MICHIGAN STATE UNIVERSITY
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A Workshop on Future Directions in Fractional Calculus Research and Applications

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Fast Numerical Methods and Mathematical Analysis of Fractional Partial Differential Equations

Abstract

Fractional partial differential equations (FPDEs) provide a powerful tool for modeling challenging phenomena including anomalous transport, and long range time memory or spatial interactions. However, FPDEs present new mathematical and numerical difficulties that have not been encountered in the context of integer-order PDEs.

Computationally, because of the nonlocal property of fractional differential operators, the numerical methods for FPDEs often generate dense stiffness matrices. Traditionally, direct methods were used to solve these problems, which require $O(N^3)$ computations (per time step) and $O(N^2)$ mememy, where N is the number of unknowns (at the current time step for time-dependent problems). In addition, the numerical schemes may also present long tails in the time direction in the context of space-time fractional PDEs, which further increases the computational cost and memory requirement. This renders the three-dimensional FPDE modeling and simulations computationally intractable.

Mathematically, we demonstrate the following issues: (i) Elliptic FPDEs posed on a smooth domain with smooth data may have nonsmooth (not in Sobolev space H^1) solutions, so any Nitsche-lift based proof of optimal-order L^2 error estimates of their numerical approximations in the literature is invalid; (ii) The Galerkin formulation of an elliptic FPDE, which is proved coercive for constant diffusivity coefficient, may lose its coercivity for variable diffusivity coefficients. Consequently, the corresponding numerical methods may fail to converge. (iii) The inhomogeneous Dirichlet boundary-value problems of conservative Caputo and Riemann-Liouville diffusion FPDEs, which are identical for homogeneous boundary conditions, were proved to well posed or has no weak solution, respectively.

We go over the recent development of accurate and efficient numerical methods for space-time fractional PDEs, which has an optimal order storage and almost linear computational complexity. We will also present our recent work in the mathematical analysis of FPDEs. Finally, we will address open problems and our future direction of research.