waiting times between events, defined in Laskin (2003) and Mainardi, Gorenflo and Scalas (2004)

$$N_\alpha(t) = \max\{n \geq 0 : T_1 + \ldots + T_n \leq t\}, \quad t \geq 0, \alpha \in (0,1)$$

where $\{T_j\}$, $j \geq 1$ are independent identically distributed random variables with the Mittag-Leffler distribution function

$$F_\alpha(x) = \mathbb{P}[T_j \leq x] = 1 - \mathcal{E}_\alpha(-\lambda x^\alpha), \quad x \geq 0, \ \alpha \in (0,1),$$

and $F_\alpha(x) = 0$ for $x < 0$. The density function of the Mittag-Leffler distribution

$$f(x) = \frac{d}{dx} F_\alpha(x) = \lambda x^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(-\lambda x^\alpha) \qquad x \geq 0,$$

where

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \qquad z \in \mathbb{C}, \ \alpha > 0, \ \beta > 0$$

is two parameter Mittag-Leffler function.

Differential equations

Beghin and Orsingher (2009) show that the probability mass function of the fractional Poisson process satisfies the system of fractional differential equations

$$D_t^\alpha q_0(t) = -\lambda q_0(t)$$
$$D_t^\alpha q_n(t) = \lambda(q_{n-1}(t) - q_n(t))$$

with the initial condition $q_0(0) = 1$, $q_n(0) = 0$, $n \geq 1$.

Meerschaert, Nane and Vellaisamy (2011) show that the definition of the fractional Poisson process as a renewal process with Mittag-Leffler distribution of inter-arrival times is equivalent to the time change definition

$$N_\alpha(t) = N_1(E(t)),$$

where $N_1(t)$, $t \geq 0$ is a homogeneous Poisson process and $E(t)$, $t \geq 0$ is the inverse stable subordinator independent of $N_1(t)$.

Fractional Skellam process of type I

<u>Definition.</u> Let $N^{(1)}(t)$ and $N^{(2)}(t)$ be two independent homogeneous Poisson processes with intensities $\lambda^{(1)} > 0$ and $\lambda^{(1)} > 0$. Let $E^{(1)}(t)$ and $E^{(2)}(t)$ be two independent inverse stable subordinators with indices $\alpha^{(1)} \in (0,1)$ and $\alpha^{(2)} \in (0,1)$ respectively, which are also independent of the two Poisson processes. The stochastic process

$$X(t) = N^{(1)}(E^{(1)}(t)) - N^{(2)}(E^{(2)}(t))$$

is called a *fractional Skellam process of type I*.

A fractional Skellam process of type I $X(t)$ has marginal laws of fractional Skellam type I denoted by $X(t) \sim fSk(t; \lambda^{(1)}, \alpha^{(1)}, \lambda^{(2)}, \alpha^{(2)})$, which is a new distribution.

Properties of the fSk process of type I

The probability mass function of fSk process of type I is given in terms of the three-parameter Mittag-Leffler function

$$\mathbb{P}(X(t)=k) = \left(\lambda^{(1)} t^{\alpha^{(1)}}\right)^k \sum_{n=0}^{\infty} \left(\lambda^{(1)}\lambda^{(2)} t^{\alpha^{(1)}+\alpha^{(2)}}\right)^n \times \mathcal{E}^{n+k+1}_{\alpha^{(1)},\alpha^{(1)}(n+k)+1}\left(-\lambda^{(1)} t^{\alpha^{(1)}}\right) \mathcal{E}^{n+1}_{\alpha^{(2)},\alpha^{(2)}n+1}\left(-\lambda^{(2)} t^{\alpha^{(2)}}\right)$$

for $k \in \mathbb{Z}$, $k \geq 0$ and when $k < 0$

$$\mathbb{P}(X(t)=k) = \left(\lambda^{(2)} t^{\alpha^{(2)}}\right)^{|k|} \sum_{n=0}^{\infty} \left(\lambda^{(1)}\lambda^{(2)} t^{\alpha^{(1)}+\alpha^{(2)}}\right)^n \times \mathcal{E}^{n+|k|+1}_{\alpha^{(2)},\alpha^{(2)}(n+|k|)+1}\left(-\lambda^{(2)} t^{\alpha^{(2)}}\right) \mathcal{E}^{n+1}_{\alpha^{(1)},\alpha^{(1)}n+1}\left(-\lambda^{(1)} t^{\alpha^{(1)}}\right).$$

The moment generating function is

$$\mathbb{E}\left[e^{\theta X(t)}\right] = \mathcal{E}_{\alpha^{(1)}}\left(\lambda^{(1)} t^{\alpha^{(1)}}(e^{\theta}-1)\right) \mathcal{E}_{\alpha^{(2)}}\left(\lambda^{(2)} t^{\alpha^{(2)}}(e^{-\theta}-1)\right), \ \theta \in \mathbb{R}.$$

Fractional Skellam process of type II

Definition. Let $S(t) = N^{(1)}(t) - N^{(2)}(t)$, $t \geq 0$ be a Skellam process. Let $E(t)$, $t \geq 0$ be an inverse stable subordinator of exponent $\alpha \in (0, 1)$ independent of $N^{(1)}(t)$ and $N^{(2)}(t)$. The stochastic process

$$Y(t) = S(E(t)) = N^{(1)}(E(t)) - N^{(2)}(E(t))$$

is called a *fractional Skellam process of type II*.

Fractional Skellam process of type II $Y(t)$ has marginal laws of fractional Skellam type II, for which we shall write $Y(t) \sim fSk(t; \lambda^{(1)}, \lambda^{(2)}, \alpha)$.

Properties of the fSk process of type II

Let $Y(t) = S(E(t))$ be fractional Skellam process of type II, and let $r_k(t) = P(Y(t) = k)$, $k \in \mathbb{Z}$. The marginal distribution is given by

$$r_k(t) = \frac{1}{t^\alpha}\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)^{k/2} \int_0^\infty e^{-u(\lambda^{(1)}+\lambda^{(2)})} I_{|k|}\left(2u\sqrt{\lambda^{(1)}\lambda^{(2)}}\right) \Phi_\alpha\left(\frac{u}{t^\alpha}\right) du,$$

where

$$\Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 - n\alpha - \alpha)}, \quad 0 < \alpha < 1$$

is the Wright function, also known as the Mainardi function, and $I_{|k|}$ is the modified Bessel function of the first kind.

Differential equations

The marginal distribution of the fSk process of type II satisfies the following system of fractional differential equations:

$$D_t^\alpha r_k(t) = \lambda^{(1)}(r_{k-1}(t) - r_k(t)) - \lambda^{(2)}(r_k(t) - r_{k+1}(t))$$

with the initial conditions $r_0(0) = 1$ and $r_k(0) = 0$ for $k \neq 0$.

The moment generating function $L(\theta, t) = \mathbb{E}e^{\theta X(t)}$ is

$$L(\theta, t) = \mathcal{E}_\alpha(-(\lambda^{(1)} + \lambda^{(2)} - \lambda^{(1)}e^\theta - \lambda^{(2)}e^{-\theta})t^\alpha),$$

and for every $\theta \in \mathbb{R}$ it satisfies the fractional differential equation

$$D_t^\alpha L(\theta, t) = (\lambda^{(1)}(e^\theta - 1) + \lambda^{(2)}(e^{-\theta} - 1))L(\theta, t)$$

with the initial condition $L(\theta, 0) = 1$.

Properties of the FSk process of type II

From the general result from Leonenko, Meerschaert, Schilling and Sikorskii (2014) the fSk process of type II has

$$\mathbb{E}[Y(t)] = \frac{t^{\alpha}(\lambda^{(1)} - \lambda^{(2)})}{\Gamma(1+\alpha)},$$

$$\mathbb{V}\mathrm{ar}[Y(t)] = \frac{t^{\alpha}(\lambda^{(1)} + \lambda^{(2)})}{\Gamma(1+\alpha)} + \left(\lambda^{(1)} - \lambda^{(2)}\right)^2 t^{2\alpha} \left[\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma(1+\alpha)^2}\right]$$

and for $0 \le s \le t$

$$\mathbb{C}\mathrm{ov}[Y(t), Y(s)] = \frac{s^{\alpha}(\lambda^{(1)} + \lambda^{(2)})}{\Gamma(1+\alpha)} + \left(\lambda^{(1)} - \lambda^{(2)}\right)^2 \mathbb{C}\mathrm{ov}[E(t), E(s)],$$

where the covariance function for the inverse stable subordinator was given earlier.

Empirical investigation

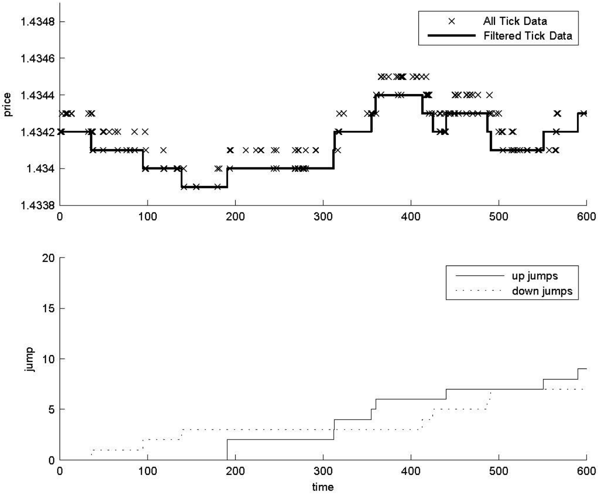
- We consider transaction records for the September 2011 Eurofx over a three month horizon from the 22nd June until expiration on the 22nd September 2011.
- The Eurofx is a type of forward asset known as a future, a contract to buy (or sell) as asset at predetermined future date and price.
- Unlike options that give the right to buy or sell, futures are obligations.
- The data set was obtained directly from the Chicago Mercantile Exchange. The market is open from 12pm Sunday evening until Friday at 5pm with a one hour close each day between 4pm and 5pm.

Data

- The price of the forward asset at time t is denoted by $F(t)$, $t = 1, 2, ..., N$.
- The data contain the transacted price along with timestamps binned to the nearest second.
- For specified period there are $N = 5,465,779$ timestamped transactions recorded.
- Of these records, 71% of transactions get completed at the previous trade price.
- No tick change from one trade to the next, and single tick price changes account for 98% of all transactions.
- Close symmetry between negative and positive tick jumps of the same magnitude is seen.
- For example, the count for jumps of three ticks up or down is 1,411 and 1,419 respectively.

Price path plot

Data were filtered to record transactions only outside of bid-ask spread. The filtered transaction chain still contains 5, 465, 779 records.



Separation of up and down jumps

Next we consider the up and down jump processes in two models for the spot prices. First is the Skellam model of Barndorff-Nielsen, Pollard and Shephard (2011). It is known that jumps of two independent Lévy processes are not simultaneous almost surely.

The second model is fSk process of type I model to model the price movements. As follows from the lemma below, absence of simultaneous jumps also holds for two components in fractional Skellam process of type I.

Lemma. Let $X(t) = N^{(1)}(E^{(1)}(t)) - N^{(2)}(E^{(2)}(t))$ be the fractional Skellam process of type I. The processes $N^{(1)}(E^{(1)}(t))$ and $N^{(2)}(E^{(2)}(t))$ have no common points of discontinuity almost surely. The parameters of the up and down processes are estimated using methods discussed in Cahoy, Uchaikin and Woyczynski (2013).

Statistical analysis

Let T be a random variable with Mittag-Leffler distribution and $T_1, \ldots, T_n$ iid sample. The moment estimators for the parameters are

$$\hat{\alpha} = \frac{2\pi}{\sqrt{2(6\widehat{\mathbb{V}\mathrm{ar}[\log(T)]} + \pi^2)}}, \quad \text{and} \quad \hat{\lambda} = \exp\{-\hat{\alpha}(\mathbb{E}[\widehat{\log(T)}] + \gamma)\},$$

where γ is Euler's constant and

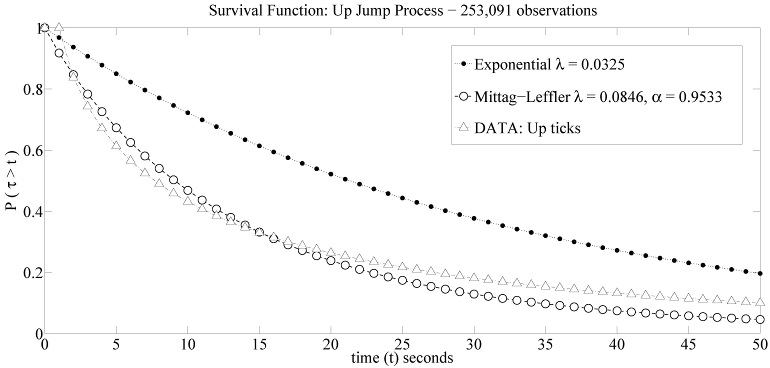
$$\mathbb{E}[\widehat{\log(T)}] := \frac{1}{n}\sum_{i=1}^{n} \log T_i, \ \mathbb{V}\mathrm{ar}[\widehat{\log(T)}] := \frac{1}{n}\sum_{i=1}^{n}(\log T_i - \mathbb{E}[\widehat{\log(T)}])^2.$$

The estimator for α is asymptotically normal as as $n \to \infty$:

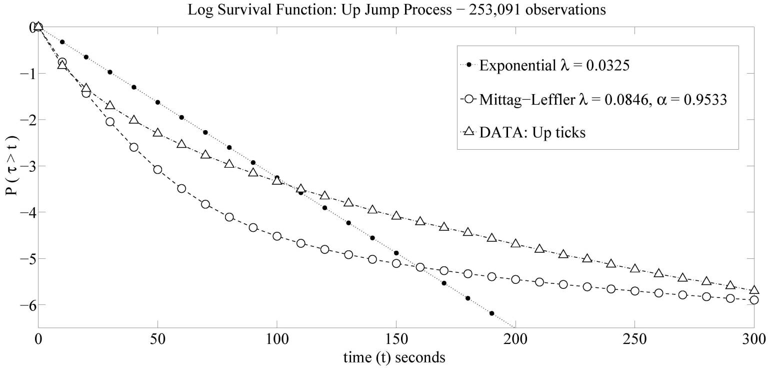
$$\sqrt{n}(\hat{\alpha} - \alpha) \longrightarrow N\left[0, \frac{\alpha^2(32 - 20\alpha^2 - \alpha^4)}{40}\right],$$

and we obtained an asymptotic $(1-\epsilon)100\%$ confidence interval for α.

Survival function of the up jumps



Log of the survival function of the up jump process



Summary and future work

- We have shown that the inter-arrival times between the jumps in both the positive and negative jump processes are clearly not exponential.
- The Mittag-Leffler law provides a closer fit to the data, however the fit is not perfect and even with the added flexibility of an additional parameter, the Mittag-Leffler does not seem to provide tails that are as heavy as the market suggests.
- Although the magnitude of 98% percent of jumps is a single tick, there is the case to extend the models further to allow for jumps greater than one tick.
- One possibility is to consider models that include the difference of fractional compound Poisson processes.

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