Space-Time Duality and Medical Ultrasound

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Time-Fractional vs. Space-Fractional

Time-Fractional PDEs Models sub-diffusion via long waiting times ("hold-ups")

$$\left(\frac{\partial}{\partial t}\right)^{\gamma} C = A_x C$$

$$C(x,t) = \int_0^{\infty} h_{\gamma}(x,u) g(u,t)\, du$$

where $\partial_t g = A_x g$ and $h_{\gamma}(x,u)$ is the inverse stable subordinator density.

Space-Fractional PDEs: Models super-diffusion via long particle jumps ("fast-paths")

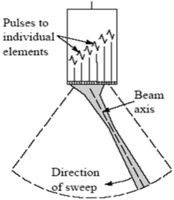
$$\frac{\partial}{\partial t} C(x,t) = \frac{\partial^{\alpha}}{\partial x^{\alpha}} C(x,t)$$

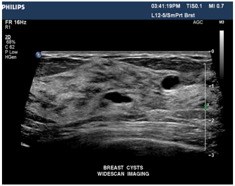
$$C(x,t) = f_{\alpha,1}(x,t)$$

where $f_{\alpha,\beta}(x,t)$ is a stable density (with scaling parameter t).

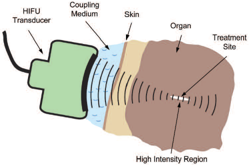
Ultrasound: Two Applications

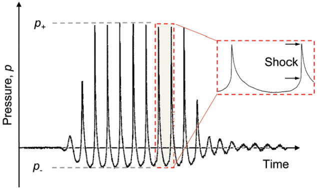
B-Mode Ultrasound Imaging (Webb, 2003)





Histotripsy (Maxwell, 2012)





Ultrasound: Power Law Attenuation

- Ultrasound waves attenuate as they travel through tissue.
- Limits maximum imaging depth for B-mode imaging.
- Influences maximum focal pressure for histotripsy.
- Attenuation coefficient $\alpha(\omega)$ fits a power-law

$$\alpha(\omega) = \alpha_0 |\omega|^y .$$

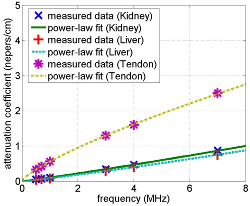


Figure: Measured attenuation (Goss, 1979)

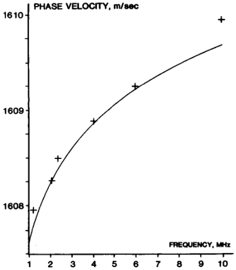


Figure: Meassured dispersion (Gurumurthy and Arthur, 1982)

Stokes Wave Equation

- Stokes Wave Equation (1845) is a classical PDE model:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \tau \frac{\partial}{\partial t}\nabla^2 p = 0.$$

- Wave attenuation is proportional to *relaxation time* τ [μs].
- Take FTs with respect to space and time, yielding

$$\left[-k^2 + \omega^2/c_0^2 + i\omega\tau k^2\right]\overline{p}(k,\omega) = 0.$$

- *Dispersion relationship* is $k(\omega) = \omega/c_0(1 - i\omega\tau)^{-1/2}$.
- Attenuation is $\alpha(\omega) = \mathrm{Im}k(\omega) \sim \tau/(2c_0)\omega^2$ for $\omega\tau \ll 1$.
- Phase velocity $c(\omega)$ is constant for $\omega\tau \ll 1$ (no dispersion).

Models for Attenuation in Ultrasound

- Early models (Gurumurthy and Arthur, 1982) modeled attenuation/dispersion in the frequency domain.
- Szabo (1994) proposed a phenomenological model for ultrasound in power law media ($0 \leq y \leq 2$).

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0 \cos(\pi y/2)}\frac{\partial^{y+1} p}{\partial t^{y+1}} = 0.$$

- Interpolates between the integer-ordered telegrapher's equation ($y = 0$) and the (viscous) Blackstock (1967) equation ($y = 2$) using a time-fractional derivative. Invalid for $y = 1$.

Power Law Wave Equation (PLWE)

Assume a dispersion relationship:

$$k(\omega) = \frac{\omega}{c_0} - \frac{\alpha_0(-i)^{y+1}\omega^y}{\cos(\pi y/2)}$$

for $\omega \geq 0$ and $k(-\omega) = k^*(\omega)$ to ensure real solutions. Imaginary part of the dispersion relationship is

$$\alpha(\omega) = \alpha_0|\omega|^y.$$

Compute the phase speed as

$$\frac{1}{c(\omega)} = \frac{\text{Re }k(\omega)}{\omega} = \frac{1}{c_0} + \alpha_0 \tan\left(\frac{\pi y}{2}\right)|\omega|^{y-1},$$

which is predicted by the Kramers-Krönig relationships and supported by measurements.

Square the dispersion relationship and multiply by FT $\overline{p}(\mathbf{k},\omega)$

$$\left[-k^2 + \frac{\omega^2}{c_0^2} - \frac{2\alpha_0(-i\omega)^{y+1}}{c_0\cos(\pi y/2)} - \frac{\alpha_0^2(-i\omega)^{2y}}{\cos^2(\pi y/2)}\right]\overline{p}(\mathbf{k},\omega) = 0.$$

Perform an inverse FTs (space and time), yielding the PLWE (Kelly et. al., 2008)

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{2\alpha_0}{c_0\cos(\pi y/2)}\frac{\partial^{y+1} p}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\pi y/2)}\frac{\partial^{2y} p}{\partial t^{2y}} = 0,$$

which satisfies the dispersion relationship *exactly* for $y \neq 1$. For mammalian tissue, power-law exponent y is very close to one!

PLWE: 3D Green's Function (1)

Solve PLWE subject to an impulse point-source with zero initial conditions in free-space

$$\nabla^2 g - \frac{1}{c_0^2}\frac{\partial^2 g}{\partial t^2} - \frac{2\alpha_0}{c_0\cos(\pi y/2)}\frac{\partial^{y+1} g}{\partial t^{y+1}} - \frac{\alpha_0^2}{\cos^2(\pi y/2)}\frac{\partial^{2y} g}{\partial t^{2y}} = -\delta(\mathbf{R})\delta(t),$$

where $\mathbf{R}$ is the relative displacement between the source and the observer and $R = |\mathbf{R}|$. Take Fourier transform wrt t

$$\nabla^2 \hat{g} + k^2(\omega)\hat{g} = -\delta(\mathbf{R}),$$

where $k(\omega)$ is our dispersion relationship. The Green's function for this Helmholtz equation is a spherical wave

$$\hat{g}(R,\omega) = \frac{e^{ik(\omega)R}}{4\pi R}.$$

PLWE: 3D Green's Function (2)

Inserting the dispersion relationship into the spherical wave solution yields

$$\hat{g}(R,\omega) = \left[\frac{\exp(i\omega R/c_0)}{4\pi R}\right]\left[\exp\left(-\alpha_0 R(|\omega|^y - i\tan(\pi y/2)\omega|\omega|^{y-1})\right)\right],$$

where the first factor solves the lossless Helmholtz equation. Evaluate inverse Fourier transform and apply the convolution theorem, yielding

$$g(R,t) = \mathcal{F}^{-1}\left[\hat{g}(R,\omega)\right].$$

$$g(R,t) = g_D(R,t) * g_L(R,t)$$

where $g_D(R,t) = \delta(t - R/c_0)/4\pi R$ is is the Green's function (transient spherical wave) for the lossless wave equation.

Interlude: Stable Parameterizations

1. ST parameterization (Samoradnitsky and Taqqu, 1994):

$$f_{\alpha,\beta}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} \exp\left(i\mu k + \sigma^{\alpha}\psi_{\alpha,\beta}(k)\right)\, dk$$

$$\psi_{\alpha,\beta}(k) = -|k|^{\alpha}\left(1 - i\beta \mathrm{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)\right) \text{ for } \alpha \neq 1$$

$$\psi_{\alpha,\beta}(k) = -|k|\left(1 + \frac{2i\,\mathrm{sgn}(k)}{\pi}\ln|k|\right) \text{ for } \alpha = 1$$

$$\sigma^{\alpha} = |\cos(\pi\alpha/2)|$$

2. Zolotarev C-Parameterization (for duality)

$$p_{\alpha}(x;\eta,b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x} \exp\left[-b|\lambda|^{\alpha} \exp\left(-\frac{i\pi\eta\lambda}{2|\lambda|}\right)\right] d\lambda$$

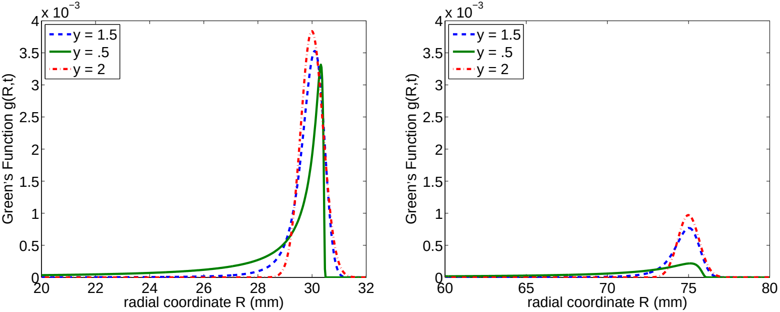
Loss Function

The second term is a loss function defined as

$$g_L(R,t) = \mathcal{F}^{-1}\left[\exp\left(-\alpha_0 R(|\omega|^y - i\tan(\pi y/2)\omega|\omega|^{y-1})\right)\right]$$
$$= \frac{1}{(\alpha_0 R)^{1/y}} f_{y,1}\left(\frac{t}{(\alpha_0 R)^{1/y}}\right).$$

- Nice for engineers, since stable PDFs may be numerically evaluated using STABLE toolbox (Nolan, 1997) or MATLAB 2016a.
- Solution of a time-fractional equation involves a stable density, not an *inverse* stable density. Not what we expected!
- Solution involves a PDF: What is the random variable?

Numerical Results: Green's Functions



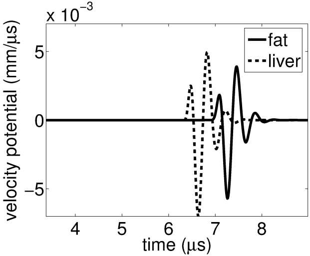
(a) $t = 20\ \mu s$ (b) $t = 50\ \mu s$

Figure: Snapshots of the 3D power law Green's function for $y = 0.5$, 1.5, and 2.0 for $\alpha_0 = 0.05\ \text{mm}^{-1}\text{MHz}^{-y}$. Snapshots of the Green's function are shown for $t = 20$ and 50 μs.

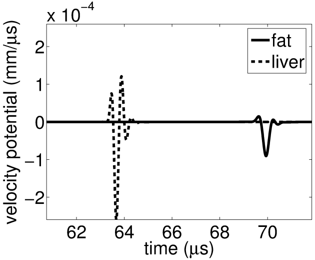
Ultrasound Pulse Propagation in Tissue

Given an input pulse $v(t)$, the velocity potential $\phi(\mathbf{r}, t)$ is

$$\phi(\mathbf{r}, t) = v(t) * g(\mathbf{r}, t)$$



(a) $R = 10$ mm



(b) $R = 100$ mm

Pulse experiences a frequency downshift and distortion as depth increases.

Causality

Physics demands causality. The Green's function is causal if $g(R,t) = 0$ for all $t < 0$. PLWE Green's function is

$$g(R,t) = \frac{1}{4\pi R} \frac{1}{(\alpha_0 R)^{1/y}} f_{y,1}\left(\frac{t - R/c_0}{(\alpha_0 R)^{1/y}} \right).$$

- If $y < 1$, then $f_{y,1}(z) = 0$ is $z < 0$. Then $g(R,t) = 0$ if $t < R/c_0$, implying causality.
- If $y \geq 1$, then $f_{y,1}(z) > 0$ for all z. Then $g(R,t) > 0$ for $t < 0$, violating causality!
- However, $f_{y,1}(z)$ decays with *exponential* order for $t \to -\infty$:

  $$f_{y,1}(z) \approx A|z|^{\nu} \exp\left(-B|z|^{\mu}\right),$$

  where A, B, μ, and ν are functions of y only.
- For observation points only one wavelength from the radiating source, the relative magnitude of $g(R,t)$ is less than -136 dB for all $1 < y \leq 2$.

Many Models: Which is the "Right" One?

The Szabo (1994) wave equation

$$\nabla^2 p = \frac{1}{c_0^2}\partial_t^2 p + \frac{2\alpha_0}{c_0 b}\partial_t^{y+1} p$$

is a simplified PLWE. Chen and Holm (2004) recommend

$$\nabla^2 p + \alpha_0 \partial_t (-\nabla^2)^{y/2} p = \frac{1}{c_0^2}\partial_t^2 p$$

using a fractional Laplacian. Caputo (1967) and Wismer (2006) propose

$$\nabla^2 p = \frac{1}{c_0^2}\partial_t^2 p + \tau^{y-1}\partial_t^{y-1}\nabla^2 p$$

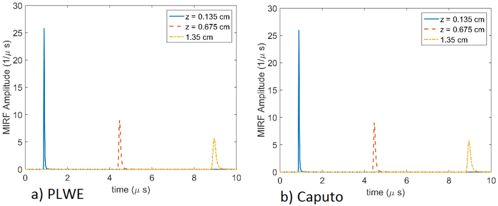
while Treeby and Cox (2010) consider

$$\nabla^2 p + \alpha_0 \partial_t (-\nabla^2)^{y/2} p + \alpha_1 \partial_t \nabla^{(\beta+1)/2} p = \frac{1}{c_0^2}\partial_t^2 p.$$

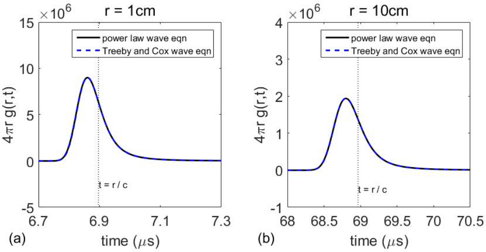
All exhibit power law attenuation $\alpha(\omega) = \alpha_0 |\omega|^\beta$ for $1 < y < 2$.

Solutions

Analytical comparison (Kelly and McGough, 2016)



Numerical comparison (Zhao and McGough, 2016)



Model Unification and Duality

- Many models proposed that agree with experiments, but who's right, and who's wrong?
- The idea of duality: two ways of looking at the same thing (Atiyah, 2008).
- Famous Example: wave-particle duality of light. Light behaves like a particle (Democritus) and a wave (Descartes). These contrary viewpoints were unified by quantum mechanics.
- Perhaps this duality principle can resolve (and unify) these alternative time-fractional and space-fractional models?

A Brief History of Space-Time Duality

- Zolotarev (1961) noted an equivalence between stable densities of index α and $1/\alpha$ in the C parameterization:

Theorem

(Duality Principle) For any pairs of admissible parameters $\alpha \geq 1$, θ and any $u > 0$

$$p_\alpha (u; \eta, 1) = u^{-(1+\alpha)} p_{\alpha^*} \left(u^{-\alpha}; \eta^*, 1 \right),$$

where $\alpha^* = 1/\alpha$ and $1 + \eta^* = \alpha(1 + \theta)$.

- Feller (1971) gave a simplified proof of this "curious by-product" using infinite series
- Baeumer et. al. (2009) recognized that (negatively skewed) space-fractional diffusion equations are solved by inverse stable densities, while time-fractional diffusion equations are solved by stable densities:

$$f_{\alpha,-1}(x, t) = \gamma h_\gamma(x, t) \text{ where } \gamma = 1/\alpha.$$

A Heuristic Argument

Let $1 < \alpha \leq 2$ and $1/2 \leq \gamma = 1/\alpha < 1$. Consider negatively skewed FDE:

$$\frac{\partial C_0}{\partial t} = \frac{\partial^\alpha C_0}{\partial(-x)^\alpha}.$$

Apply the Fourier transform in both variables

$$[(i\omega) - (-ik)^\alpha]\hat{C}_0 = 0.$$

Dispersion relationship: $i\omega - (-ik)^\alpha = 0$. Dual dispersion relationship:$(i\omega)^\gamma = (-ik)$. Inverting the FTs leads to the dual equation

$$\frac{\partial^\gamma C_0}{\partial t^\gamma} = -\frac{\partial C_0}{\partial x}.$$

- Heaviside (1871) noted this relationship for the classical diffusion equation ($\alpha = 2$).
- Baeumer et. al. (2009) noted this equivalence from $f_{\alpha,-1}(x,t) = \gamma h_\gamma(x,t)$.

Some New Results

1. New proof of duality using Fourier-Laplace transforms (FLTs).
2. Duality principle assumes $x > 0$. We extend duality to $x < 0$, thereby covering the real line.
3. Consider problems with drift: fractional advection dispersion equation (FADE)

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^\alpha C}{\partial(-x)^\alpha}.$$

FLT Approach

Cauchy problem for fractional diffusion/dispersion equation (FDE)

$$\frac{\partial C_0}{\partial t} = \frac{\partial^\alpha C_0}{\partial(-x)^\alpha} \text{ subject to } C(x,0) = \delta(x).$$

Apply the Fourier-Laplace transform (FLT)

$$\overline{C_0}(k,s) = \int_0^\infty \int_{-\infty}^\infty e^{-st} e^{-ikx} C_0(x,t)\, dx\, dt$$

to get $s\overline{C_0}(k,s) - 1 = (-ik)^\alpha \overline{C_0}(k,s)$. Rearrange as

$$\overline{C_0}(k,s) = \frac{1}{s - (-ik)^\alpha}.$$

Apply an inverse LT followed to inverse FT:

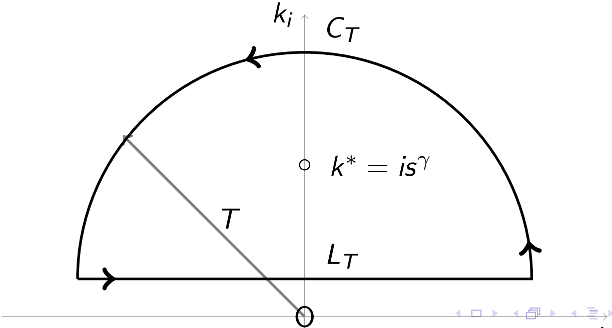
$$C_0(x,t) = \frac{1}{t^{1/\alpha}} f_{\alpha,-1}\left(\frac{x}{t^{1/\alpha}}\right).$$

FLT Approach (Cont'd)

Why not apply the inverse FT first? The inverse FT can be expressed as (Morse and Feschbach, 1953)

$$\tilde{C}_0(x,s) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T+i\tau}^{T+i\tau} \frac{e^{ikx}}{s-(-ik)^{\alpha}}\, dk,$$

where $\tau > 0$ is chosen to avoid the branch cut along the negative real axis. Integrand has a single pole at $k^* = is^{1/\alpha}$ and remains analytic for all other points in the upper half-plane.



FLT Approach (Cont'd)

Evaluate the contour integral, yielding

$$\tilde{C}_0(x,s) = \gamma s^{\gamma-1} \exp\left(-xs^{\gamma}\right) \quad \text{for } x > 0,$$

where $\gamma = 1/\alpha \in [1/2, 1)$. Invert using

$$\tilde{h}_{\gamma,+}(x,s) = s^{\gamma-1} \exp\left(-xs^{\gamma}\right)$$

for the LT of the inverse stable subordinator density (see Meerschaert and Sikorskii, 2012)

$$h_{\gamma,+}(x,t) = \frac{t}{\gamma x^{1+1/\gamma}} f_{\gamma,1}\left(tx^{-1/\gamma}\right).$$

Compare and use the uniqueness of the LT to get

$$C_0(x,t) = \gamma h_{\gamma,+}(x,t) \quad \text{for all } x > 0.$$

For $x > 0$, the negatively skewed diffusion (dispersion) equation is solved by a positively skewed stable PDF with index $\gamma = 1/\alpha$.

FLT Approach (Cont'd)

Take FT of $\tilde{h}_{\gamma,+}(x,s) = H(x)s^{\gamma-1}\exp(-xs^{\gamma})$, yielding

$$\bar{h}_{\gamma,+}(k,s) = \frac{s^{\gamma-1}}{ik + s^{\gamma}}.$$

Rewrite $s^{\gamma}\bar{h}_{\gamma,+}(k,s) - s^{\gamma-1} = -(ik)\bar{h}_{\gamma,+}(k,s)$ and invert

$$\left(\frac{\partial}{\partial t}\right)^{\gamma} h_{\gamma,+}(x,t) = -\frac{\partial}{\partial x} h_{+}(x,t); \quad h_{\gamma,+}(x,0) = \delta(x).$$

Since $C_0(x,t)$ is proportional to $h_{\gamma,+}(x,t)$ for all $x > 0$ and $t > 0$,

$$\left(\frac{\partial}{\partial t}\right)^{\gamma} C_0(x,t) = -\frac{\partial}{\partial x} C_0(x,t) \quad \text{for } x > 0 \text{ and } t > 0.$$

Agrees with heuristic argument and Baeumer et. al. (2009) result.

Duality for $x < 0$

Apply the *reflection property* $p_\alpha(-x; \eta, b, 0) = p_\alpha(x; -\eta, b, 0)$ for stable densities for $x < 0$:

$$\begin{aligned} p_\alpha(x; \eta, 1, 0) &= p_\alpha(-|x|; \eta, 1, 0) \\ &= p_\alpha(|x|; -\eta, 1, 0) \\ &= |x|^{-1-\alpha} p_\gamma\left(|x|^{-\alpha}; \eta^*, 1, 0\right) \end{aligned}$$

with $\gamma = 1/\alpha$ and $\eta^* = 2 - 3\gamma$. In ST parameterization

$$f_{\alpha,-1}(x, 0) = |x|^{-1-1/\gamma} f_{\gamma,\beta^*}\left(|x|^{-1/\gamma}\right).$$

Hence, $C_0(x, t) = \gamma h_{-,\gamma}(-x, t)$ for $x < 0$ where

$$h_{\gamma,-}(x, t) = \frac{t}{\gamma x^{1+1/\gamma}} f_{\gamma,\beta^*}\left(t x^{-1/\gamma}\right) H(x).$$

Duality for FADE

Consider the negatively-skewed FADE (Benson et. al., 2000)

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^\alpha C}{\partial(-x)^\alpha}$$

on the real line. Then $C(x,t)$ has a traveling wave solution

$$C(x,t) = C_0(x - vt, Dt)$$

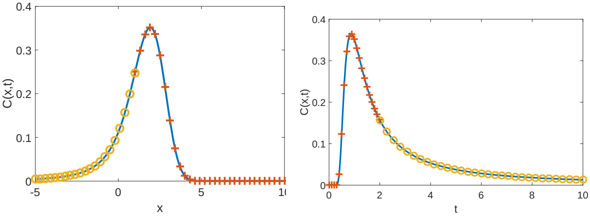
where $C_0(x,t)$ solves the FDE. Apply duality on the positive and negative axes:

$$C(x,t) = \gamma h_{\gamma,+}(x-vt, Dt)H(x-vt) + \gamma h_{\gamma,-}(x-vt, Dt)H(vt-x).$$

where $H(x)$ is the Heaviside function.

Dual Solution for FADE

$$C(x,t) = \gamma h_{\gamma,+}(x - vt, Dt)H(x - vt) + \gamma h_{\gamma,-}(x - vt, Dt)H(vt - x).$$



(a) snapshot (b) breakthrough curve

Figure: Comparison of FADE solution (solid) with dual solution (markers) with parameters are $\alpha = 3/2$, $v = 1$, $t = 2$, and $D = 1$.

The Governing Equation

For $x > vt$, we can show the FLT relationship (Kelly and Meerschaert, 2016)

$$\bar{C}(k,s) = \frac{\gamma(s+ikv)^{\gamma-1}}{D^\gamma ik + (s+ikv)^\gamma}.$$

Invert using the FLT formula (Meerschaert et. al., 2002)

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^\gamma f(x,t) \mapsto (s+ikv)^\gamma \bar{f}(k,s)$$

and the LT formula $t^{-\gamma}/\Gamma(1-\gamma) \mapsto s^{\gamma-1}$, yielding a *coupled space-time fractional governing equation* for $x > vt$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^\gamma C(x,t) = -D^\gamma \frac{\partial}{\partial x} C(x,t) + \gamma\delta(x-vt)\frac{t^{-\gamma}}{\Gamma(1-\gamma)}.$$

This space-time operator is a *fractional material derivative* (Sokolov and Metzler, 2003).

Physical Explanation

- Negatively skewed FADE models large negative (upstream) jumps. Zhang (2009) noted this is unphysical!
- The dual space-time fractional equation resolves this problem. Consider the fractional material derivative:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^{\gamma}$$

- Material derivative is the time-rate of change in a moving coordinate system.
- The Caputo derivative models waiting times (retention) in this moving frame.

Space-Time Duality: Generalizations

Can we extend these results?

1. Space-fractional PDEs: $\partial_t u(x,t) = p\partial_x^\alpha u(x,t) + q\partial_{-x}^\alpha u(x,t)$.
2. Tempered FDEs: $\partial_t u(x,t) = \partial_{-x}^{\alpha,\lambda} u(x,t)$, where $\partial_{-x}^{\alpha,\lambda}$ is the tempered fractional RL derivative (Baeumer and Meerschaert, 2010) and (Li et. al, 2015).
3. FDEs with boundary conditions: $\partial_t u(x,t) = \partial_{-x}^\alpha u(x,t)$ on $x > 0$ with $\partial_{-x}^{\alpha-1} u(0,t) = 0$.

General space-FDEs

Consider FPDE for $x > 0$

$$\frac{\partial}{\partial t} C(x,t) = p \frac{\partial^\alpha}{\partial x^\alpha} C(x,t) + q \frac{\partial^\alpha}{\partial (-x)^\alpha} C(x,t), \tag{1}$$

where $p + q = 1$, $\beta = p - q$, and the fractional derivatives are Riemann-Liouville. Solution is

$$C(x,t) = \frac{1}{t^{1/\alpha}} f_{\alpha,\beta} \left(\frac{x}{t^{1/\alpha}} \right). \tag{2}$$

Rewrite in Zolotarev's C parameterization, apply duality, and transform back to ST parameterization:

$$C(x,t) = \frac{t}{x^{1+1/\gamma}} \frac{1}{x^{1/\gamma}} f_{\gamma,\beta^*} \left(\frac{t}{x^{1/\gamma}} \right) H(x) \tag{3}$$

with $\gamma = 1/\alpha$ and skew $\beta^* = \beta^*(\beta, \alpha)$.

General space-FDEs

This dual solution solves a time-fractional PDE:

$$p^* \frac{\partial^\gamma}{\partial t^\gamma} C(x,t) + q^* \frac{\partial^\gamma}{\partial (-t)^\gamma} C(x,t) = -\frac{\partial}{\partial x} C(x,t) + p^* \delta(x) b(t),$$

where $\beta^* = p^* - q^*$ and $b(t)$ is a source term. Several questions:

1. Is this time-fractional equation the scaling limit of some CTRW? For example, a time-reversed subordinator? (Lorick Huang)
2. Is it possible to transform only the negative jumps into a positive time-fractional derivative, yielding a governing equation without the negatively-skewed time-fractional derivative?

Tempered FDEs

Truncated power-laws can be modeled using with tempered time derivatives or tempered space derivatives. Consider

$$\partial_t u = \partial_{-x}^{\alpha,\lambda} u \text{ where } u(x,0) = \delta(x).$$

where $\partial_{-x}^{\alpha,\lambda}$ has Fourier symbol $\psi(k) = (\lambda - ik)^\alpha - \lambda^\alpha$, $1 < \alpha \leq 2$, and $\lambda > 0$. Solve using FLTs and apply Zolotarev duality, yielding

$$\begin{aligned} u(x,t) =& e^{\lambda x} e^{-\lambda^\alpha t} f_{\alpha,-1}(x,t) \\ =& \gamma e^{\lambda x} e^{-\lambda^\alpha t} h_\gamma(x,t), \end{aligned}$$

Solves

$$\left(\frac{\partial}{\partial t}\right)^{\gamma,\lambda} u(x,t) = -\partial_x u(x,t) + b(x,t)$$

FDEs with boundary conditions

- Boundary-value problems for space-fractional PDEs are difficult.
- Is it possible to *transform* a space FDE with boundary conditions to an equivalent time-fractional FDE with boundary conditions?
- $$\partial_t u(x,t) = \partial^{\alpha}_{-x} u(x,t)$$

  on $x > 0$ subject to a *fractional flux boundary condition*

  $$\partial^{\alpha-1}_{-x} u(0,t) = 0$$

“Fractional Derivative” of order 1

What is the governing equation of Lévy motion of order one and skewness one? Define an operator

$$D_+^1 f(x) = \mathcal{F}^{-1}\left[\psi_{1,1}(-k)\hat{f}(k)\right],$$

where $\psi_{1,1}(k)$ is the log characteristic function of a stable law with $\alpha = 1$ and $\beta = 1$:

$$\psi_{1,1}(k) = -|k|\left(1 + \frac{2i\,\mathrm{sgn}(k)}{\pi}\ln|k|\right).$$

By Lemma 7.3.9 in (Meerschaert and Scheffler, 2001)

$$\psi_{1,1}(k) = \frac{2}{\pi}\int_0^\infty \left(e^{iky} - 1 - ik\sin y\right) y^{-2}\,dy.$$

Invert FT, yielding the *generator form*:

$$\mathcal{D}_+^1 f(x) = \int_0^\infty \left(f(x-y) - f(x) + f'(x)\sin y\right) y^{-2}\,dy.$$

Caputo Form and an example

Integrate by parts with $u = f(x-y) - f(x) + f'(x)\sin y$ and $dv = y^{-2}dy$, to yield the *Caputo form*

$$\mathcal{D}_+^1 f(x) = \frac{2}{\pi}\int_0^\infty \left[f'(x)\cos y - f'(x-y)\right] y^{-1}\, dy.$$

Example

Let $f(x) = e^{\lambda x}$, where $\lambda > 0$.

$$\begin{aligned}\mathcal{D}_+^1 f(x) =& \frac{2}{\pi}\int_0^\infty \left[\lambda e^{\lambda x}\cos y - \lambda e^{\lambda(x-y)}\right] y^{-1}\, dy \\ =& \frac{2\lambda}{\pi} e^{\lambda x}\int_0^\infty \left(\cos y - e^{-\lambda y}\right) y^{-1}\, dy \\ =& \frac{2}{\pi}\lambda \ln \lambda e^{\lambda x}\end{aligned}$$

If $\lambda = 1$, this "derivative" is zero!

Summary

- Fractional wave equations (e.g. PLWE) are used to model attenuation and dispersion in biomedical ultrasound.
- Both TF and SF power-law models exist, prompting the question: "What is the correct model?"
- Space-time duality, which links SF and TF PDEs, allows models to be unified.
- We have applied duality to the negatively-skewed FDE and the spatial FADE.
- Many questions remain regarding general FDEs, FDEs with boundary conditions, etc.