A Workshop on Future Directions in Fractional Calculus Research and Applications

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The Richness of Fractional Integro-Differential Operators Defined by Convolution with the Lévy Measure

Abstract

We start with the view that the operators in a Fokker-Planck equation are defined by the underlying natural process: Brownian motion defines the diffusion equation because the Fourier picture of its Gaussian density function defines a second-order PDE. A natural generalization of Brownian motion is Lévy motion, which is the limit of random walks with infinite variance in at least one direction. Concentrating on the infinite-variance portion only, the density P of a multi-dimensional (multi-scaling) Lévy motion follows a generalized diffusion equation that can be written $P_t = D\nabla_M^{H^{-1}} P$, where D is a scalar diffusion coefficient, H^{-1} is the inverse of the matrix of Hurst coefficients or growth rates of the density, and $M(\theta)$ is a measure of the jump propensity in all directions θ. This general (matrix-order) fractional differential operator is defined by convolution with the Lévy measure. A few examples illustrate the construction: For Brownian motion, the growth rate is isotropic, so $H = diag(1/2)$, $H^{-1} = 2I$, $DM(\theta)$ reverts to the covariance matrix, and the second-order diffusion equation is recovered. In 1-d, there is only one growth rate $H = 1/\alpha$ so $H^{-1} = \alpha$, and $M(\theta) = p(-1) + q(1)$. In this case, $P_t = Dp(\partial^\alpha P/\partial x^\alpha) + Dq(\partial^\alpha P/\partial(-x)^\alpha)$. The relative weights $p + q = 1$ describe skewness in 1-d. The inverse of the multi-order fractional differential operator just described can be used to generalize fractional Brownian motion to have different Hurst coefficients in different directions, along with completely general weights of convolutions in all directions via $M(\theta)$. This talk will show examples of solutions of all of these PDEs and integral equations with emphasis on pollution migration in granular and fractured aquifer material.