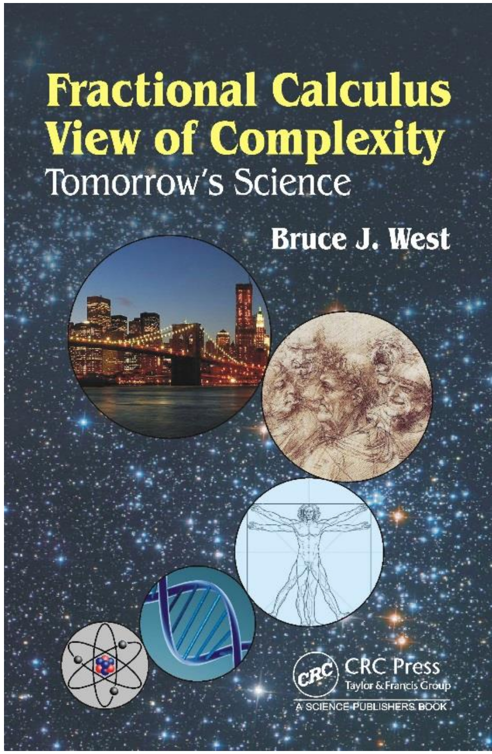
Complexity Science and Fractional Calculus

presented at

Future Applications of Fractional Calculus

October 17, 2016



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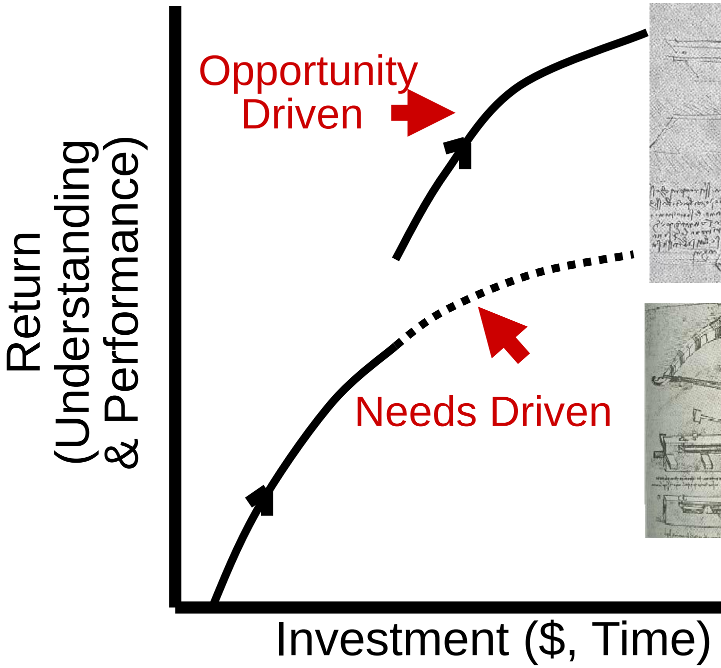
Outline of Talk

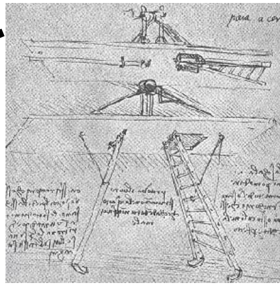
- **What will we talk about?**
  - Research thrusts of Mathematics Division, Army Research Office (ARO)
    - How does complexity enter the picture?
    - What does that imply for fractional calculus?
  - Research campaigns in US Army Research Laboratory (ARL)
  - Potential applications of fractional calculus in ARL research campaigns

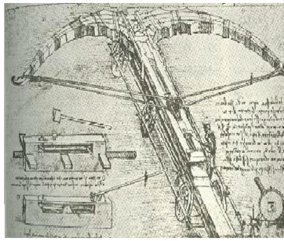
ARL/AROInvestment Strategy

Opportunity Driven vs. Need Driven Research

Goal is a Balance Between Near- and Long-Term Investments







Need Driven Research: improve specific capabilities or overcome identified technology barriers (evolutionary: ripples)

Opportunity Driven Research: develop and exploit scientific breakthroughs to produce revolutionary new capabilities (revolutionary: tsunami)

And

Selective Investing: Investing where we can make a difference
Coordinated and complementary; not duplicative
Potential to advance the discipline, and ultimately the Army

ARO Mathematics Division

- Probability and Statistics (Dr. Andrew Vlasic SETA Contractor)
  - Tales of tails
  - Non-commutative probabilities
- Modeling of Complex Systems (Dr. Joseph Myers)
  - Social and sociolinguistic modeling
  - Geometric and topological modeling
- Computational Mathematics (Dr. Joseph Myers)
  - Multi-scale methods
  - Fractional PDE methods
- Biomathematics (Dr. Virginia Pasour)
  - Fundamental laws of biology
  - Inverse problems

Of what use is fractional calculus?

Fractional calculus can help physical, life and social scientists to understand problems that are otherwise too;

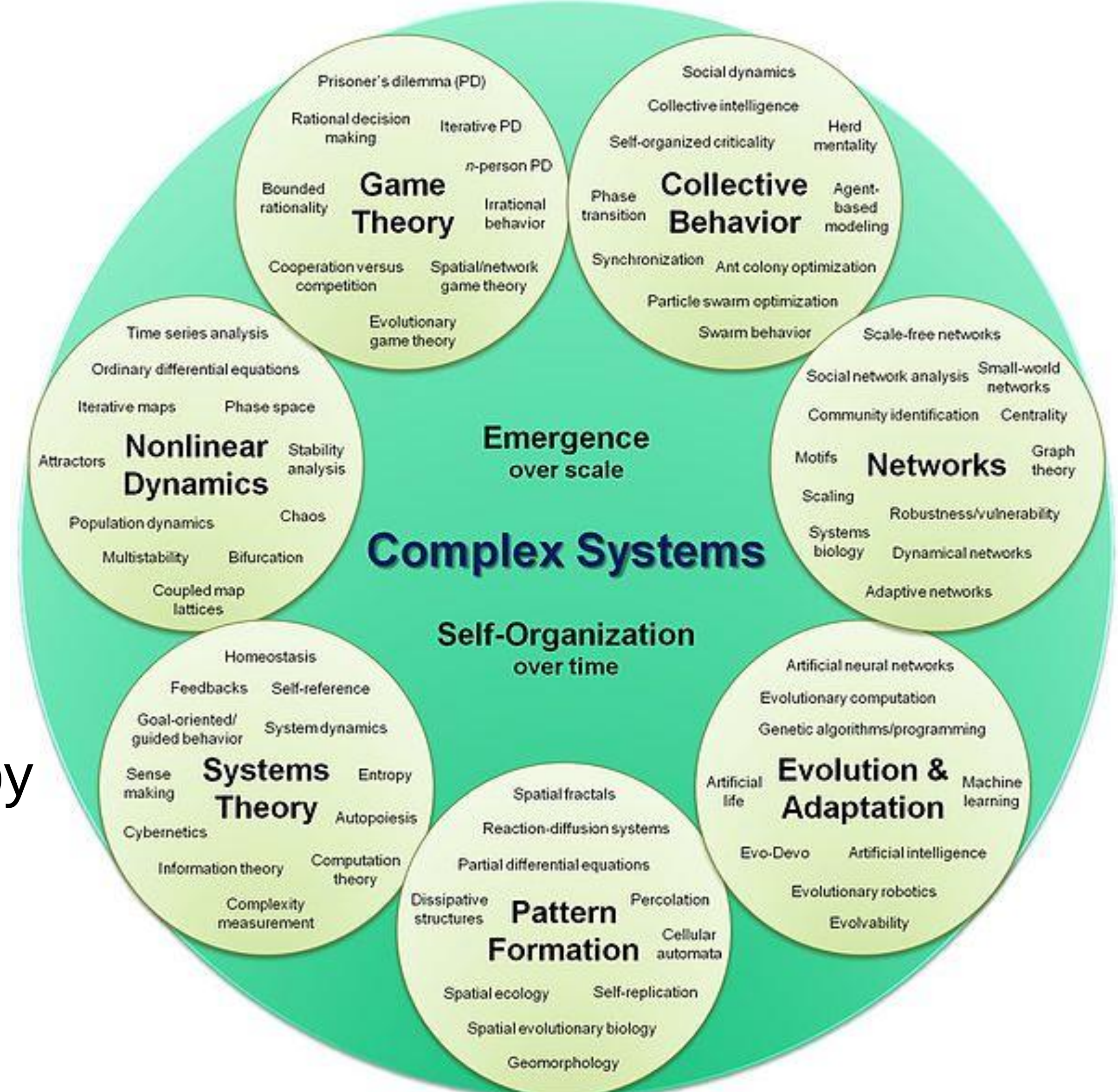
• Big	(the bio- and social-spheres)
• Small	(molecular structure, individuals)
• Slow	(macroevolution of species, societies..)
• Fast	(photosynthesis, phase transitions)
• Remote in time	(early extinctions, genetics, memory)
• Remote in space	(life at extremes, heterogeneity)
• Complex	(human brain, IoBT)
• Dangerous or unethical	(infectious agents, cyber fog)

Modeling of Complex Systems

- **Complexity-induced barriers to understanding**
  - Heterogeneity in space
  - Non-locality in time
  - No characteristic time scales (fractals, scaling)
    - Geometrical, statistical, dynamical
  - Dynamics
    - Strange attractors
    - Non-integrable Hamiltonians
    - Fractional differential equations
  - Trajectories are chaotic
    - Ensembles of chaotic trajectories
    - Scaling ensemble probability distribution functions
    - Fractional probability calculus
  - How do we begin to build a coherent picture?

Complex Systems

- Can these be synthesized?
- Is the fractional calculus entailed by complexity?



Complex Phenomena

MATHEMATICAL MODELING

Analytic functions: Taylor series

$$F(x) = \sum_{j=0}^{\infty} C_j \, (x-\varepsilon)^{\,j}$$

$$F^{(1)}(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = C_1$$

Ordinary calculus
Newton's Laws

Bessel
Hermite
Jacobi
Legendre
,
,

non-analytic functions: gen. Taylor series

$$G(x) = \sum_{j=0}^{\infty} C_j (x-\varepsilon)^{j\alpha} \; ; 0 < \alpha \leq 1$$

$$G^{(\beta)}(x) = \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x^{\beta}} = \lim_{\Delta x \to 0} C_1 \Delta x^{\alpha - \beta} \begin{Bmatrix} \infty & \alpha < \beta \leq 1 \\ & C_1 \delta_{\alpha\beta} \end{Bmatrix}$$

Fractional calculus
new laws

Jacobi
poly-fractonomial
Zayernouri & Karniadakis
J. Comp. Phys. **252** (2013)

Empirical Power Laws

Discipline	Law's name	Form of law
Anthropology		
1913 [4]	Auerbach	Pr(city size rank r) $\propto 1/r$
1998 [65]	War	Pr(intensity $> I$) $\propto 1/I^{\alpha}$
1978 [86]	$1/f$ Music	Spectrum(f) $\propto 1/f$
Biology		
1992 [87]	DNA sequence	Symbol spectrum(frequency f) $\propto 1/f^{\alpha}$
2000 [49]	Ecological web	Pr(k species connections) $\propto 1/k^{1.1}$
2001 [35]	Protein	Pr(k connections) $\propto 1/k^{2.4}$
2000 [34]	Metabolism	Pr(k connections) $\propto 1/k^{2.2}$
2001 [40]	Sexual relations	Pr(k relations) $\propto 1/k^{\alpha}$
Botany		
1883 [64]	da Vinci	Branching; $d_0^{\alpha} = d_1^{\alpha} + d_2^{\alpha}$
1922 [101]	Willis	No. of genera(No. of species N) $\propto 1/N^{\alpha}$
1927 [51]	Murray	$d_0^{2.5} = d_1^{2.5} + d_2^{2.5}$
Economics		
1897 [56]	Pareto	Pr(income x) $\propto 1/x^{1.5}$
1998 [24]	Price variations	Pr(stock price variations x) $\propto 1/x^3$
Geophysics		
1894 [55]	Omori	Pr(aftershocks in time t) $\propto 1/t$
1933 [67]	Rosen–Rammler	Pr(No. of ore fragments $<$ size r) $\propto r^{\alpha}$
1938 [44]	Korčak	Pr(island area $A > a$) $\propto 1/a^{\alpha}$
1945 [31]	Horton	No. of segments at n/No. of segments at $n+1$ = constant
1954 [26]	Gutenberg–Richter	Pr(earthquake magnitude $< x$) $\propto 1/x^{\alpha}$
1957 [27]	Hack	River length $\propto$ (basin area)$^{\alpha}$
1977 [44]	Richardson	Length of coastline $\propto$ 1/(ruler size)$^{\alpha}$
2004 [84]	Forest fires	Frequency density(burned area A) $\propto 1/A^{1.38}$
Information theory		
1999 [32]	World Wide Web	Pr(k connections) $\propto 1/k^{1.94}$
1999 [19]	Internet	Pr(k connections) $\propto 1/k^{\alpha}$

Discipline	Law's name	Form of law
Physics		
1918 [70]	$1/f$ noise	Spectrum(f) $\propto 1/f$
2002 [25]	Solar flares	Pr(time between flares t) $\propto 1/t^{2.14}$
2003 [69]	Temperature anomalies	Pr(time between events t) $\propto 1/t^{2.14}$
Physiology		
1959 [61]	Rall	Neurons; $d_0^{1.5} = d_1^{1.5} + d_2^{1.5}$
1963 [76]	Mammalian vascular network	Veins and arteries; $d_0^{2.7} = d_1^{2.7} + d_2^{2.7}$
1963 [90]	Bronchial tree	$d_0^3 = d_1^3 + d_2^3$
1973 [48]	McMahon	Metabolic rate(body mass M) $\propto M^{0.75}$
1976 [103]	Radioactive clearance	Pr(isotope expelled in time t) $\propto 1/t^{\alpha}$
1987 [93]	West–Goldberger	Airway diameter(generation n) $\propto 1/n^{1.25}$
1991 [30]	Mammalian brain	Surface area $\propto$ volume$^{0.90}$
1992 [77]	Interbreath variability	No. of breaths(interbreath time t) $\propto 1/t^{2.16}$
1993 [58]	Heartbeat variability	Power spectrum(frequency f) $\propto f$
2007 [23]	EEG	Pr(time between EEG events) $\propto 1/t^{1.61}$
2007 [13]	Motivation and addiction	Pr(k behavior connections) $\propto 1/k^{\alpha}$
Psychology		
1957 [75]	Psychophysics	Perceived response(stimulus intensity x) $\propto x^{\alpha}$
1963 [71]	Trial and error	Reaction time(trial N) $\propto 1/N^{0.91}$
1961 [29]	Decision making	utility(delay time t) $\propto 1/t^{\alpha}$
1991 [3]	Forgetting	Percentage correct recall(time t) $\propto 1/t^{\alpha}$
2001 [20]	Cognition	Response spectrum(frequency f) $\propto 1/f^{\alpha}$
2009 [37]	Neurophysiology	Pr(phase-locked interval $< \tau$) $\propto 1/\tau^{\alpha}$
Sociology		
1926 [41]	Lotka	Pr(No. of papers published rank r) $\propto 1/r^2$
1949 [104]	Zipf	Pr(word has rank r) $\propto 1/r$
1963 [16]	Price	Pr(citation rank r) $\propto 1/r^3$
1994 [8]	Urban growth	Population density(radius R) $\propto 1/R^{\alpha}$
1998 [88]	Actors	Pr(k connections) $\propto 1/k^{2.3}$

Complex Webs: Anticipating the Improbable, B.J. West and P. Grigolini, Cambridge (2011).

ARL Research Campaigns

- Human Sciences
- Information Sciences
- Computational Sciences
- Science-for-Maneuver
- Materials Research
- Science for Lethality and Protection
- Assessment and Analysis

Each campaign has complexity-induced research barriers

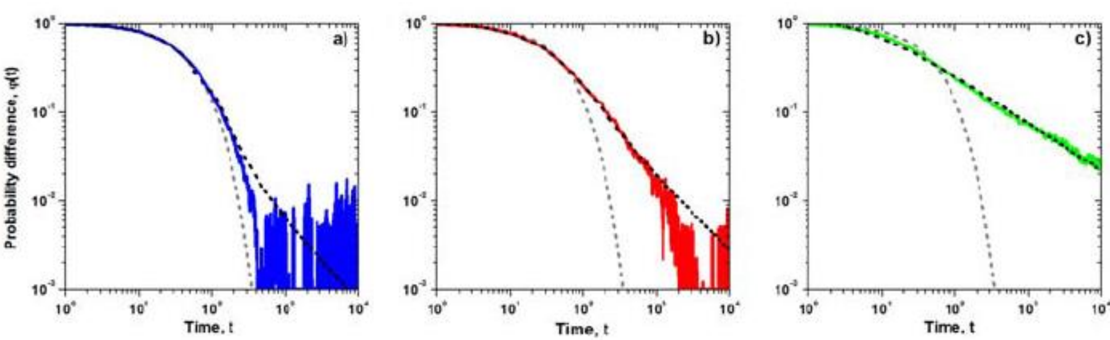
Human Sciences is focused on gaining a greater understanding of individual physical, perceptual, and cognitive performance through Human-Physical Interface, Human-Human Interface, and Human-Technology Interface.

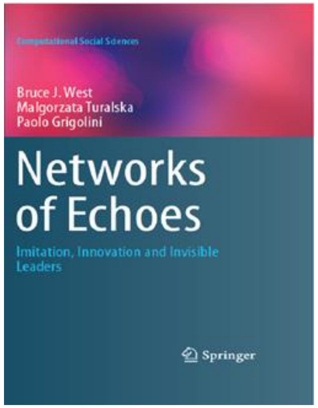
- The application of the fractional calculus appears at the nexus of physics and physiology in the measurement of healthy tissue and tumors. For example, Magin *et al*., *J. Mag. Res.* **190**, 255 (2008) generalize the Bloch-Torrey equations used in modeling nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI):

$$\tau^{\alpha-1}\partial_t^{\alpha}[M_{\pm}(\boldsymbol{r},t)] = \gamma(\boldsymbol{r},t)M_{\pm}(\boldsymbol{r},t) + K_{\beta}\nabla^{2\beta}[M_{\pm}(\boldsymbol{r},t)]$$

- The influence of a complex social network on the behavior of an individual within the network was determined, using renormalization group arguments, to be the fractional rate equation, with a Mittag-Leffler function solution:

$$\partial_t^{\alpha}[P(t)] = -\mu P(t) \rightarrow P(t) = E_{\alpha}(-\mu t^{\alpha})$$



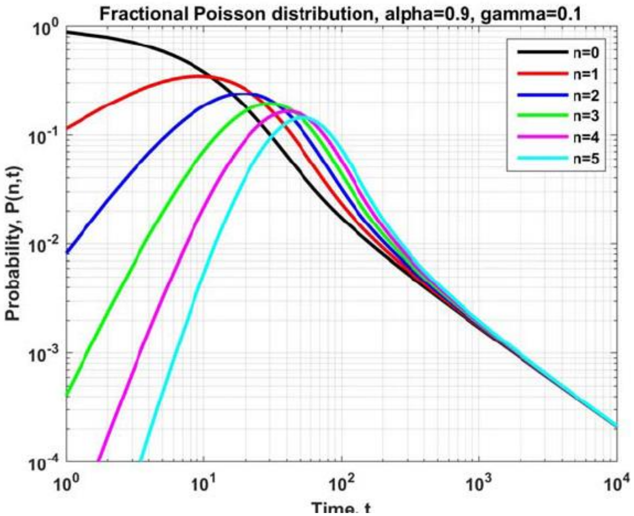


Information sciences is focused on gaining a greater understanding of emerging technology opportunities that support intelligent information systems that perform acquisition, analysis, reasoning, decision-making, collaborative communication, and assurance of information and knowledge through Sensing and Effecting, System Intelligence and Intelligent Systems, Human and Information Interaction, Networks and Communications, and Cyber Security.

- A direct generalization of the Poisson distribution for the arrival time of information packets was made by Laskin, *Comm. Non. Sci. & Num. Sim*. **8**, 201 (2003) to a fractional Poisson process (FPP) expressed in terms of derivatives of the Mittag-Leffler function:

$$P(N,t) = \frac{(-z)^N}{N!} E_\alpha^{(N)}(z), \quad at \quad z = -(\gamma t)^\alpha$$

- At early times the messages look nearly Poisson.
- At late times the messages have IPL statistics, whose time series occurs in bursts.



Computational Sciences Campaign

Computational sciences is focused on advancing the fundamentals of Predictive Simulation Sciences, Data Intensive Sciences, Computing Sciences, and emerging Computing Architectures to transform the future of complex Army applications.

- The two-scale model of Brownian motion has a fractional stochastic differential representation: Basset, *A Treatise on Hydrodynamics, Vol. 2* (1888):

$$\frac{dV(t)}{dt} = -\gamma V(t) - U'(Q) - C\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\frac{dV(\tau)}{d\tau} + f(t)$$

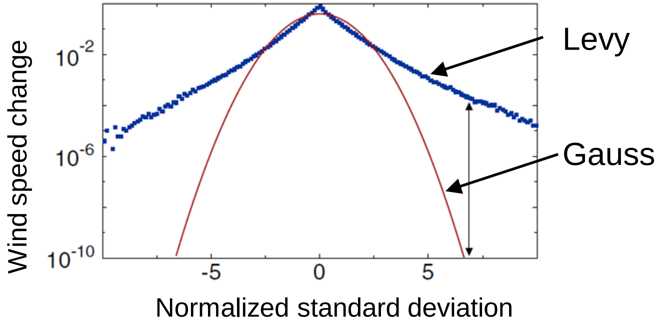
Mainardi and Pironi, *Extracta Mathematicae* **11**, 140 (1996), identified the integral as a fractional derivative of order ½; experiments have experimentally verified the solution in the last few years.

- A fundamental barrier to active control of aerodynamic surfaces is turbulence and our failure to understand complex fluid dynamics. Boettcher *et al.*, *Boundry-Layer Metero* **108** (2003)

$$\partial_t^{\alpha}[P(x,t)] = K_\beta D_{|x|}^{\beta}[P(x,t)]$$

$$P(x,t) = \frac{1}{t^\mu}F\left(\frac{x}{t^\mu}\right) \; ; \; \mu = \frac{\alpha}{\beta}$$

$$E(k) \propto k^{-\frac{9-2\beta}{3}} \; ; \; Kolmogorov - Levy\ scaling$$

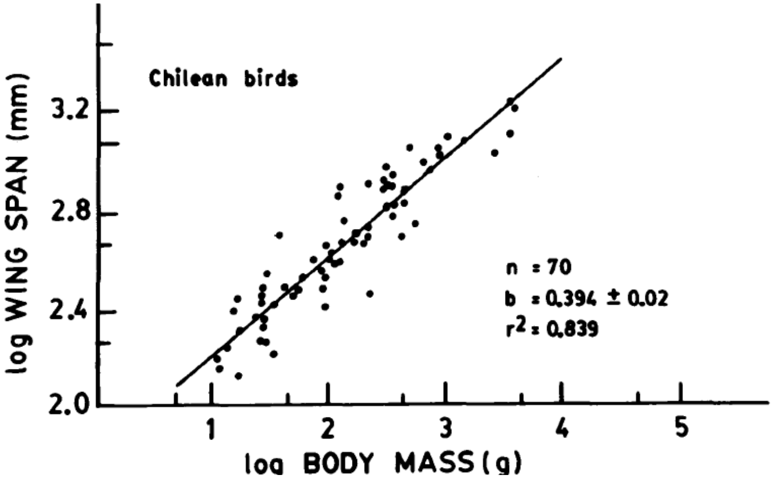


Science-for-Maneuver Campaign

Sciences-for-Maneuver is focused on gaining a greater fundamental understanding of advanced mobility systems and their supporting architectures -- critical to the future Army's movement, sustainment, and maneuverability through Energy and Propulsion, Platform Mechanics, Platform Intelligence, and Logistics and Sustainability.

- One interface between the complexity of the future battle space and science involves the physical restrictions on the maneuverability of unmanned aerial vehicles (UAV).
- The science limitations are two-fold: the lack of mathematical tools and our restricted understanding of low Reynolds number aerodynamics.
- One of the principles that guides the design of micro-UAVs are allometry relations, between a system's functionality and size, for example, between the wing size and body mass of flying animals.
- The ubiquity of allometry relations has been explained by West & West, *Frac. Calc. & App. Anal.* **15** (2012) in terms of the scaling PDF quoted In the Computational Science Campaign:

$$\langle Y \rangle = a \langle M \rangle^b$$



Materials Research Campaign

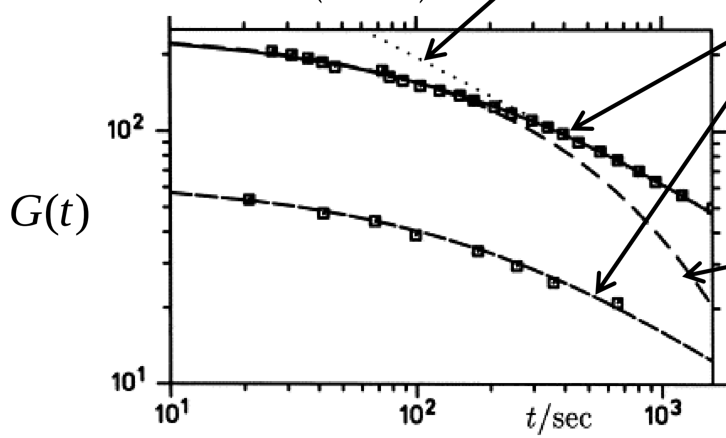
Materials Research is focused on fundamental research for scientific discovery and innovative problem-solving to provide superior materials and devices needed to achieve lasting strategic land power dominance through Structural Materials, Electronics, Photonics, Energy and Power, Biotechnology and Bioinspired, High Strain and Ballistic Materials, and Manufacturing Science, Processing and Sustainment.

- The relaxation of disturbances in complex materials such as taffy, tar, or polymers, are not described by simple rate equations and have been shown to require a fractional relaxation equation to describe their dynamics. The solution to such equations are given by the Mittag-Leffler function by Glöckle & Nonnenmacher, *J. Stat. Phys*. **71** (1993).

$$\partial_t^\alpha\left[G(t)\right] = -\lambda G(t) \Rightarrow G(t) = G(0)E_\alpha\left(-\lambda t^\alpha\right)$$

Nutting $t^{-\alpha}$

Mittag-Leffler function (MLF)



Kohlrausch-Williams-Watts

$e^{-\lambda t^\alpha}$

Sciences for Lethality & Protection Campaign

Sciences-for-Lethality and Protection is focused on gaining a greater understanding of emerging technologies that support weapon systems, protection systems, and the mechanisms of injury affecting the warfigther through Lethality Research for Soldiers and Army Platforms, Protection Research for Soldiers and Army Platforms, and Battlefield Injury Mechanisms.

- The research into disruptive energetics is focused on the exploration of formulations of energetic materials, which seeks to understand very high energy density storage and release on desired time scales. For anomalous diffusion the fractional reaction-diffusion equation is more appropriate

$$\partial_t[\varphi] = \gamma \partial_x^{\beta}[\varphi] + \mathrm{F}(\varphi)$$

See, for example, del-Castillo-Negrete, Carreras & Lynch, *PRL* **91** (2003); Baeumer, Kovacs & Meerschaert, *Comp. & Math. with Appl*. **55** (2008).

- Viscoelastic properties of soft biological tissues provide information useful in medical diagnosis and the treatment of wounds. Meral, Royston & Magin, *Comm. Nonl. Sci. Nu. Sim*. **15** (2010), explain that non-invasive elasticity imaging techniques, reconstruct viscoelastic material properties from dynamic displacement images and the reconstruction algorithms are sensitive to the viscoelastic models chosen.

Assessment and Analysis Campaign

Assessment and Analysis is focused on supporting evaluators, PMs, and decision makers; modernizing the Army's capabilities in engineering-level analyses of technologies and systems; and leveraging those strengths to create fundamentally new capabilities through Assessment of Science and Technology, Science and Technology of Assessment, Assessing Mission Capability of Material, and Material Capable of Assessing Mission Capability.

- Two threads that run through the forecasting theme, whether it is the impact of scientific or technologic discovery, assessing mission capability of systems, or system failure, is the significance of uncertainty and the sensitivity of system dynamics to that uncertainty.

- As the process being considered became more complex, the theory of failure, necessarily, became correspondingly more complex. In the Information Science Campaign we introduced the fractional Poisson process (FPP). This should be the backbone of a new theory of failure.

- The world is filled with stochastic vibrations; the aircraft response to clear air turbulence, bridge cables to strong winds and the sway of tall buildings to earthquake tremors. In order to determine the probability the structures will survive their response, the reliability of the stochastic fractional dynamic system must be determined.

I have talked long enough.

Do you have any questions for me?