Probabilistic Generalisation of Fractional Derivatives and Related Differential Equations

Lorenzo Toniazzi[1]

A Workshop on Future Directions in Fractional Calculus Research and Applications, MSU October 2016

[1]Dep. of Mathematics, University of Warwick, Coventry CV4 7AL, UK; Email: l.toniazzi@warwick.ac.uk

Introduction I

From the point of view of stochastic analysis the Caputo and Riemann-Liouville derivatives of order $\beta \in (0, 2)$ can be viewed as (regularized) generators of stable Lévy motions 'interrupted' on crossing a boundary.

—

This interpretation naturally suggests fully mixed, two-sided or even multidimensional generalizations of these derivatives, as well as a probabilistic approach to the analysis of the related equations.

Introduction II

In this talk we will present current theory for such generalised operators, and address questions concerning the role of this theory in the literature.

—

Key highlights:

- Provide a *unified* treatment of a variety of differential equations of fractional type within a probabilistic framework.
- Provide *stochastic representations* for the solutions to a variety boundary value problems for a variety of fractional differential operators (useful for numerics).

Introduction III

This talk is based on the following papers

- '*On fully mixed and multidimensional extensions of the Caputo and Riemann-Liouville derivatives, related Markov processes and fractional differential equations*'
  V. N. Kolokoltsov, Fract. Calc. Anal. Appl., Vol 18, Issue 4 (Aug 2015)
- '*On the probabilistic approach to the solution of generalized fractional differential equations of Caputo and Riemann-Liouville type*'
  M. E. Hernández-Hernández, V. N. Kolokoltsov, Journal of Fractional Calculus and Applications, 7 (1) 2016.

Introduction IV

- '*Probabilistic solutions to nonlinear fractional differential equations of generalised Caputo and Riemann-Liouville type*'
  M. E. Hernández-Hernández, V. N. Kolokoltsov.
  Submitted to "Stochastics. An International Journal of Probability and Stochastic Processes".
- '*On the solution of two-sided fractional ordinary differential equations of Caputo type*'
  M. E. Hernández-Hernández, V. N. Kolokoltsov.
  Submitted to the " Fractional Calculus and Applied Analysis".
- Part of the speaker's future PhD thesis.

Plan of the talk 1/2

In the first half of the talk we present a toy-version of the theory to show the key ideas.
The steps are

- Introduce $D^{(\nu)}_{a+*}$, the generalised (right) Caputo operator for $\beta \in (0,1)$, $a \in \mathbb{R}$.
- Theorem: $D^{(\nu)}_{a+*}$ generates a Feller semigroup on $C_\infty[a,\infty)$.
- Application: well-posedness for a (generalised fractional) ordinary differential equation and stochastic representation of its solution.

Keep in mind that a variety of generalised operators is available (R-L, $\beta \in (1,2)$, left/right versions, linear combinations on bounded domains, multidimensional extensions,...).

Plan of the talk 2/2

In the second half of the talk we present other results and address some open questions for other classes of fractional derivatives and BVPs.
Namely

- Other results:
  - Left/right, two-sided, multidimensional, $\beta \in (1, 2)$ generalised operators.
  - some related wellposedness results.
- Future directions:
  - Cauchy problems and time changes.
  - Wellposedness of fractional differential equations on bounded domains.

Probabilistic Intuition I

Consider the generator $D_{+}^{(\nu)}$ of an $\mathbb{R}$-valued jump-type Feller process (with negative jumps) of the form

$$D_{+}^{(\nu)}f(x) := \int_0^\infty (f(x-y)-f(x))\nu(x,\mathrm{d}y), \quad x \in \mathbb{R}, \qquad (1)$$

for a Lévy kernel $\nu(\cdot,\cdot)$.
Now let us heuristically force all jumps that fall below a to fall exactly on a by modifying $D^{(\nu)}$ as follows

$$\begin{aligned} D_{a+*}^{(\nu)}f(x) :=& \int_0^{x-a} (f(x-y)-f(x))\nu(x,\mathrm{d}y) \\ &+ (f(a)-f(x))\int_{x-a}^\infty \nu(x,\mathrm{d}y), \quad x > a. \end{aligned} \qquad (2)$$

Probabilistic Intuition II

Questions:

- how is operator $D_{a+*}^{(\nu)}$ related to a right Caputo derivative (of order $\beta \in (0,1)$, $a \in \mathbb{R}$)?
- Under what conditions (on ν) is $D_{a+*}^{(\nu)}$ the generator of a Feller process on $[a, \infty)$?

Caputo derivative in Generator form

Consider a right Caputo fractional derivative of order $\beta \in (0,1)$, $a \in \mathbb{R}$

$$D^{\beta}_{a+*}f(x) := \frac{1}{\Gamma(1-\beta)} \int_a^x f'(y)(x-y)^{-\beta} \mathrm{d}y, \quad x > a. \quad (3)$$

For f 'nice' (say C^1), we can rewrite $D^{\beta}_{a+*}f$ as

$$D^{\beta}_{a+*}f(x) = \int_0^{x-a} (f(x-y) - f(x)) \frac{\beta}{\Gamma(1-\beta)} \frac{\mathrm{d}y}{y^{1+\beta}} + (f(a) - f(x)) \int_{x-a}^{\infty} \frac{\beta}{\Gamma(1-\beta)} \frac{\mathrm{d}y}{y^{1+\beta}}, \quad x > a. \quad (4)$$

which equals $D^{(\nu)}_{a+*}f$ if $\nu(x, \mathrm{d}y) = \frac{\beta}{\Gamma(1-\beta)} \frac{\mathrm{d}y}{y^{1+\beta}}$.

Definition of a Generalised derivative

The previous suggests the following
Definition 1.
For a Lévy kernel $\nu(\cdot,\cdot)$ such that

$$\int_{\mathbb{R}} \min\{1,|y|\}\nu(x,\mathrm{d}y) < \infty, \;\; x \in [a,\infty), \tag{5}$$

We call $D^{(\nu)}_{a+*}$ the *generalised right Caputo fractional derivative of order* $\beta \in (0,1)$ for $a \in \mathbb{R}$, where

$$\begin{aligned} D^{(\nu)}_{a+*}f(x) := &\int_0^{x-a} (f(x-y)-f(x))\nu(x,\mathrm{d}y) \\ &+ (f(a)-f(x))\int_{x-a}^{\infty}\nu(x,\mathrm{d}y), \quad x>a. \end{aligned} \tag{6}$$

Examples: Lévy Kernels

Which Lévy kernels ν are included in our theory?

$$\nu(x,y) = \begin{cases} cy^{-(1+\beta)} \\ \sum_{n=0}^{N-1} y^{-(1+\beta_n)} \\ y^{-(1+\beta(x))} \\ w(x)y^{-(1+\beta)} \\ e^{-\lambda y} y^{-(1+\beta)} \end{cases} \tag{7}$$

where c is a positive constant, $\lambda > 0$, w a non-negative function, $\beta : [a, \infty) \to (0,1)$, such that the assumptions of Theorem 1 (which we present next) are satisfied.

Well-posedness Theorem for $D_{a+*}^{(\nu)}$

Theorem 1.(Kolokoltsov '15)
Assume that $\nu(x, dy)$ has a density $\nu(x, y)$ which is a continuous function of two variables, continuously differentiable in the x-variable and has the following uniform bounds and tightness property

$$\sup_x \int 1 \wedge |y| \nu(x, y)\, dy < \infty, \quad \sup_x \int 1 \wedge |y| \left| \frac{\partial}{\partial x} \nu(x, y) \right| dy < \infty,$$

and

$$\lim_{\delta \to 0} \sup_x \int_{|y| \le \delta} |y| \nu(x, y)\, dy = 0.$$

$\Rightarrow$ the operator $D_{a+*}^{(\nu)}$ generates a Feller process on $[a, \infty)$ and a Feller semigroup on $C_\infty[a, \infty)$ with invariant core $C_\infty^1[a, \infty)$.

Well-posedness Theorem for $D^{(\nu)}_{a+*}$ II

Theorem 1.(continued)
Moreover, if

$$\int_{-\infty}^{0} \min(|y|, \epsilon)\nu(a, y)\, dy > C\epsilon^{r}$$

for some $C > 0$, $r \in (0, 1)$, then the point a is regular in expectation for the process above.
A process X_t^x is *regular in expectation at a* if

$$\mathbf{E}[\tau_a^{x,(\nu)}] < \infty,\ x \in [a, \infty),\ \&\ \lim_{x \to a} \mathbf{E}[\tau_a^{x,(\nu)}] = 0.$$

where $\tau_a^{x,(\nu)}$ is the first hitting time of X_t^x of $\{a\}$.

Well-posedness Theorem for $D^{(\nu)}_{a+*}$ III

Remarks about the proof:
- Approximate $D^{(\nu)}_{a+*}$ with $D^{(\nu),h}_{a+*}$ for the existence of a Feller semigroup.

Fix $h > 0$. Define the operator

$$D^{(\nu),h}_{a+*} f(x) := \int_h^{x-a} (f(x-y) - f(x))\nu(x,y)\mathrm{d}y + (f(a) - f(x)) \int_{(x-a)\vee h}^{\infty} \nu(x,y)\mathrm{d}y. \tag{8}$$

Then $D^{(\nu),h}_{a+*}$ is bounded and by perturbation theory this operator generates a Feller semigroup $\{T^h_t\}_{t\in\mathbb{R}^+}$ on $C_\infty[a,\infty)$.

Well-posedness Theorem for $D^{(\nu)}_{a+*}$ IV

By considering $\partial_x D^{(\nu),h}_{a+*} f(x)$ we can recover bounds uniform in $h > 0$ for $\partial_x T^h_t f(x)$ for appropriate class of functions f then we obtain that $T^h_t \to T_t$ using the equality

$$(T^h_t - T^{h'}_t)f = \int_0^t (T^h_{t-s}((D^{(\nu),h}_{b-*} - D^{(\nu),h'}_{b-*})T^{h'}_s)f\,\mathrm{d}s, \qquad (9)$$

for some semigroup T_t which turns out to be Feller. Then we show that the generator of T_t agrees with $D^{(\nu)}_{a+*}$ on $C^1_\infty[a,\infty)$ and that $C^1_\infty[a,\infty)$ is an invariant core.

Well-posedness Theorem for $D^{(\nu)}_{a+*}$ V

Remarks about the proof:
- Lyapunov functions for regularity in expectation.
If $Lf(x) \leq -c$, $c > 0$ and $f(a) = 0$ $f \geq 0$ by Dynkin martingale and Optional Stopping we obtain

$$-f(x) \leq \mathbf{E}[f(X^x_{\tau^x_a})] - f(x) = \mathbf{E}\left[\int_0^{\tau^x_a} Lf(X^x_s) \mathrm{d}s\right] \leq -c\mathbf{E}[\tau^x_a] \tag{10}$$

and so

$$\mathbf{E}[\tau^x_a] \leq \frac{f(x)}{c} \to 0, \ x \to a. \tag{11}$$

Examples: generalised operators on bounded domains

With a similar procedure we can define operators/generators of the type

$$D^{(\nu)}_{ab*}f(x) := (D^{(\nu)}_{a+*} + D^{(\nu)}_{b-*})f(x), \ x \in (a, b),$$

acting on a subset of $C[a, b]$, where $D^{(\nu)}_{b-*}$ is a left version of the generalised Caputo operator $D^{(\nu)}_{a+*}$ (more details later). Notice that this theory offers a general framework for the treatment of fractional diffusions on bounded domains.

Application: An Initial Value problem I

Consider the Initial Value Problem (IVP)

$$D^{(\nu)}_{a+*}u(x) = \lambda u(x) + g(x), \; x \in (a, b], \; u(a) = u_a, \qquad (12)$$

for $\lambda > 0$, $g \in B[a, b]$ and $\nu(\cdot, \cdot)$ satisfying the assumptions of Theorem 1.

Suppose $g \in C([a, b])$, then (given Theorem 1) we have a natural notion of solution for (12) without the boundary condition, given by the resolvent operator.

$$u(x) = R^{\lambda}g(x) = \mathbf{E}\left[\int_0^{\infty} e^{-\lambda s} g(X_s^{x,(\nu)})\mathrm{d}s\right], \qquad (13)$$

where the expectation is taken with respect to the probability measure corresponding to the semigroup generated by $D^{(\nu)}_{a+*}$.

We call such a solution u *a solution in the domain of the generator.*

Application: An Initial Value problem II

Exploiting the specific properties of the form of the generator/generalised fractional derivative $D_{a+*}^{(\nu)}$ we have that the underlying process is monotone so that u takes the representation

$$u(x) = \frac{g(a)}{\lambda}\mathbf{E}\left[e^{-\lambda\tau_a^{x,(\nu)}}\right] + \mathbf{E}\left[\int_0^{\tau_a^{x,(\nu)}} e^{-\lambda s} g(X_s^{x,(\nu)}) \mathrm{d}s\right], \quad (14)$$

where $\tau_a^{x,(\nu)}$ is the first time of $X_t^{x,(\nu)}$ hits the set $\{a\}$.

Application: An Initial Value problem III

Consider the following solution concept for the initial value problem (12).

Definition 2. For a given $g \in B([a,b])$ $u_a \in \mathbb{R}$, $u \in B[a,b]$ is a *generalised solution to the ivp (12)* if $\forall \{g_n\}_{n\in\mathbf{N}} \in C([a,b])$ such that

$$\sup_n ||g_n||_\infty < \infty, \; g_n \to g \text{ a.e.}, \; g_n(a) = \lambda u_a \; \forall n \in \mathbf{N},$$

we have $u_n \to u$ a.e.,
where u_n is the (unique) solution in the domain of the generator to the ivp (12) for g_n.

Application: An Initial Value problem IV

Consider a sequence $\{g_n\}_{n\in\mathbf{N}}$ as above.
Given the boundedness and convergence properties of the sequence $\{g_n\}_{n\in\mathbf{N}}$ along with the stochastic representation of u_n we obtain by Dominated Convergence

$$u(x) = \lim_{n\to\infty} u_n(x) = u_a \mathbf{E}\left[e^{-\lambda\tau_a^{x,(\nu)}}\right] + \mathbf{E}\left[\int_0^{\tau_a^{x,(\nu)}} e^{-\lambda s} g(X_s^{x,(\nu)}) \mathrm{d}s\right], \tag{15}$$

from which uniqueness of the generalised solution follows along with continuity on the initial conditions.

Application: An Initial Value problem V

The following Theorem summarises the above.

Theorem 2.(Hernandez-Hernandez, Kolokoltsov '16)
Let ν be a Lévy kernel satisfying the conditions of Theorem 1 and suppose $\lambda > 0$.
(i) If $g \in C([a, b])$, then the ivp (12) has a unique solution in the domain of the generator given by

$$u = R^{\lambda} g$$

the resolvent operator for λ.
(ii) For any $g \in B([a, b])$ and $u_a \in \mathbb{R}$, the IVP (**??**) is well-posed in the generalized sense and the solution admits the stochastic representation

$$u(x) = u_a \mathbf{E}\left[e^{-\lambda \tau_a^{x,(\nu)}}\right] + \mathbf{E}\left[\int_0^{\tau_a^{x,(\nu)}} e^{-\lambda} g(X_s^{x,(\nu)}) \mathrm{d}s\right], \quad (16)$$

for the definitions given above, wiht continuous dependence on initial conditions.

Application: An Initial Value problem VI

Theorem 2.(continued)
(iii) Moreover, if additionally ν satisfies condition (C) below, then

$$u(x) = u_a \int_0^\infty e^{-\lambda s} \mu_a^{x,(\nu)}(s) \mathrm{d}s + \int_0^{x-a} g(x-r) \left(\int_0^\infty e^{-\lambda s} p_s^{+(\nu)}(x, x-r) \mathrm{d}s \right) \mathrm{d}r \tag{17}$$

where $\mu_a^{x,(\nu)}(s)$ is the density of $\tau_a^{x,(\nu)}$.
(C)- The transition probabilities of the process X are absolutely continuous w.r.t. Lebesgue & the transition density function $p_s^{+(\nu)}(r, y)$, the density of X, is continuously differentiable in the variable s.

Application: An Initial Value problem VII

Now set $\nu(x,y) = \frac{\beta}{\Gamma(1-\beta)} \frac{1}{y^{1+\beta}}$ $\beta \in (0,1)$. Then we obtain the following formula.

Corollary 1.(Hernandez-Hernandez, Kolokoltsov '16)
Let $x \in (a,b]$ and $\lambda > 0$. Then the Laplace transform of $\tau_a^{x,\beta}$, the first exit time from $(a,b]$ for the inverted β-stable subordinator started at x is given by

$$\mathbf{E}[e^{-\lambda \tau_a^{x,\beta}}] = E_\beta(\lambda(x-a)^\beta)$$

where $E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n+1)}$ is the (one parameter) Mittag-Laffler function, and

$$\mathbf{E}[e^{-\lambda \tau_a^{x,(\nu)}}] = \frac{x-a}{\beta} \int_0^\infty e^{\lambda s} s^{-\frac{1}{\beta}-1} \omega_\beta((x-a)s^{-\frac{1}{\beta}}); 1; 1) ds,$$

for $\omega_\beta(\cdot;\cdot;\cdot)$ the density of the inverted β-stable subordinator.

Key features of the theory (again)

- Unified probabilistic framework to deal with fractional differential equations.
- Obtaining stochastic representations for solutions (good for numerics).
- Giving natural probabilistic generalisations to many fractional differential and integral operators on intervals and multi-dimensional bounded domains.

Second part of the Talk

- Other results:
  - Left/right, two-sided, multidimensional, $\beta \in (1, 2)$ generalised operators.
  - some related wellposedness results.
- Future directions:
  - Cauchy problems and time changes.
  - Wellposedness of fractional differential equations on bounded domains.

Definitions: Generalised Fractional differential operators (1-dimension) $\beta \in (0,1)$

Generalised right R-L fractional derivative of order $\beta \in (0,1)$, $a \in \mathbb{R}$

$$D^{(\nu)}_{a+} f(x) := \int_0^{x-a} (f(x-y) - f(x))\nu(x, \mathrm{d}y) \\ - f(x) \int_{x-a}^{\infty} \nu(x, \mathrm{d}y), \quad x > a. \tag{18}$$

Generalised right Caputo fractional derivative of order $\beta \in (0,1)$, $a \in \mathbb{R}$ (same as the previous operator)

$$D^{(\nu)}_{a+*} f(x) := \int_0^{x-a} (f(x-y) - f(x))\nu(x, \mathrm{d}y) \\ (f(a) - f(x)) \int_{x-a}^{\infty} \nu(x, \mathrm{d}y), \quad x > a. \tag{19}$$

Definitions: Generalised Fractional differential operators (1-dimension) $\beta \in (0,1)$

Generalised left R-L fractional derivative of order $\beta \in (0,1)$, $b \in \mathbb{R}$

$$D^{(\nu)}_{b-} f(x) := \int_0^{b-x} (f(x+y) - f(x))\nu(x, \mathrm{d}y) - f(x) \int_{b-x}^{\infty} \nu(x, \mathrm{d}y), \quad x < b. \tag{20}$$

Generalised left Caputo fractional derivative of order $\beta \in (0,1)$, $b \in \mathbb{R}$

$$D^{(\nu)}_{b-*} f(x) := \int_0^{b-x} (f(x+y) - f(x))\nu(x, \mathrm{d}y) + (f(b) - f(x)) \int_{b-x}^{\infty} \nu(x, \mathrm{d}y), \quad x < b. \tag{21}$$

Definition of Generalised Fractional differential operators (two-sided) $\beta \in (0,1)$

Generalised two-sided Caputo fractional derivative of order $\beta \in (0,1)$, $a, b \in \mathbb{R}$, $a < b$

$$D^{(\nu)}_{ab*}f(x) := (D^{(\nu)}_{a+*} + D^{(\nu)}_{b-*})f(x), \quad x \in (a,b). \tag{22}$$

Generalised two-sided R-L fractional derivative of order $\beta \in (0,1)$, $a, b \in \mathbb{R}$, $a < b$

$$D^{(\nu)}_{ab}f(x) := (D^{(\nu)}_{a+} + D^{(\nu)}_{b-})f(x), \quad x \in (a,b). \tag{23}$$

Well-posedness theorem for the two-sided operator $D^{(\nu)}_{ab*}$ I

Theorem 4.(Hernández-Hernández, Kolokoltsov '16)
Suppose $\nu(x, y)$ satisfy the conditions of Theorem 1.
Suppose that $\gamma \in C_0^3[a, b]$, $\alpha \in C^3[a, b]$ with first derivative $\alpha' \in C_0[a, b]$, $\alpha > 0$.
$\Rightarrow A_{ab*} := \gamma \frac{d}{dx} + \alpha \frac{d^2}{dx^2} + D^{(\nu)}_{ab*}$ generates a Feller semigroup on $C[a, b]$ such that $\{f \in C^2[a, b] : \ f' \in C_0[a, b]\} \subset \mathrm{Dom}(A_{ab*})$.
Moreover the points $\{a, b\}$ are regular in expectation for the process generated by $(A_{ab*}, \mathrm{Dom}(A_{ab*}))$ on $C[a, b]$.

GFODE for the two-sided operator $D_{ab}^{(\nu)}$ (R-L on bounded domain)

Theorem 5.(Hernández-Hernández, Kolokoltsov '16)
Suppose ν satisfies the conditions of Theorem 4 then

$$D_{ab}^{(\nu)}u(x) = \lambda u(x) + g(x), \; x \in (a, b), u(a) = u(b) = 0, \quad (24)$$

- if $g \in C_0[a, b]$ bvp (24) has a unique solution in the domain of the generator and
- if $g \in B[a, b]$ bvp (24) has a unique generalised solution with the stochastic representation

$$u(x) = \mathbf{E}\left[\int_0^{\tau_{a,b}^x} e^{-\lambda s} g(X_s^x) \mathrm{d}s\right] \qquad (25)$$

where $\lambda > 0$ and X_t^x is the process induced by the semigroup generated by $D_{ab}^{(\nu)}$ on $C_0[a, b]$.

Definition: Generalised operators $\beta \in (1,2)$ (1-dimension)

Let $\nu(\cdot,\cdot)$ be a Lévy kernel such that

$$\int_{\mathbb{R}} |y| \wedge |y|^2 \nu(x, \mathrm{d}y) < \infty, \;\; x \in \mathbb{R}. \tag{26}$$

Generalised right R-L fractional derivative of order $\beta \in (1,2)$, $a \in \mathbb{R}$

$$\begin{aligned} {}^{2}D^{(\nu)}_{a+} f(x) := &\int_0^{x-a} (f(x-y) - f(x) + y f'(x)) \nu(x, \mathrm{d}y) \\ &- f(x) \int_{x-a}^{\infty} \nu(x, \mathrm{d}y) + f'(x) \int_{x-a}^{\infty} y \nu(x, \mathrm{d}y), \quad x > a. \end{aligned} \tag{27}$$

Definition: Generalised operators $\beta \in (1,2)$ (1-dimension) II

Generalised right Caputo fractional derivative of order $\beta \in (1,2)$, $b \in \mathbb{R}$

$$
{}^2D^{(\nu)}_{b-*}f(x) := \int_0^{b-x} (f(x-y) - f(x) + yf'(x))\nu(x,\mathrm{d}y)
+ (f(a) - f(x)) \int_{x-a}^{\infty} \nu(x,\mathrm{d}y) + f'(x) \int_{x-a}^{\infty} y\nu(x,\mathrm{d}y), \tag{28}
$$

$x > a$.

Definition: Generalised Fractional differential operators $\beta \in (1,2)$ (1-dimension) III

Generalised left R-L fractional derivative of order $\beta \in (1,2)$, $b \in \mathbb{R}$

$$
{}^{2}D_{b-}^{(\nu)} f(x) := \int_0^{b-x} (f(x+y) - f(x) - yf'(x))\nu(x, \mathrm{d}y) \\
- f(x) \int_{b-x}^{\infty} \nu(x, \mathrm{d}y) - f'(x) \int_{x-a}^{\infty} y\nu(x, \mathrm{d}y), \quad x < b. \tag{29}
$$

Definition: Generalised Fractional differential operators $\beta \in (1,2)$ (1-dimension) IV

Generalised left Caputo fractional derivative of order $\beta \in (1,2)$, $b \in \mathbb{R}$

$$
{}^{2}D^{(\nu)}_{b-*} f(x) := \int_0^{b-x} (f(x+y) - f(x) - y f'(x)) \nu(x, \mathrm{d}y) + (f(a) - f(x)) \int_{b-x}^{\infty} \nu(x, \mathrm{d}y) - f'(x) \int_{x-a}^{\infty} y \nu(x, \mathrm{d}y), \tag{30}
$$

$x < b$.

Definition: Generalised Fractional differential operators $\beta \in (1, 2)$ (1-dimension) V

Generalised two-sided R-L fractional derivative of order $\beta \in (1, 2)$, $a, b \in \mathbb{R}$

$$ {}^{2}D_{ab}^{(\nu)} f(x) := ({}^{2}D_{a+}^{(\nu)} + {}^{2}D_{b-}^{(\nu)}) f(x), \quad a < x < b. \tag{31} $$

Generalised two-sided Caputo fractional derivative of order $\beta \in (1, 2)$, $a, b \in \mathbb{R}$

$$ {}^{2}D_{ab*}^{(\nu)} f(x) := ({}^{2}D_{a+*}^{(\nu)} + {}^{2}D_{b-*}^{(\nu)}) f(x), \quad a < x < b. \tag{32} $$

Definition: Generalised Fractional differential operators $\beta \in (1,2)$ (1-dimension) VI

Remark. Fix $\nu(x, \mathrm{d}y) = \frac{1-\beta}{\Gamma(2-\beta)} \frac{\mathrm{d}y}{y^{1+\beta}}$.

The (right) Caputo fractional derivative for $\beta \in (1,2)$, ${}^C D^\beta_{a+}$ when rewritten in Generator/Marchaud/Itô form equals

$$
\begin{aligned}
{}^C D^\beta_{a+} f(x) := \int_0^{x-a} (f(x-y) - f(x) + y f'(x)) \nu(x, \mathrm{d}y) \\
(f(a) - f(x)) \int_{x-a}^\infty \nu(x, \mathrm{d}y) \\
+ (f'(a) - f'(x)) \int_{x-a}^\infty y \nu(x, \mathrm{d}y), \quad x > a,
\end{aligned}
\tag{33}
$$

The extra term $f'(a)$ is incompatible with a Markov generator structure. Notice also that so

$$
{}^C D^\beta_{a+*} f(x) =^2 D^\beta_{a+*} f(x) \iff f'(a) = 0.
$$

Existence Theorems for two-sided operator $\beta \in (1,2)$

Theorem 5.(Kolokoltsov, Toniazzi '16)
Let ν be a Lévy kernel with density $\nu(x,y)$ continuous in x and y and $\nu(\cdot,y) \in C^2$ all y,

$$\sup_x \int_{\mathbb{R}} |y| \wedge |y|^2 \nu(x,y) \mathrm{d}y < \infty, \quad \sup_x \int_{\mathbb{R}} |y| \wedge |y|^2 |\nu_x(x,y)| \mathrm{d}y < \infty,$$

$$\sup_x \int_{\mathbb{R}} |y|^2 |\nu_{xx}(x,y)| \mathrm{d}y < \infty,$$

where $\nu_x(x,y)$ and $\nu_{xx}(x,y)$ are the first and second derivatives (in the x variable) of $\nu(x,y)$, the following boundary regularity conditions holds

$$\int_{x-a}^{\infty} \nu(x,y) \mathrm{d}y = \mathcal{O}_a((x-a)^{-\beta}), \quad \int_{x-a}^{\infty} y\nu(x,y) \mathrm{d}y = \mathcal{O}_a((x-a)^{1-\beta}),$$

for some $\beta \in (1,2)$ and similarly for b.

Existence Theorems for two-sided operator $\beta \in (1, 2)$ II

(...Theorem 5 continued)
and (monotonicity condition) for any $h > 0$

$$\int_h^{\infty} \nu_x(x, y) \mathrm{d}y \geq 0, \quad \int_{-\infty}^{-h} \nu_x(x, y) \mathrm{d}y \leq 0. \qquad (34)$$

$\Rightarrow {}^2D_{ab*}^{(\nu)}$ generates a Feller semigroup on $C([a, b])$ such that $\{f \in C^2[a, b] : f' \in C_0[a, b]\} \subset \mathrm{Dom}({}^2D_{ab*}^{(\nu)})$.
Moreover ${}^2D_{ab}^{(\nu)}$ generates a sub-Feller semigroup on $C_0([a, b])$ such that
$\{f \in C^2[a, b] \cap C_0([a, b]) : f' \in C_0[a, b]\} \subset \mathrm{Dom}({}^2D_{ab}^{(\nu)})$.

Existence Theorems for two-sided operator $\beta \in (1, 2)$

(...Theorem 5 continued)
If in addition ν satisfies the following conditions: $\exists\ \omega < 1$ s.t.

$$(b-x)^{\omega} \int_{b-x}^{\infty} \left(\omega \frac{y}{(b-x)} - 1 \right) \nu(x,y)\mathrm{d}y \to -\infty \text{ as } x \to b, \tag{35}$$

and $\exists\ \omega' < 1$ such that

$$(x-a)^{\omega'} \int_{x-a}^{\infty} \left(\omega' \frac{y}{(x-a)} - 1 \right) \nu(x,y)\mathrm{d}y \to -\infty \text{ as } x \to a, \tag{36}$$

then the boundary points for the processes above are regular in expectations.

Examples of Concrete operators I

$$
\begin{aligned}
{}^{2}D^{\text{temp},\beta}_{a+*}f(x) := \int_0^{x-a} \left(f(x-y) - f(x) + yf'(x)\right)\frac{e^{-\lambda y}}{y^{1+\beta}}\mathrm{d}y \\
+ \left(f(a) - f(x)\right)\int_0^{x-a} \frac{e^{-\lambda y}}{y^{1+\beta}}\mathrm{d}y + \\
+ f'(x)\int_0^{x-a} y\frac{e^{-\lambda y}}{y^{1+\beta}}\mathrm{d}y, \quad x > a.
\end{aligned}
\tag{37}
$$

Examples of Concrete operators II

$$ {}^2D_{ab*}^{\text{temp}} f(x) := \sum_{n=1}^{N} {}^2D_{ab*}^{\text{temp},\beta_i} f(x), \quad x \in (a,b) \tag{38} $$

where

$$ D_{ab*}^{\text{temp},\beta_i} f(x) := ({}^2D_{a+*}^{\text{temp},\beta_i} +^2 D_{b-*}^{\text{temp},\beta_i}) \tag{39} $$

for $\lambda > 0$, $\beta_i \in (1,2)$ for all $i \leq N$.

Application: Generalised fractional Cauchy problem $\beta \in (1, 2)$

Theorem 6. Consider Cauchy problem

$$\begin{aligned} \partial_t u(t,x) &= {}_x D^{(\nu)}_{ab*} u(t,x), \; x \in (a,b), \\ u(0,x) &= f(x), \; x \in [a,b], \end{aligned} \tag{40}$$

where ν satisfies the conditions of Theorem 5 and $f \in \mathrm{Dom}(D^{(\nu)}_{ab*})$ we require, then the Cauchy problem (40) is wellposed in the sense of semigroup theory.

Generalised fractional Cauchy problem $\beta \in (1,2)$ with Dirichlet boundary conditions

Theorem 7. Consider Cauchy problem

$$\begin{aligned} \partial_t u(t,x) &=_x D_{ab}^{(\nu)} u(t,x), \ x \in (a,b), \\ u(0,x) &= f(x), \ x \in [a,b], \\ 0 = f(x) &= u(t,x), \ (t,x) \in \mathbb{R}^+ \times \{a,b\}, \end{aligned} \tag{41}$$

where ν satisfies the conditions of Theorem 5 and $f \in \mathrm{Dom}(D_{ab}^{(\nu)})$, then the Cauchy problem is wellposed in the sense of semigroup theory.

Definition of Generalised Fractional differential operators (multidimensional) $\beta \in (0, 1)$ I

Let us now turn to the multidimensional extension of this interruption procedure.
The analog of RL derivative arising from a process in $\mathbb{R}^d$ and a domain $D \subset \mathbb{R}^d$ is the generator of the process killed on leaving D.
For Caputo version this is more subtle, as we have to specify a point where a jump crosses the boundary. The most natural model assumes that a trajectory of a jump follows shortest path (a straight line in Euclidean case).

Definition of Generalised Fractional differential operators (multidimensional) $\beta \in (0, 1)$ II

Suppose $L^{(\nu)}$ is a generator of a Feller process $X_t(x)$ in $\mathbb{R}^d$ with the generator of type

$$L^{(\nu)} f(x) = (\gamma(x), \nabla) f(x) + \int_{\mathbb{R}^d} (f(x+y) - f(x)) \nu(x, dy)$$

with a kernel $\nu(x, .)$ on $\mathbb{R}^d \setminus \{0\}$ such that

$$\sup_x \int_{\mathbb{R}^d} \min(1, |y|) \nu(x, dy) < \infty,$$

that is, a generator or order at most one.

Multidimensional versions, III

Let D be an open convex subset of $\mathbb{R}^d$ with $B = \mathbb{R}^d \setminus D$. For $x \in \mathbb{R}^d$, let

$$D(x) = \{y \in \mathbb{R}^d : x + \lambda y \in D \quad \text{for all sufficiently small} \quad \lambda\}.$$

In particular, $D(x) = \mathbb{R}^d$ for all $x \in D$. Let

$$\lambda(x, y/|y|) = \min\{R > 0 : x + Ry/|y| \in B\},$$

$$R_D(x,y) = \begin{cases} x + y, & \text{if } |y| \leq \lambda(x, y/|y|) \\ x + \lambda(x, y/|y|)y/|y|, & \text{if } |y| \geq \lambda(x, y/|y|) \end{cases}$$

Multidimensional versions, IV

The process $X_t(x)$ with jumps interrupted on crossing B can be defined by the generator

$$D_{D*}^{(\nu)} f(x) := (\gamma(x), \nabla) f(x) + \int_{D(x)} [f(R_D(x,y)) - f(x)] \nu(x, dy),$$

which represents an analog of the Caputo-type boundary operator, that is the modification of the process on $\mathbb{R}^d$ obtained by interrupting jumps on an attempt to cross B.

Mixed BVP (FPDE) (multidimensional) $\beta \in (0, 1)$

Consider the GFPDE

$$
\begin{aligned}
{}_{x_1}D_{0+}^{(\nu)} u(x_1, x_2) + {}_{x_2}D_{0+*}^{(\nu)} u(x_1, x_2) =& \lambda u(x_1, x_2) + g(x_1, x_2), \\
& (x_1, x_2) \in (0, b_2) \times (0, b_2), \\
u(0, x_2) =& 0, \;\; x_2 \in [0, b_2] \\
u(x_1, 0) =& \phi(x_1), \;\; x_1 \in [0, b_1].
\end{aligned}
\tag{42}
$$

Theorem 8. (Hernandez-Hernandez, Kolokoltsov '16)
Suppose that $\lambda > 0$, $\nu = (\nu_1, \nu_2)$ where ν_1, ν_2 are both Lévy kernels satisfying the conditions of Theorem 1 and satisfies the conditions of Theorem 1 and $\phi \in C_0[0, b_1]$.
(i) If $g \in C[\mathbf{0}, \mathbf{b}]$ satisfies $g(\cdot, 0) = \lambda \phi(\cdot)$ then $\exists!$ solution in the domain of the generator to GFPVP (42) of the process $Y^{(x_1,x_2),(\nu)} := (X^{x_1,(\nu_1)}, X^{x_2,(\nu_2)})$ given by the resolvent operator.

Mixed BVP (FPDE) (multidimensional) $\beta \in (0,1)$ II

(...Theorem 8 continued)
(ii) For any $g \in B[\mathbf{0}, \mathbf{b}]$ mixed linear problem is wellposed in the generalised sense and the solution admits the stochastic representation

$$
\begin{aligned}
u(x_1, x_2) =& \mathbf{E}\left[e^{-\lambda \tau_0^{x_2,(\nu_2)}} \phi(X^{x_1,(\nu_1)}(\tau_0^{x_2,(\nu_2)})) 1(\tau_0^{x_2,(\nu_2)} < \tau_0^{x_1,(\nu_1)})\right] \\
& \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{\mathbf{x},(\nu)}} e^{-\lambda s} g(X^{x_1,(\nu_1)}(s), X^{x_2,(\nu_2)}(s)) \mathrm{d}s\right] \\
& + 0.
\end{aligned} \tag{43}
$$

Variable coefficient IVP $\beta \in (0,1)$

$$D^{(\nu)}_{a+*}u(x) = \lambda(x)u(x) + g(x),\ x \in (a,b),\ u(a) = u_a. \quad (44)$$

Theorem 9. (Hernandez-Hernandez, Kolokoltsov '16)
Suppose that ν satisfies the conditions of Theorem 1 and $\lambda \in C[a,b]$ is a positive function.
(i) If $g \in C[a,b]$ and $g(a) = u_a\lambda(a)$ then $\exists!$ solution in the domain of the generator.
(ii) For any $g \in B[a,b]$ and $u_a \in \mathbb{R}$, the linear problem 44 has a unique generalised solution given by the Feynman-Kac type formula

$$\begin{aligned} u(x) =& u_a \mathbf{E}\left[\exp\left\{-\int_0^{\tau_a^{x,\nu}} \lambda(X_r^{x,\nu})\mathrm{d}r\right\}\right] \\ & \mathbf{E}\left[\int_0^{\tau_a^{x,\nu}} \exp\left\{-\int_0^s \lambda(X_r^{x,\nu})\mathrm{d}r\right\} g(X_s^{x,\nu})\mathrm{d}s\right]. \end{aligned} \quad (45)$$

Summary of Boundary Value Problems already approached

The following BVPs have already been studied:

- $$D^{(\nu)}_{a+*}u(x) = \lambda(x)u(x) + g(x), \ x \in (a, \infty], \tag{46}$$
- $$D^{(\nu)}_{ab*}u(x) = \lambda u(x) + g(x), \ x \in (a, b), \tag{47}$$
- $$\sum_{i=1}^{N} {}_{x_i}D^{(\nu)}_{a*}u(x) = \lambda(x)u(x) + g(x), \ x \in D \subset \mathbb{R}^N, \ \text{(PDE)} \tag{48}$$
- $$^{2}D^{(\nu)}_{ab*}u(x) = \lambda u(x) + g(x), \ x \in (a, b), \tag{49}$$

for $\lambda \geq 0$ with the corresponding boundary conditions (along with the R-L versions).

Summary of Boundary Value Problems already approached

The following Cauchy problems (along with the R-L versions):

- $$\begin{aligned} \partial_t u(t,x) =& {}_x D^{(\nu)}_{a+*} u(t,x),\ (t,x) \in \mathbb{R}^+ \times [a,\infty), \\ f(x) =& u(0,x)\ \forall x \in [a,\infty), \end{aligned} \tag{50}$$

- $$\begin{aligned} \partial_t u(t,x) =& {}_x D^{(\nu)}_{ab*} u(t,x),\ (t,x) \in \mathbb{R}^+ \times [a,b], \\ f(x) =& u(0,x)\ \forall x \in [a,b], \end{aligned} \tag{51}$$

and the non-linear problem

- $$D^{(\nu)}_{ab*} u(x) = F(u)(x),\ x \in (a,b) \tag{52}$$

Future Directions

- Cauchy problems and time changes (a Dynkin martingale approach).
- Wellposedness of Fractional differential equations on bounded domains.

Cauchy problems and time changes

One main **Question**:

- Can we obtain unified generalised time changes results with respect to stochastic representations of solutions to fractional boundary value problems? (through the application of Dynkin martingale).

Differential equations that can be approached I

Let $\beta \in (0,1)$, $\alpha \in (1,2)$. Some possible equations to be considered are:

- Fractional in (space and time) diffusion equations

$$ {}_tD^{\beta}_{0+*}u(t,x) = -|_x\Delta|^{\alpha}u(t,x), \ (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{53} $$

- Fractional in (space and time) diffusion equations on bounded domains

$$ {}_tD^{\beta}_{a+*}u(t,x) =^2_x D^{\alpha}_{ab*}u(t,x), \ (t,x) \in (0,\infty) \times (a,b), \tag{54} $$

with the boundary conditions $u = \phi$ on $\{0\} \times \mathbb{R}^n$ in (40) and $u = \phi'$ on $(\{0\} \times [a,b]) \cup ([0,\infty) \times \{a,b\})$ in (41), and $-|_x\Delta|^{\alpha}$ is the fractional Laplacian.

Differential equations that can be approached

... and their generalised versions

- Generalised Fractional in (space and time) diffusion equations

$$ {}_tD^{(\nu)}_{-\infty *}u(t,x) = {}^2_x D^{(\nu)}u(t,x), \ (t,x) \in (0,\infty)\times\mathbb{R}^n, \quad (55) $$

- Generalised Fractional in (space and time) diffusion equations on bounded domains

$$ {}_tD^{(\nu)}_{0+*}u(t,x) = {}^2_x D^{(\nu')}_{ab*}u(t,x), \ (t,x) \in (0,\infty)\times(a,b), \quad (56) $$

along with their boundary conditions.

Stochastic representation for solutions and time-change arguements to Cauchy problems (through Dynkin martingale)

Consider the problem

$$-{}_tD^{(\nu)}_{0+*}u(t,x) = Lu(t,x),\ (t,x)\in(0,\infty)\times\mathbb{R}^n, u=\phi \text{ on } \{0\}\times\mathbb{R}^d, \tag{57}$$

where L is the generator of an $\mathbb{R}^n$-valued Feller process.
By Dynkin Martingale (if a solution in the domain of the generator exists) the solution u to the above problem (57) has the representation

$$u(t,x) = \mathbf{E}\left[\phi\left(X^{x,L}(\tau_0^{t,(\nu)})\right)\right], \tag{58}$$

where $\tau_0^{t,(\nu)}$ is the first time the Feller process generated by $D^{(\nu)}_{0+*}$ hits 0.

Stochastic representation for solutions and time-change arguements to Cauchy problems (through Dynkin martingale) II

Formal argument: rewrite problem the previous differential equation in stationary form $(D^{(\nu)}_{0+*} + L)u = 0$. Through the (successful) application of Dynkin martingale

$$u(t,x) = \mathbf{E}\left[u(T^{t,(\nu)}_s, X^{x,L}_s) - \int_0^s (D^{(\nu)}_{0+*} + L)u(T^{t,(\nu)}_r, X^{x,L}_r)\mathrm{d}r\right], \tag{59}$$

for all $s \in \mathbb{R}^+$, where $T^{t,(\nu)}$ is the R^+-valued process generated by $D^{(\nu)}_{0+*}$ started at t, $X^{x,L}$ is the R^d-valued process generated by L started at x and the two processes are independent.

Stochastic representation for solutions and time-change arguements to Cauchy problems (through Dynkin martingale) III

Now the first time $Y_s^{t,x} := (T^{t,(\nu)}(s), X^{x,L}(s))$ hits the boundary $\{0\} \times R^d$ equals the first time $T_s^{t,(\nu)}$ hits 0, call it $\tau_0^{t,(\nu)} = Z_t$. By Optional Stopping

$$\begin{aligned} u(t,x) =& \mathbf{E}\left[u(T^{t,(\nu)}(\tau_0^{t,(\nu)}), X^{x,L}(\tau_0^{t,(\nu)}))\right] = \mathbf{E}\left[u(0, X^{x,L}(\tau_0^{t,(\nu)}))\right] \\ =& \mathbf{E}\left[\phi(X^{x,L}(Z_t))\right]. \end{aligned} \tag{60}$$

Insurance-relevant example

Consider the problem

$$({}_xD^{\beta}_{0+*} + {}_yD^{\alpha}_{0+*})u(x,y) = 0 \text{ on } \Omega := (0,\infty)\times(0,\infty),\ \phi = u \text{ on } \partial\Omega. \tag{61}$$

The stochastic representation of the solution of this BVP under a Dynkin martingale example is given by

$$\begin{aligned} u(x,y) =& \mathbf{E}\left[u(X^{x,\beta}(\tau_0^{y,\alpha}),0)1(\tau_0^{x,\beta} > \tau_0^{y,\alpha})\right] \\ &+ \mathbf{E}\left[u(0,(X^{y,\alpha}(\tau_0^{x,\beta}))1(\tau_0^{x,\beta} < \tau_0^{y,\alpha})\right] \end{aligned} \tag{62}$$

More general case

$$Au = g, \ \mathbf{x} \in \Omega, u = \phi \text{ on } \partial\Omega, \ \Omega \subset \mathbb{R}^N, \tag{63}$$

where $A := \sum_{i=1}^{N} {}_{x_i}L_i$ then

$$u(\mathbf{x}) = \mathbf{E}[\phi(X^{x_1,L_1}_{\tau^{\mathbf{x},A}_{\partial\Omega}}, ..., X^{x_N,L_N}_{\tau^{\mathbf{x},A}_{\partial\Omega}})] - \mathbf{E}\left[\int_0^{\tau^{\mathbf{x},A}_{\partial\Omega}} g(X^{x_1,L_1}_s, ..., X^{x_N,L_N}_s) \mathrm{d}s\right]. \tag{64}$$

A cross-dependency example

Consider the problem

$$({}_xD^{\beta(y)}_{a+*} + {}_yD^{\alpha(x)}_{b+*})u(x,y) = 0 \text{ on } \Omega := (a,\infty)\times(b,\infty),\ \phi = u \text{ on } \partial\Omega. \tag{65}$$

The solution of this problem under a Dynkin martingale example is given by

$$\begin{aligned} u(x,y) =& \mathbf{E}\left[u(X^{x,\beta(y)}(\tau_b^{y,\alpha(x)}), b)1(\tau_a^{x,\beta(y)} > \tau_b^{y,\alpha(x)})\right] \\ &+ \mathbf{E}\left[u(a, X^{y,\alpha(x)}(\tau_a^{x,\beta(y)}))1(\tau_a^{x,\beta(y)} < \tau_b^{y,\alpha(x)})\right] \end{aligned} \tag{66}$$

Well-posedness of Fractional differential equations on bounded domains I

Two main topics

- Clarify which processes are related to the Caputo (generator) in terms of the free generator and its domain (interrupted, stopped, reflected, censored processes).
- Probabilistic wellposedness of Cauchy problems on bounded domains with non-zero-Dirichlet boundary conditions.

Well-posedness of Fractional differential equations on bounded domains II

For recent research on well-posedness fractional (diffusion) equations on bounded domains and their probabilistic counterpart see for example:

- '*Fractional diffusion on bounded domains*'
  O. Defterli, M. D'Elia, Q. Du, M. Gunzburger, R. Lehoucq, M.M. Meerschaert, Frac. Cal. and Appl. An, Vol 18, 2, 2015.
- '*Space-time fractional Dirichlet problems*'
  B. Baeumer, T. Luks, M. M. Meerschaert, arXiv:1604.06421, 2016.

- '*Reflected spectrally negative stable processes and their governing equations*'
  B. Beaumer, M. Kovács, M.M. Meerschaert, R.L. Schilling, P. Straka. Transactions of the American Mathematical Society Vol 368, 1, Jan 2016.
- '*Fractional Cauchy problems on bounded domains*'
  M.M. Meerschaert, E. Nane. Annals of Probability Vol 37, no. 3, 2009.
- '*Space-time fractional diffusion on bounded domain*'
  Z.-Q. Chen, M.M. Meerschaert, E. Nane. Jurnal of Mathematical Analysis and Applications, Vol 393, 2012.

Recover interrupted process from Free process: the R-L case

Question 1: can we recover the process generated by $D^{(\nu)}_{ab*}$ from the free process on $\mathbb{R}$ generated by

$$D^{(\nu)}f(x) := \int_{\mathbb{R}} (f(x-y) - f(x) + yf'(x))\nu(x,y)\mathrm{d}y \quad ? \quad (67)$$

Results in [Baeumer et al. 2016] is likely to solve this issue for generalised R-L.
Roughly speaking [Baeumer et al. 2016] gives conditions for a general strong-Feller Feller process to describe the generator of the respective killed process, and they obtain the two-sided R-L fractional derivative as the generator of the killed free motion by applying their results to the kernel

$$\nu(x,y) := \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{1}{|y|^{1+\alpha}}, \quad \alpha \in (1,2).$$

Recover interrupted process from Free process: The Caputo case

[Baeumer, Kovácset al. 2016] show that the Caputo derivative of order $\alpha \in (1,2)$, D^{α}_{0+*} is the generator of the process

$$Y_t := X^{\alpha}_t - \inf_{s \le t} X^{\alpha}_s$$

on $C_\infty[a,\infty)$, where X^{α} is a spectrally negative α-stable Lévy process, and they provide a core D.
If $X^{\alpha}_{t \wedge \tau^x_0}$ is Feller process then the action of its generator L_{stop} will equal the action of D^{α}_{0+*} on functions such that

$$D^{\alpha}_{0+*} f(0) = 0,$$

which is the case if $f \in C^2[0,\infty) \cap D$, for example.
Question 2: which processes are generated by a Caputo type fractional differential operator?

Wellposedness of fractional differential equations and bounded domains III

Consider the BVP for $\alpha \in (1,2)$ with zero-Dirichlet boundary conditions

$$\partial_t u(t,x) = {}_x D^{\alpha}_{ab} u(t,x), \ (t,x) \in (0,\infty) \times (a,b), \tag{68}$$

$$f(x) = u(0,x) \ \forall x \in [a,b], u(t,a) = u(t,b) = 0 \ \forall t \geq 0,$$

where ${}_x D^{\alpha}_{ab} := {}_x D^{\alpha}_{a+} + {}_x D^{\alpha}_{b-}$ the sum of left and right R-L derivatives.

It is a non-trivial task to impose boundary conditions to BVPs like (68). As it is shown in [Baeumer et al. 2016] the probabilistic framework provides a setting to obtain wellposedness of BVP (68).

If we require $f \in \mathrm{Dom}({}_x D^{\alpha}_{ab})$, the domain of the respective killed free-process and the solution $u(t, \cdot) \in C_0[a, b]\ \forall t \geq 0$ then the solution is unique and is given by

$$\mathbf{E} f(X_t^{x,kill}), \tag{69}$$

where $X_t^{x,kill}$ is the Feller process obtained by killing the process X_t^x on the attempt of leaving (a, b).

Zero-Dirichlet boundary conditions and R-L

In our framework each Cauchy problem with with zero-Dirichlet boundary conditions involving R-L type generators

$$\begin{aligned} \partial_t u(t,x) =& {}_x D_{ab}^{(\nu)} u(t,x),\ (t,x) \in \Omega, \\ f(x) =& u(0,x),\ \forall x \in [a,b], \\ 0 = u(t,a) =& u(t,b),\ \forall t \geq 0, \end{aligned} \tag{70}$$

where $\Omega = (0,\infty) \times (a,b)$, is well-posed in the sense above and the underlying process is a killed process.

NON-Zero-Dirichlet boundary conditions and Caputo

As we already mentioned, if we consider a generalised Caputo diffusion equation we obtain a (probabilistic) well-posedness to the IVP

$$\begin{aligned} \partial_t u(t,x) =& {}_x D^{\alpha}_{a+*} u(t,x), \ (t,x) \in (0,\infty) \times (a,\infty) \\ f(x) =& u(0,x), \ \forall x \in [0,\infty), \end{aligned} \tag{71}$$

for $f \in \mathrm{Dom}({}_x D^{\alpha}_{a+*})$, but we have no control over the values at the boundary points $[0,\infty) \times \{a\}$ in the sense that we do NOT have

$$u(t,a) = \mathbf{E}[f(X^a_t)] = \text{constant}.$$

NON-Zero-Dirichlet boundary conditions and Caputo

Question 3: how do we obtain wellposedness of the problem

$$\begin{aligned} \partial_t u(t,x) =& {}_x D^{(\nu)}_{ab*} u(t,x),\ (t,x) \in \Omega, \\ f(x) =& u(0,x),\ \forall x \in [a,b], \\ f(a) =& u(t,a),\ f(b) = u(t,b),\ \forall t \geq 0. \end{aligned} \tag{72}$$

where $\Omega = (0,\infty) \times (a,b)$?
Guess: Stop the free process.
where ${}_x D^{(\nu)}_{ab*}$ is a generalised two-sided Caputo operator of order $\beta \in (1,2)$.

Reasons for the guess

- Heuristic argument: Definition and R-L case.
- Stopped Feller process that is a Feller process.
- Dynkin Martingale + Characteristic operator.
- Resolvent equation.

(Almost all the following pseudo-arguments are independent of the non-locality of the differential operators, only assumption is that the differential operators are generators of Markov processes on $[a, b]$.)

- Dynkin Martingale + Characteristic operator (not present in this slides).

Consider the problem

$$(-\partial_t + {}_xD^{(\nu)}_{0+*})u(t,x) = 0, \text{ on } \Omega := (0,\infty)\times(0,\infty), \ \phi = u \text{ on } \partial\Omega. \tag{73}$$

The stochastic representation of the solution of the Caputo type BVP under a Dynkin martingale/Optional Stopping Theorem framework the solution has the representation

$$\begin{aligned} u(t,x) =& \mathbf{E}\left[u(0, X^{x,(\nu)}(\tau_0^t))1(\tau_0^{x,(\nu)} > \tau_0^t)\right] \\ &+ \mathbf{E}\left[u(t, X^{x,(\nu)}(\tau_0^{x,(\nu)}))1(\tau_0^{x,(\nu)} < \tau_0^t)\right] \\ =& \mathbf{E}\left[u(0, X^{x,(\nu)}(t))1(\tau_0^{x,(\nu)} > t)\right] \\ &+ \mathbf{E}\left[u(t, X^{x,(\nu)}(\tau_0^{x,(\nu)}))1(\tau_0^{x,(\nu)} < t)\right] \end{aligned} \tag{74}$$

Resolvent equation

In the case of a Resolvent equation

$$ {}_xD^{\alpha}_{ab*}u(x) = \lambda u(x) + g(x),\ x \in (a,b),\ g(x) = u(x)\, x \in \{a,b\}, \tag{75}$$

if we impose the (non-local) boundary condition

$$ {}_xD^{\alpha}_{ab}u(a) =_x D^{\alpha}_{ab}u(b) = 0 \tag{76}$$

we obtain that the solution is given by the resolvent of the stopped underlying process which equals...

...

$$u(x) = R^{\lambda} g(x) = \mathbf{E}[\int_0^{\tau_{a,b}^{x,(\nu)}} e^{-\lambda s} g(X_s^{x,(\nu)})] + \frac{g(a)}{\lambda} \mathbf{E}[e^{-\lambda \tau_a^{x,(\nu)}} 1(\tau_a^{x,(\nu)} < \tau_b^{x,(\nu)})] \frac{g(b)}{\lambda} \mathbf{E}[e^{-\lambda \tau_b^{x,(\nu)}} 1(\tau_a^{x,(\nu)} > \tau_b^{x,(\nu)})] \quad (77)$$

Proposition 2. Let X^x be a Feller process with semigroup acting on $C_\infty[a,\infty)$ and $(L,\mathcal{D})$ the pair of generator and its domain. Let τ_a^x the first time X^x hits $\{a\}$ and assume it is finite. Denote $X^x_{t\wedge\tau_a^x}$ to be the stopped process and suppose it also is a Feller process with semigroup acting on $C_\infty[a,\infty)$ and denote by $(L_{stop},\mathcal{D}_{stop})$ its generator and the respective domain.
Suppose that there exists $f\in\mathcal{D}$ such that $Lf(a)=0$,
$\Rightarrow f\in\mathcal{D}_{stop}$, $Lf=L_{stop}f$ and $R^\lambda g=R^\lambda_{\text{stop}}g$.

Summary: Wellposedness Cauchy problems and bounded domains III

Questions:

- What processes are associated to this probabilistic generalised fractional derivatives in terms of the free process?
- Can we obtain a classification of the domains for stopped/interrupted/reflected/censored processes associated to Caputo-type fractional derivatives?
- How do we obtain wellposedness of the problem

$$\begin{aligned} \partial_t u(t,x) &=_x D^{(\nu)}_{ab*} u(t,x),\ (t,x) \in (0,\infty) \times (a,b), \\ f(x) &= u(0,x),\ \forall x \in [a,b], \\ f(a) &= u(t,a),\ f(b) = u(t,b),\ \forall t \geq 0, \end{aligned} \tag{78}$$

Key features of the theory (re-again)

- Unified probabilistic framework to deal with fractional differential equations.
- Obtaining stochastic representations for solutions (good for numerics).
- Giving natural probabilistic generalisations to many fractional differential and integral operators on intervals and multi-dimensional bounded domains.

End

Thank you.