A Workshop on Future Directions in Fractional Calculus Research and Applications

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Space-Time Duality and Medical Ultrasound

Abstract

Sound waves passing through many complex materials like human tissue attenuate according to a power law with respect to frequency. The wave amplitude falls off like $\exp(-\alpha x)$, where x is the distance from the source, and the attenuation coefficient alpha grows as a power y of the frequency, typically between $y = 1$ and $y = 2$. Several fractional calculus models have been proposed to capture this frequency dependent attenuation. Our power law wave equation model employs a fractional derivative in time, and solutions can be expressed using a stable probability density function. The appearance of the stable density in a solution of a time-fractional equation was a surprise, and further investigation led to a remarkable discovery: Some time-fractional partial differential equations have exactly the same solution as certain space-fractional equations, a kind of space-time duality. Now stable solutions and inverse stable solutions are compatible, and super-diffusion models can capture sub-diffusion. This discovery has implications for boundary value problems, where time-fractional models seem more tractable than their space-fractional counterparts. Other open questions include duality for tempered fractional derivatives, the interesting boundary case $y = 1$, and duality in three dimensions.