A Workshop on Future Directions in
Fractional Calculus Research and Applications
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Fractional Derivatives on Cosmic Scales

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Fractional Derivatives on Cosmic Scales

Cosmology

Calcagni, G. (2009). *Journal of High Energy Physics*, *2009*(09), 112.
Rami, E. N. A. (2007). *Romanian Reports in Physics*, *59*(3), 763-771.
Shchigolev, V. K. (2011). *Communications in Theoretical Physics*, *56*(2), 389.
Vacaru, S. I. (2012). *International Journal of Theoretical Physics*, *51*(5), 1338-1359.
Roberts, M. D. (2009). *arXiv preprint arXiv:0909.1171*.

Solar physics

Milovanov, A. V., & Zelenyi, L. M. (1998). *Astrophysics and Space Science*, *264*(1-4), 317-345.
Stanislavsky, A. A. (2010). W*orkshop on the International Heliophysical Year.*
Uchaikin, V., Sibatov, R., & Byzykchi, A. (2014). *Comm in Applied and Industrial Mathematics*, *6*(1), e-480.

Cosmic rays

Lagutin, A. A., Nikulin, Y. A., & Uchaikin, V. V. (2001). *Nuclear Physics B-Proc Supplements*,*97*(1), 267-270.
Webb, G. M., Zank, G. P., et al. (2006). *The Astrophysical Journal*, *651*(1), 211.
Uchaikin, V. V., & Sibatov, R. T. (2012). *Gravitation and Cosmology*, *18*(2), 122-126.
Ketabi, N., & Fatemi, J. (2009). *Trans B: Mech Engineering, Sharif University of Technology*, *16*(3), 269-272.
Uchaikin, V. V. E. (2013). *Physics-Uspekhi*, *56*(11), 1074.

Astronomical image processing

Sparavigna, A. C. (2009). *arXiv preprint arXiv:0910.2381*.

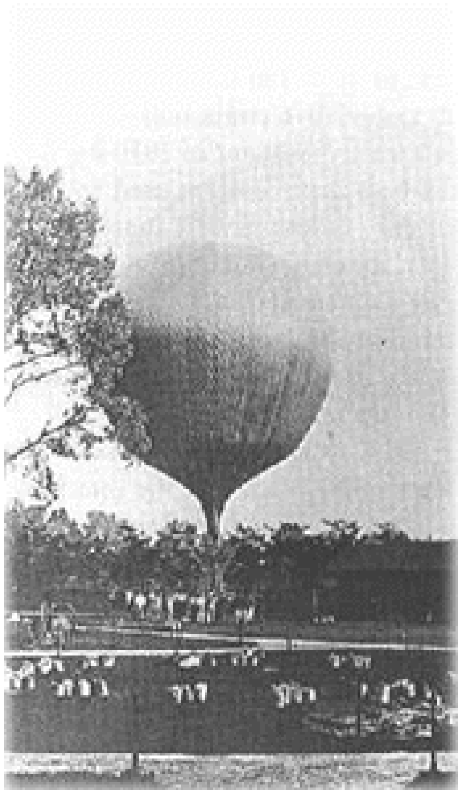
Large-scale structure of the Universe

Uchaikin, V. V. (2004). *Gravitation and Cosmology*, *10*, 5-24.

Gamma-ray astronomy
Interstellar plasma dynamics
Dark Matter

The Discovery of Cosmic Rays

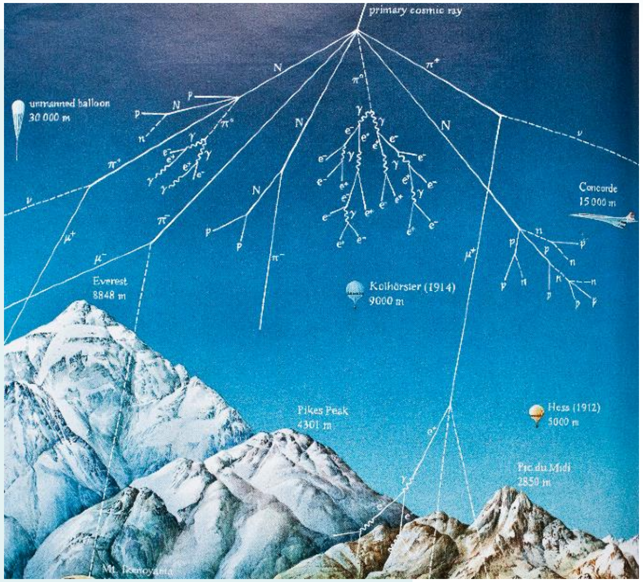
- At the beginning of the 20th century, scientists thought there was too much radioactivity than could be accounted for naturally. Where was it coming from?
  - Victor Hess decided to test the idea that the additional radiation came from outer space. In 1912, one way to do this was by BALLOON!
  - He got to about 18,000 feet (without oxygen) He noticed that the radiation steadily increased.



The discharging radiation comes from the outer space!

- COSMIC RAYS!

About the history of Cosmic Rays



1912 Discovery of cosmic rays (Victor Hess)

1929 Skobelzyn observed CRs with a cloud chamber.
Bothe and Kolhorster: tracks are curved by a magnetic field.
1928/29 Clay observed the “latitude effect”
1934 The sign of the east-west asymmetry.

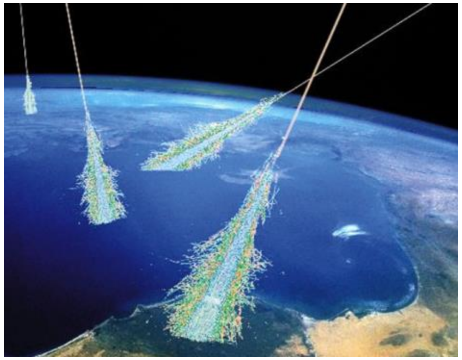
1932 Anderson discovered the positron in cosmic rays.
1936 muon (Anderson),
1947 charged pions, 1947–50 strange particles

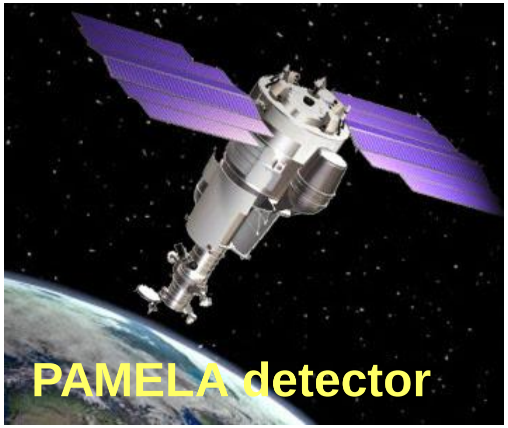
1934/38 Rossi and independently Auger discovered ”extensive air showers”
1934 The electromagnetic cascade theory (Bethe and Heitler).
1947 scaling of hadronic interactions (Zatsepin).

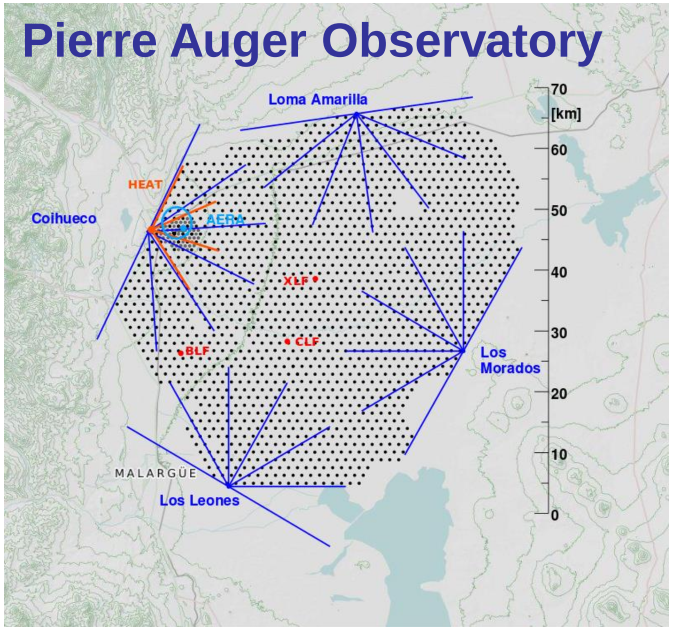
1949 acceleration mechanism (Fermi).
1952 Syrovatskii model of cosmic ray diffusion
2000 Fractional diffusion equation for galactic CR (Lagutin, Uchaikin)

1952-54 The first human accelerators reaching p >1 GeV were built.
1954 First measurements of high energy cosmic rays
1972 the start of high energy gamma astronomy.
1976 Start of the high energy neutrino astronomy

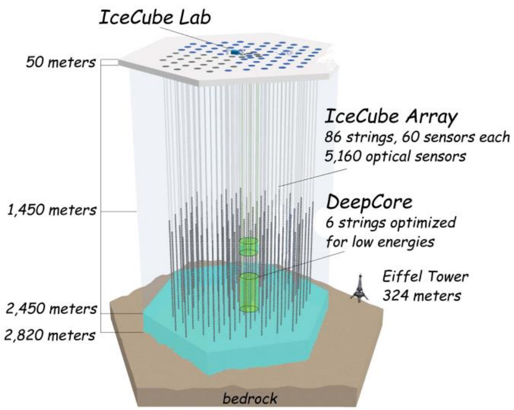
Aspiration of humankind to solve the mystery of cosmic rays



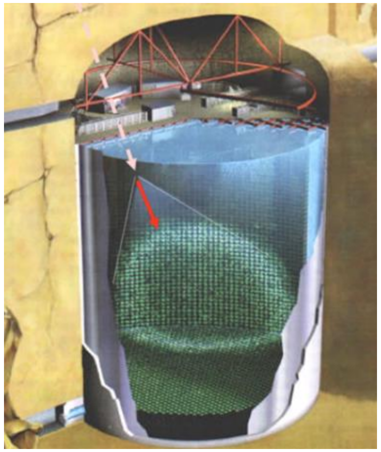




Ice Cube



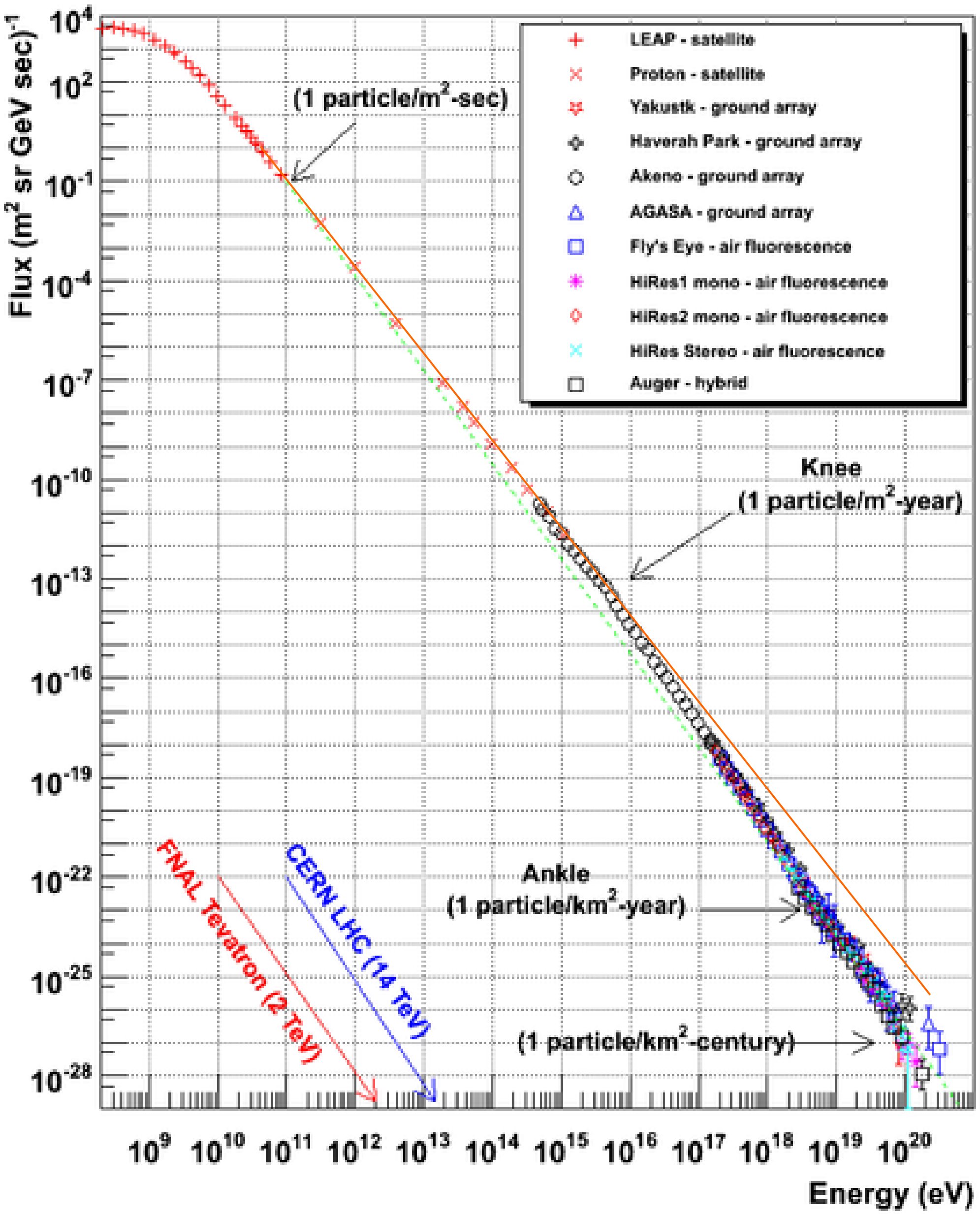
Super-K



Voyager-2

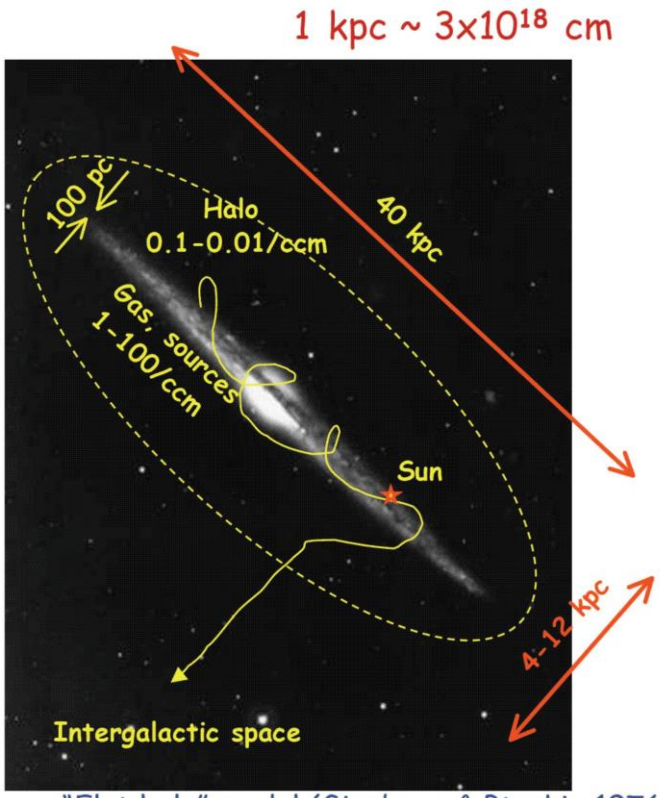
and many others …

All-particle spectrum



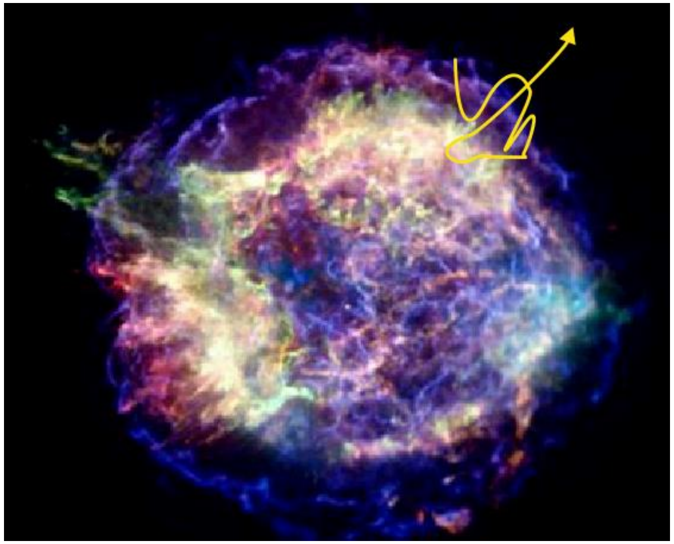
Propagation of cosmic rays in the Galaxy

Interstellar medium: dust, gas, plasma, magnetic fields – long lines, clouds, random irregular structures.



"Flat halo" model (Ginzburg & Ptuskin 1976)

Sources: bearing in supernovas bursts and accelerating on theirs remnants.



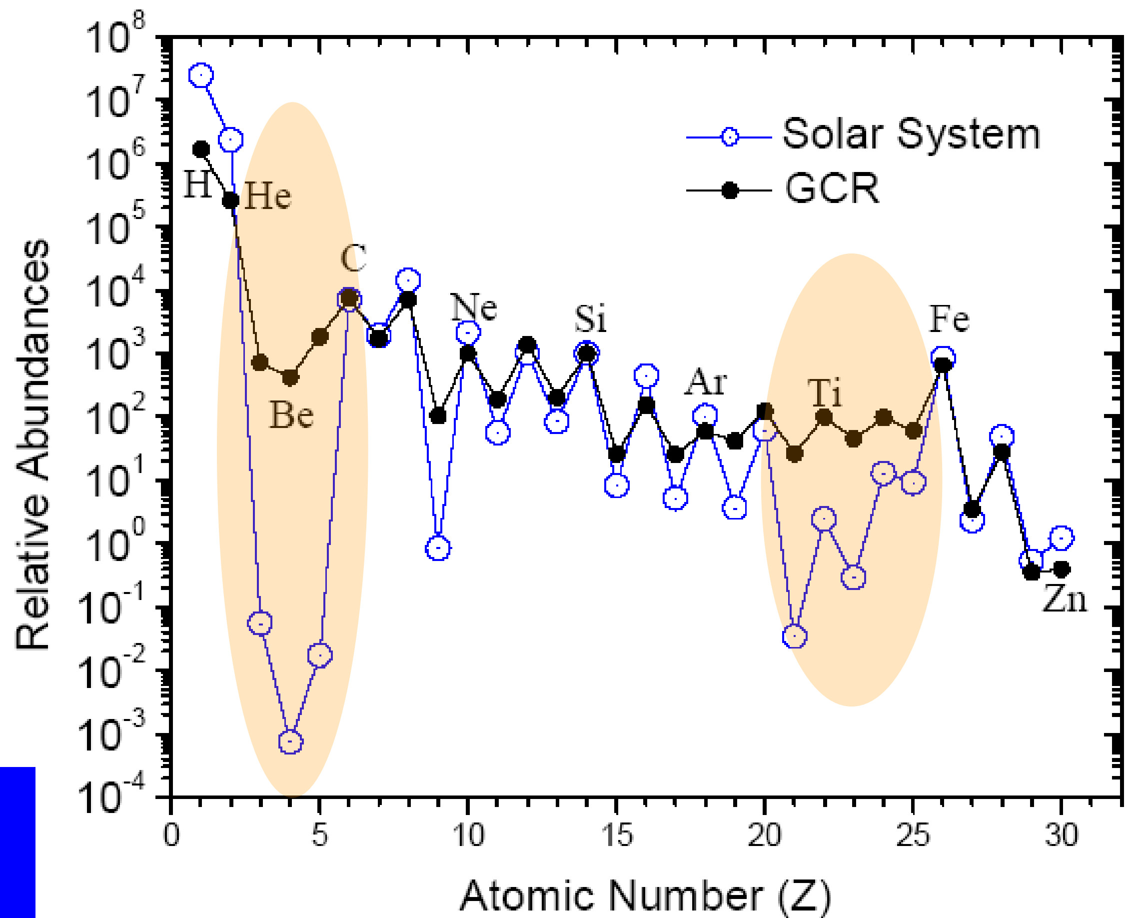
The problem is to develop such propagation model which could predict all observed characteristics of CR: mass composition, energy spectra, angular distributions.

CR move diffusively in the Galaxy

- **Secondary particle production (B/C ratio) implies CR pass through 5-10 g/cm^2 in lifetime**
  - Average Galactic plane density of 1 H atom/cm^3 implies CR traverse 1000 kpc for a lifetime of 3 x 10^6 yr
  - Ratio decreases with energy implies higher energy CR escape Galaxy more quickly

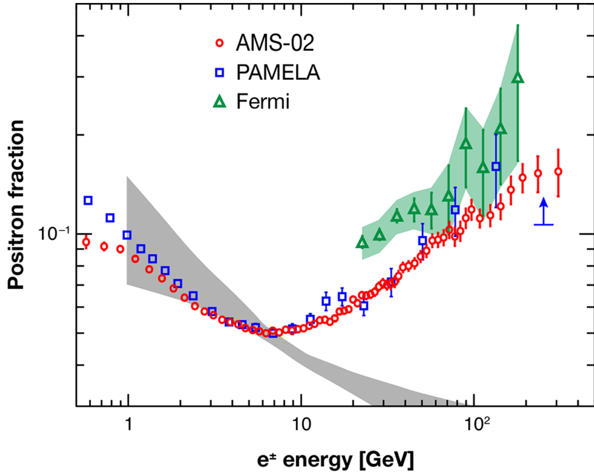
- **Long lived radioactive isotope (e.g. Be^{10}) implies even longer lifetime so CR spend considerable time in Galactic halo**

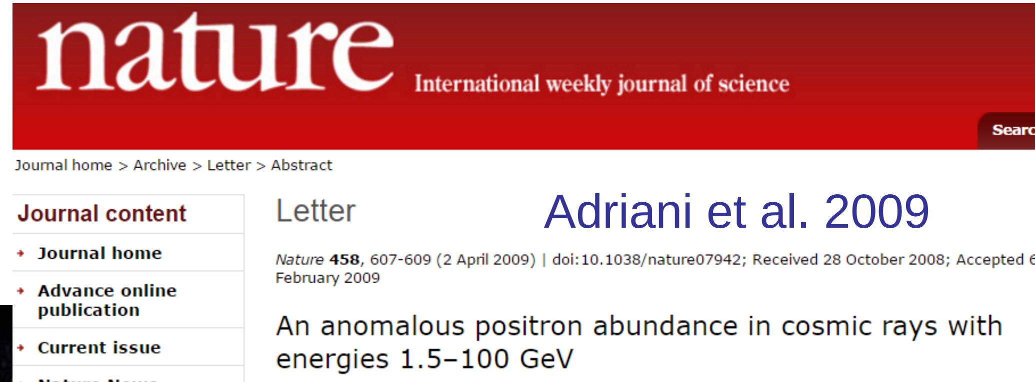
CRs STAY IN THE GALAXY FOR A FEW TENS MILLION YEARS BEFORE ESCAPING THE GALAXY



What is the diffusion regime?

Positron/electron fraction and indirect dark matter detection



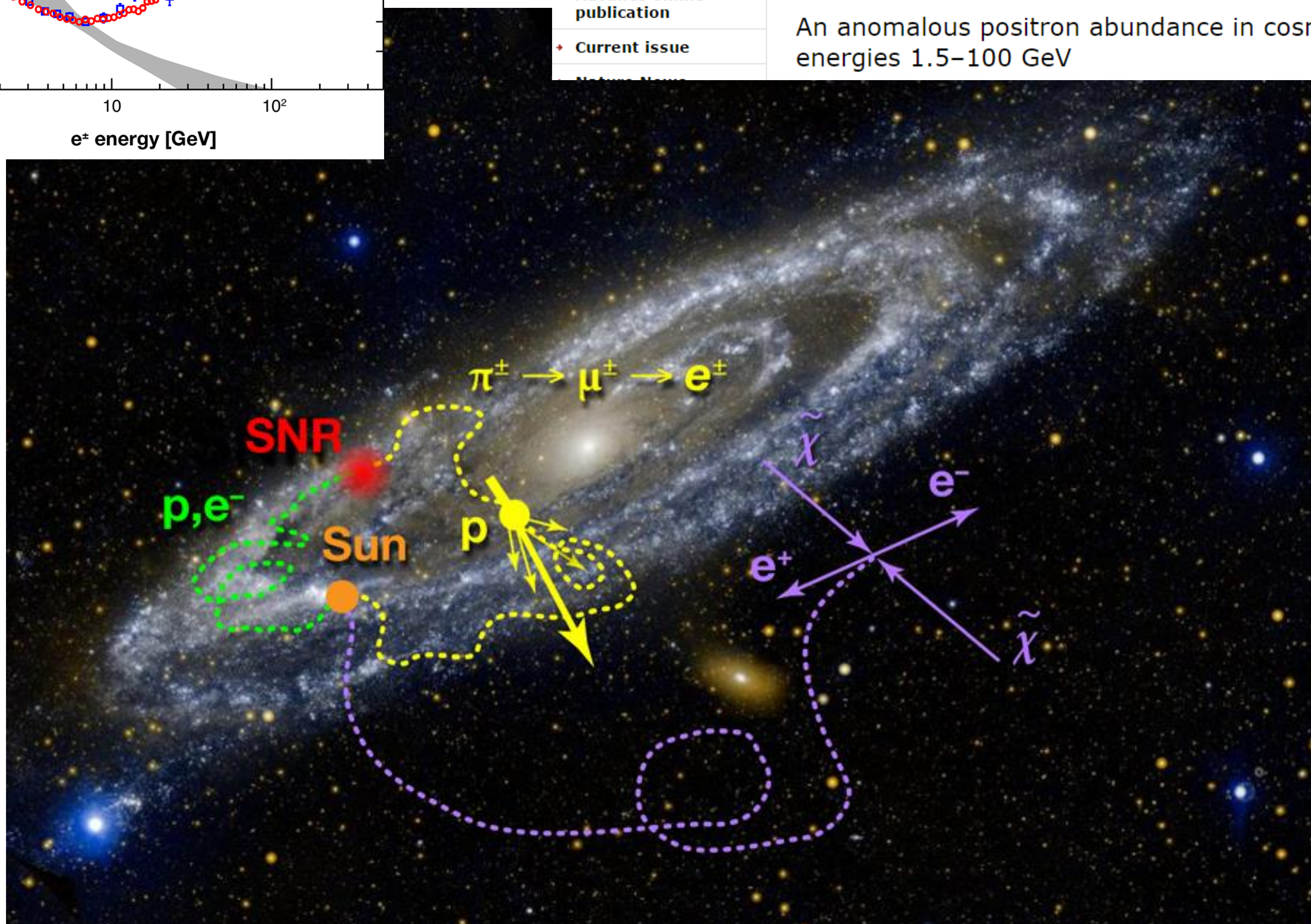


Letter

Nature **458**, 607-609 (2 April 2009) | doi:10.1038/nature07942; Received 28 October 2008; Accepted 6 February 2009

An anomalous positron abundance in cosmic rays with energies 1.5–100 GeV

Adriani et al. 2009



Transport equation and computational codes

$$\overbrace{\frac{\partial N^j}{\partial t}}^{\text{Variation}} + \overbrace{\left(-\vec{\nabla}\cdot\left(D(E,\vec{r})\vec{\nabla}\right) + \vec{\nabla}\cdot\vec{V}_c(\vec{r})\right)}^{\text{Spatial transport: diffusion+convection}} N^j + \overbrace{\frac{\partial}{\partial E}\left(b^j N^j - D_{EE}\frac{\partial N^j}{\partial E}\right)}^{\text{E losses and gains}} + \overbrace{\left(\Gamma_{\rm rad} + \Gamma_{\rm inel}\right)}^{\text{Catastrophic losses}} N^j = \overbrace{Q^j(t,E,\vec{r})}^{\text{Source}}$$

	(Semi-)analytical	*Numerical*	*Monte Carlo*
Approach	<u>Simplify the problem</u>: • keep dominant effects only • simplify the geometry	<u>Finite difference scheme</u>: • discretise the equation • scheme (e.g., Crank-Nicholson)	<u>Follow each particle</u>: • N particles at t=0 • evolve each of them to t+1 $1\mathrm{D}: \Delta z = \pm\sqrt{2D\Delta t}$
Tools	• Differential equations (Green functions, Fourier+Bessel expansions...)	• Numerical recipes/solvers (NAG, GSL libraries)	• Stochastic differential equations (Markov process) + MPI
Pros	• Useful to understand the physics • Fast (MCMC analyses "simple")	• Very simple algebra • Any new input easily included	• Statistical properties (along path) • No grid but t step (for/back)-ward
cons	• Only solve approximate model • New solution for new problem	• Slower, memory for high res. • "Less" insight in the physics	• Even slower (+ statistical errors) • Massively parallel problem
Codes and/or references	Webber (1970+) Ptuskin (1980+) Schlickeiser (1990+) **USINE (2000+)**	GALPROP (Strong et al. 1998) DRAGON (Evoli et al. 2008) PICARD (Kissmann et al., 2013)	Webber & Rockstroh (1997) Farahat et al. (2008) Kopp, Büshing et al. (2012)

Some defects of the diffusion model

It contradicts the special relativity (superluminal motion)

Aloisio, R., Berezinsky, V., Gazizov, A. (2009). The Astrophysical Journal, 693(2), 1275.
Prosekin, A. Y., Kelner, S. R., & Aharonian, F. A. (2015). *Physical Review D*, *92*(8), 083003.

Model ignores the multiscale structure

Ragot B. R., Kirk J. G. (1997). Astron. Astrophys. 327.
Lagutin A., Nikulin Yu., Uchaikin V. (2001). Nucl. Phys. B Proc. Suppl. 97
Erlykin, A. D., Lagutin, A. A., & Wolfendale, A. W. (2003). *Astroparticle Physics*, *19*(3), 351-362.

It doesn't provide transition to ballistic motion at high energies

Aloisio, R., Berezinsky, V., Gazizov, A. (2009). The Astrophysical Journal, 693(2), 1275.

Invalid near local sources and boundaries

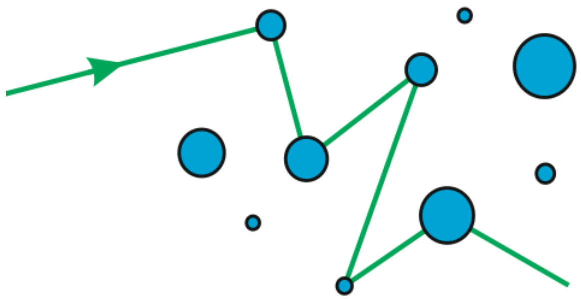
Litvinenko, Y. E., Effenberger, F., & Schlickeiser, R. (2015). The Astrophysical Journal, 806(2), 217.

Invalid at high anisotropies

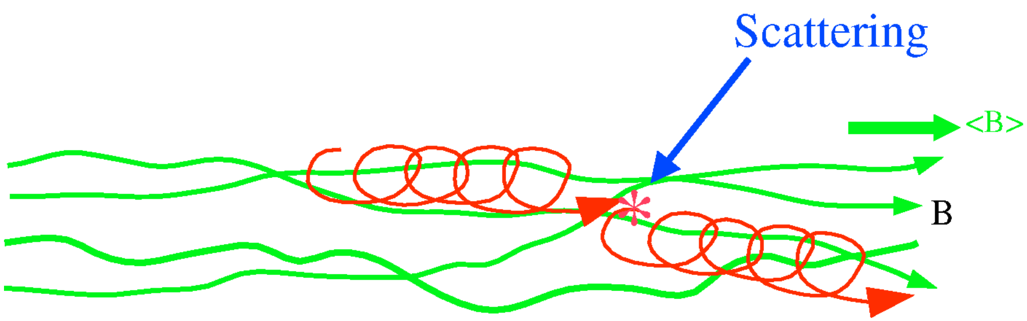
Erlykin, A. D., Sibatov, R. T., Uchaikin, V. V., & Wolfendale, A. PoS (ICRC2015) 463.

R. Carmona, W. C. Masters, and B. Simon, J. Funct. Anal. 91, 117, 1990.
J. Dunkel, P. Talkner, and P. Hänggi, Phys. Rev. D 75, 043001, 2007.
B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman, Phys. Rev. Lett. 53, 419, 1984.
J. Dunkel and P. Hänggi, Phys. Rep. 471, 1, 2009.
Baeumer, B., Meerschaert, M. M., Naber, M. *Physical Review E*, *82*(1), 011132, 2010.

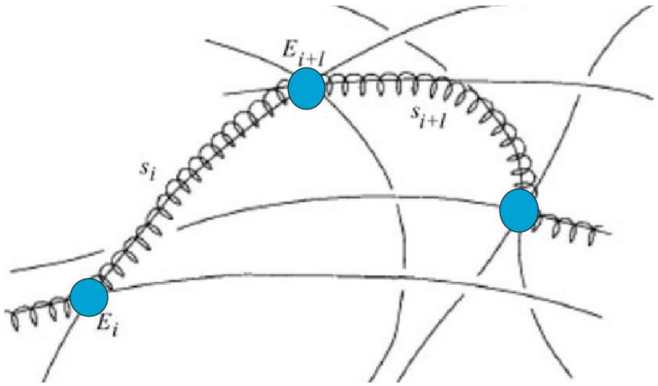
Three types of diffusion models



Isotropic diffusion



Anisotropic diffusion

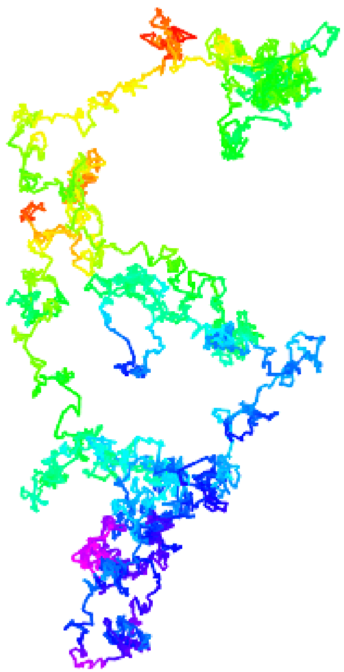


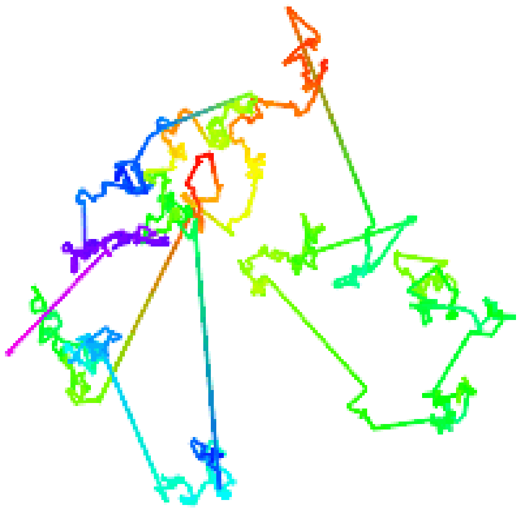
Isotropization
of compound diffusion

Nonlocal diffusion

Ragot B. R., Kirk J. G. Astron. Astrophys. 327 (1997).
Lagutin A., Nikulin Yu., Uchaikin V. Nucl. Phys. B Proc. Suppl. 97 (2001)

$$\alpha = 1.6$$





Exponential distribution of free path length

Power law distribution of free path length

$$\mathrm{Prob}(R > r) \propto r^{-\alpha}$$

Propagator of nonlocal diffusion

Fractional equation of cosmic rays diffusion

$$\frac{\partial N}{\partial t} = -D_\alpha(-\Delta)^{\alpha/2} N(\mathbf{r}, t, E) + \delta(\mathbf{r})\delta(t)S(E)$$

Solution in terms of Levy-Feldheim density

$$\boxed{N(\mathbf{r}, t, E) = [D_\alpha t]^{-3/\alpha} \Psi_3^{(\alpha)}([D_\alpha t]^{-1/\alpha} r) S(E)}$$

$$D(E) = D_0 E^\delta$$

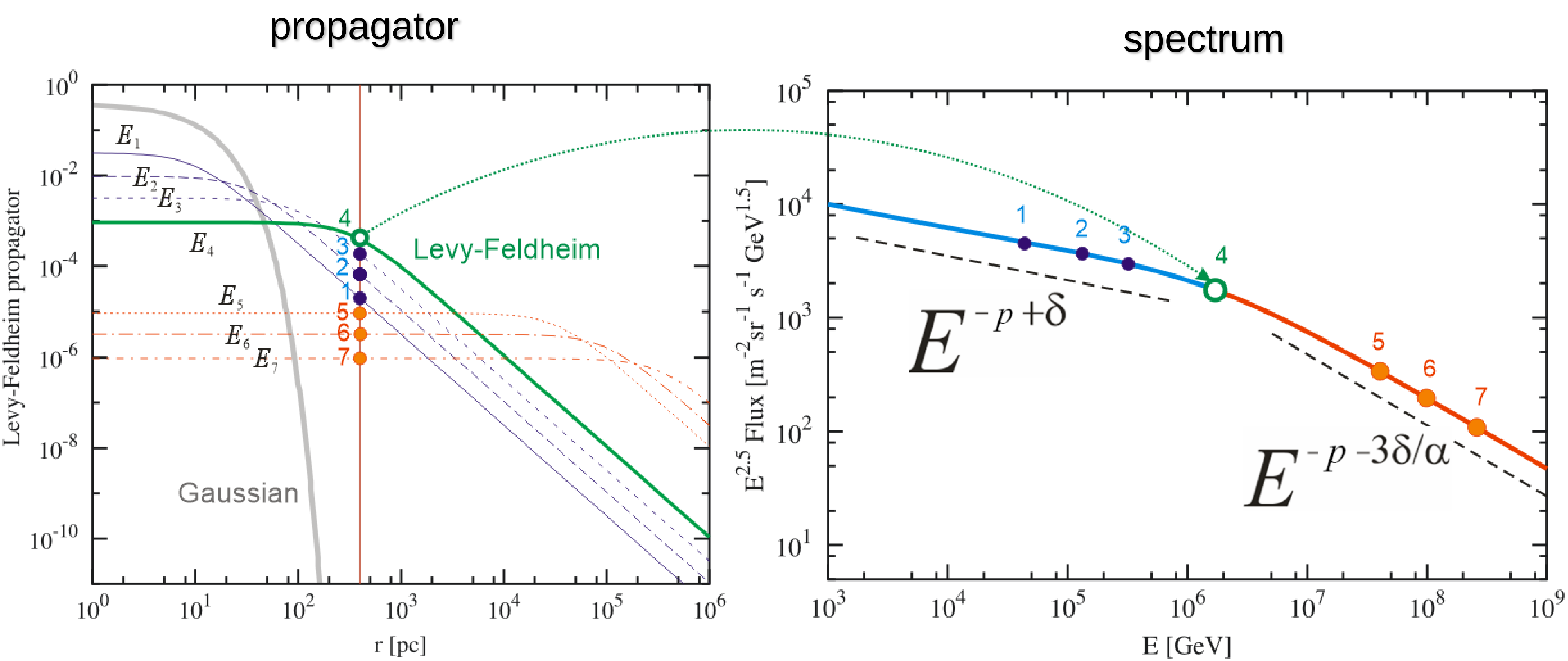
For high energies

$$N(\mathbf{r}, t, E) \sim S_0 r^{-3} E^{-p} \left[\xi^3 \Psi_3^{(\alpha)}(0)\right] \propto E^{-p-3\delta/\alpha}, \quad E \to \infty,$$

For low energies

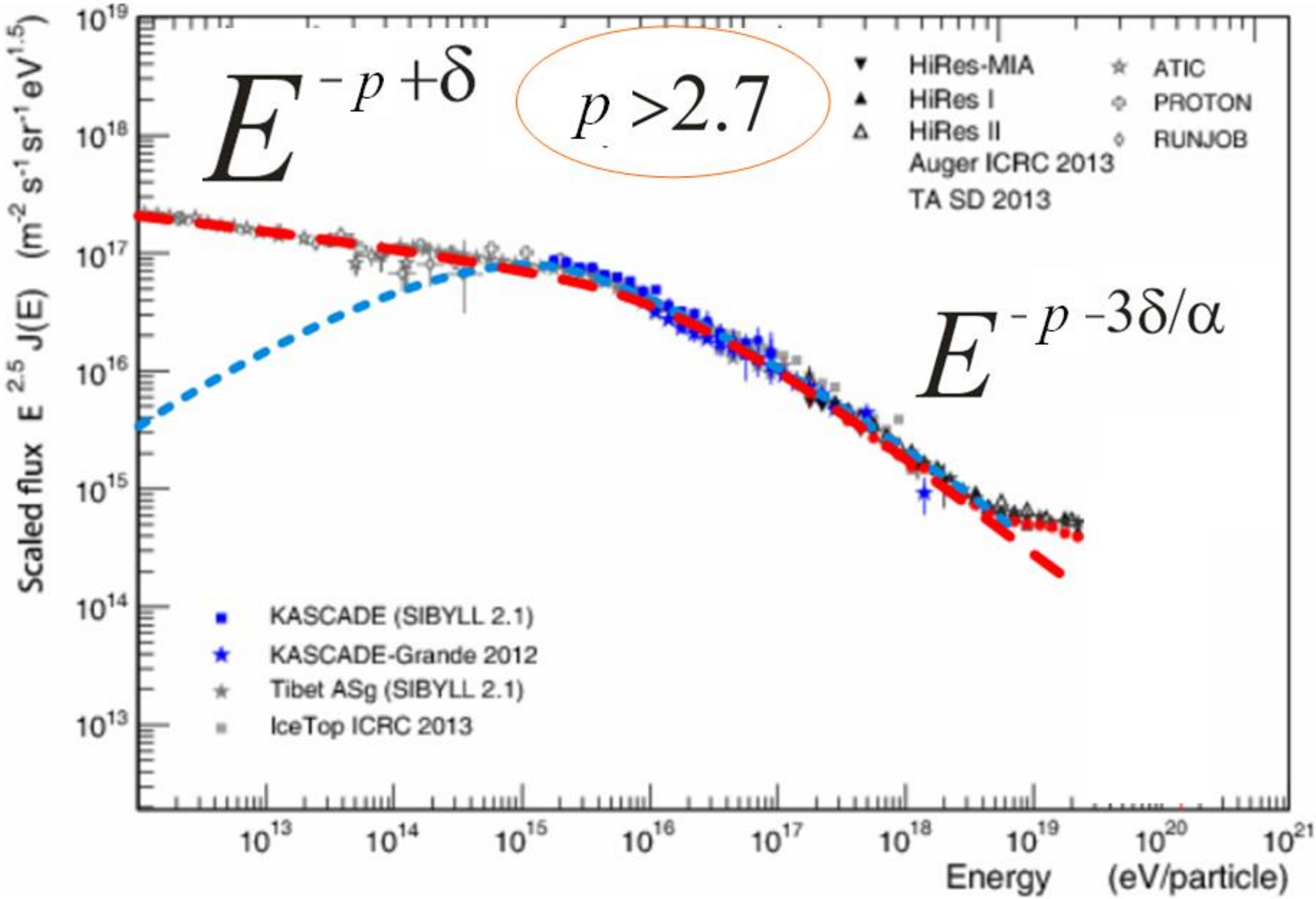
$$N(\mathbf{r}, t, E) \propto \begin{cases} E^{-(p+3\delta/2)} \exp(-\xi_1^2/4E^\delta), & \alpha = 2; \\ E^{-(p-\delta)}, & \alpha < 2. \end{cases}$$

“Knee” in the model of anomalous diffusion

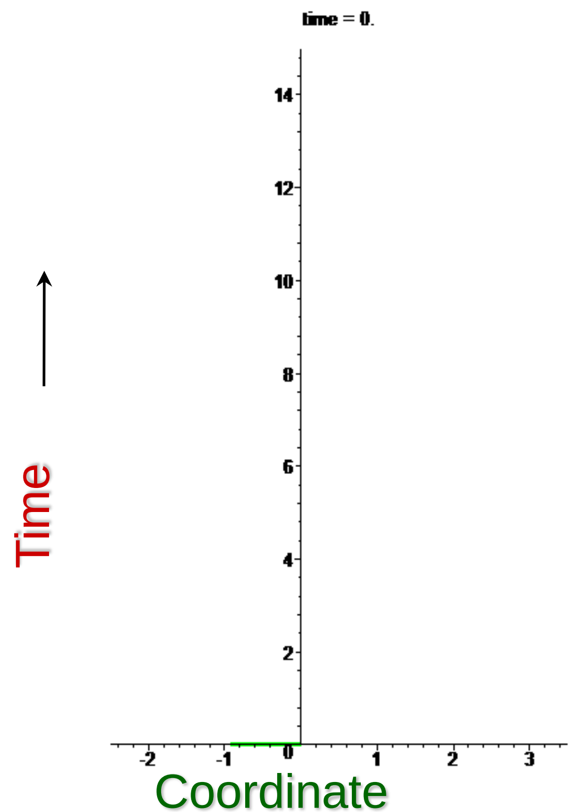


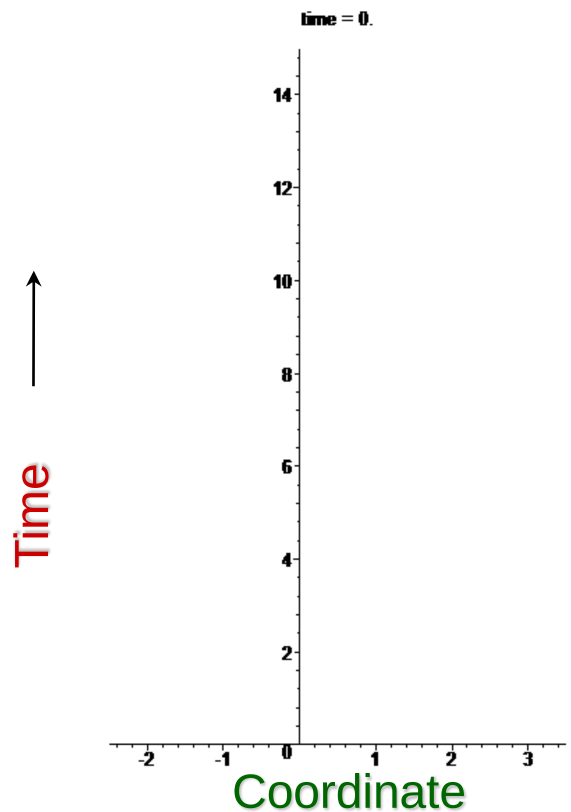
Lagutin A A, Uchaikin V V Nucl. Instrum. Meth. Phys. Res. B 201 212 (2003).
Lagutin A A, Tyumentsev A G, in Proc. of ICRC (2004).
Erlykin A D, Lagutin A A, Wolfendale A W Astropart. Phys. 19 (2003).

Spectrum in the anomalous diffusion model



‘Nonrelativistic’ and ‘relativistic’ random walk





$$\mathsf{D}_t^\alpha + D(-\Delta)^{\alpha/2} \mapsto \langle(\mathsf{D}_t + v\mathbf{\Omega}\nabla)^\alpha\rangle$$

Uchaikin, V. V. (2010). *JETP letters*, *91*(3), 105-109.

Uchaikin, V. V., & Sibatov, R. T. (2012). *Gravitation and Cosmology*, *18*(2), 122-126.

Uchaikin, V. V. (2013). *Physics-Uspekhi*, *56*(11), 1074.

Sibatov, R. T., & Uchaikin, V. V. PoS (ICRC2015) 538.

Erlykin, A. D., Sibatov, R. T., Uchaikin, V. V., & Wolfendale, A. W. PoS (ICRC2015) 463.

NoRD-model

Green function in terms of Fourier-Laplace transformation

$$\widetilde{G}(\mathbf{k},\lambda) = \frac{(1/v)\widetilde{P}(\mathbf{k},\lambda/v)}{1-\widetilde{p}(\mathbf{k},\lambda/v)}$$

$$\widetilde{P}(\mathbf{k},\lambda/v) = \int P(\mathbf{r})e^{-(\lambda/v)r}e^{i\mathbf{kr}}d\mathbf{r}$$

$$\widetilde{p}(\mathbf{k},\lambda/v) = \int p(\mathbf{r})e^{-(\lambda/v)r}e^{i\mathbf{kr}}d\mathbf{r}$$

Laplace transform of PDF of free path length

$$p_R(r) = -\frac{d\mathrm{P}(R>r)}{dr} \sim \alpha A r^{-\alpha-1}$$

$$\widehat{p}_R(s) = \int_0^\infty p_R(r)e^{-sr}dr \sim \begin{cases} 1 - A_\alpha s^\alpha, & 0<\alpha<1; \\ 1 - \langle R\rangle s + A_\alpha s^\alpha, & 1<\alpha<2; \\ 1 - \langle R\rangle s + \langle R^2/2\rangle s^2, & \alpha>2, \end{cases}$$

Transform of left hand side operator

$$1-\widetilde{p}(\mathbf{k},\lambda/v) = \int \left[1 - e^{-[\lambda/v - i\mathbf{k\Omega}]r}\right] p(\mathbf{r})d\mathbf{r} \sim$$

$$\sim \begin{cases} A_\alpha \langle(\lambda/v - i\mathbf{k\Omega})^\alpha\rangle, & 0<\alpha<1; \\ \langle R\rangle\langle(\lambda/v - i\mathbf{k\Omega})\rangle - A_\alpha\langle(\lambda/v - i\mathbf{k\Omega})^\alpha\rangle, & 1<\alpha<2; \\ \langle R\rangle\langle(\lambda/v - i\mathbf{k\Omega})\rangle - \langle R^2/2\rangle\langle(\lambda/v - i\mathbf{k\Omega})^2\rangle, & \alpha>2, \end{cases}$$

3D material derivative of fractional order

Inverse Fourier-Laplace transformation leads to

$$\left\langle \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right)^{\alpha} \right\rangle N(\mathbf{r},t) = S_{\alpha}(\mathbf{r},t), \quad 0 < \alpha < 1$$

$$\left[\frac{\partial}{\partial t} - \frac{A_{\alpha}}{v^{\alpha-1}\langle R\rangle} \left\langle \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right)^{\alpha} \right\rangle \right] N(\mathbf{r},t) = \frac{1}{\langle R\rangle} S_{\alpha}(\mathbf{r},t), \quad 1 < \alpha < 2;$$

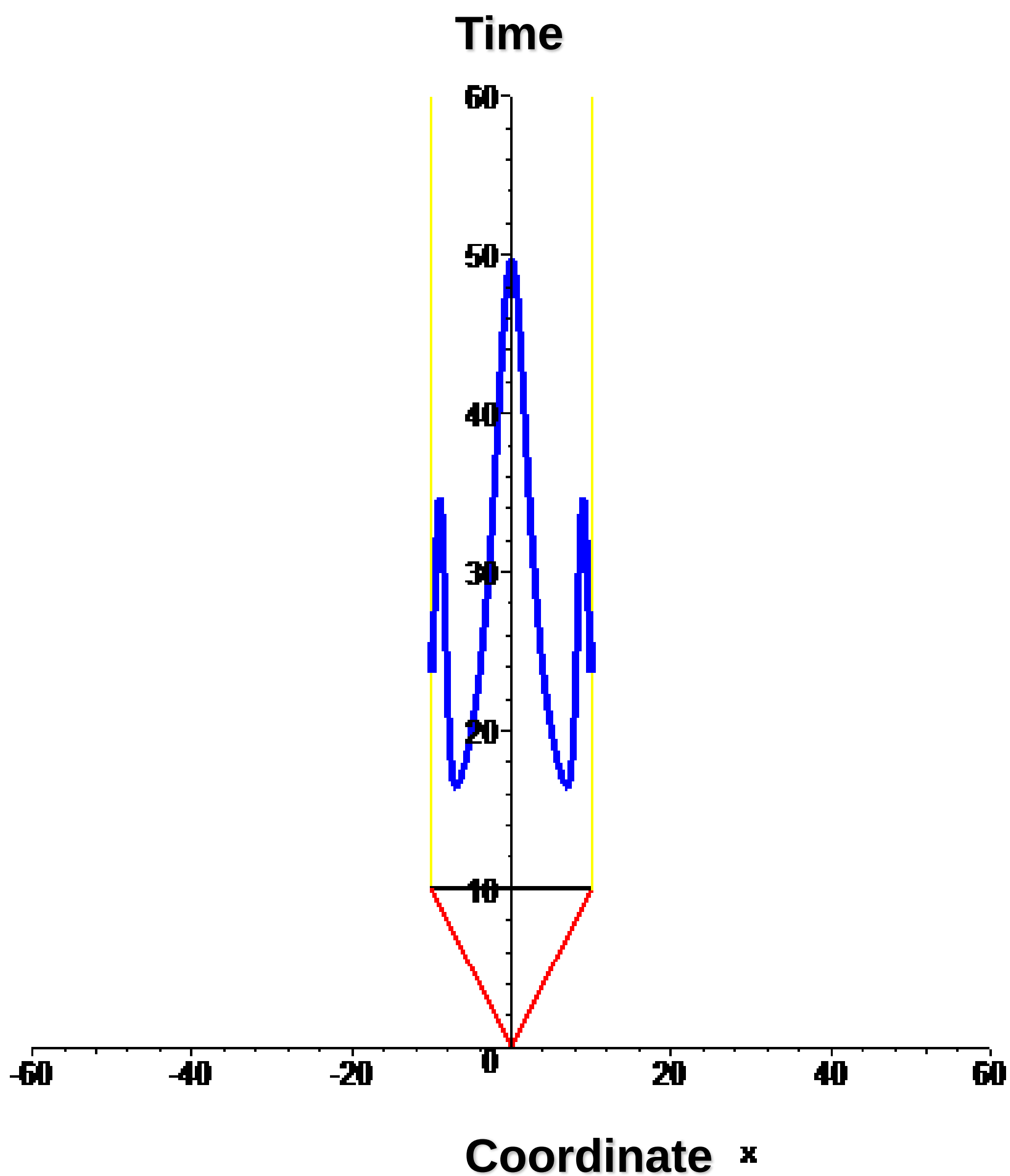
$$\left(\frac{\partial}{\partial t} - \frac{v\langle R^2\rangle}{6\langle R\rangle} \triangle \right) N(\mathbf{r},t) = \frac{1}{\langle R\rangle} S_2(\mathbf{r},t), \quad \alpha > 2$$

3D material derivative of fractional order

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right)^{\alpha} N(\mathbf{r},t) = \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right) \int\limits_0^t \frac{N(\mathbf{r} - \mathbf{v}(t-\tau), \tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}} d\tau.$$

Uchaikin, V. V. (2013). *Physics-Uspekhi*, *56*(11), 1074.
Uchaikin V.V., Sibatov R.T. Fractional Kinetics on Cosmic Scales (to be published).

Evolution of NoRD-propagator



One-dimensional case

$$0 < \alpha < 1$$

Transport equation

$$\left[\gamma_2 \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^{\alpha} + \gamma_1 \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)^{\alpha}\right] G_{\|}(x,t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}[\gamma_2\delta(x+vt) + \gamma_1\delta(x-vt)].$$

Sokolov, I. M., & Metzler, R. (2003). *Physical Review E*, *67*(1), 010101.
Uchaikin, V. V., & Sibatov, R. T. (2004). *Technical Physics Letters*, *30*(4), 316-318.

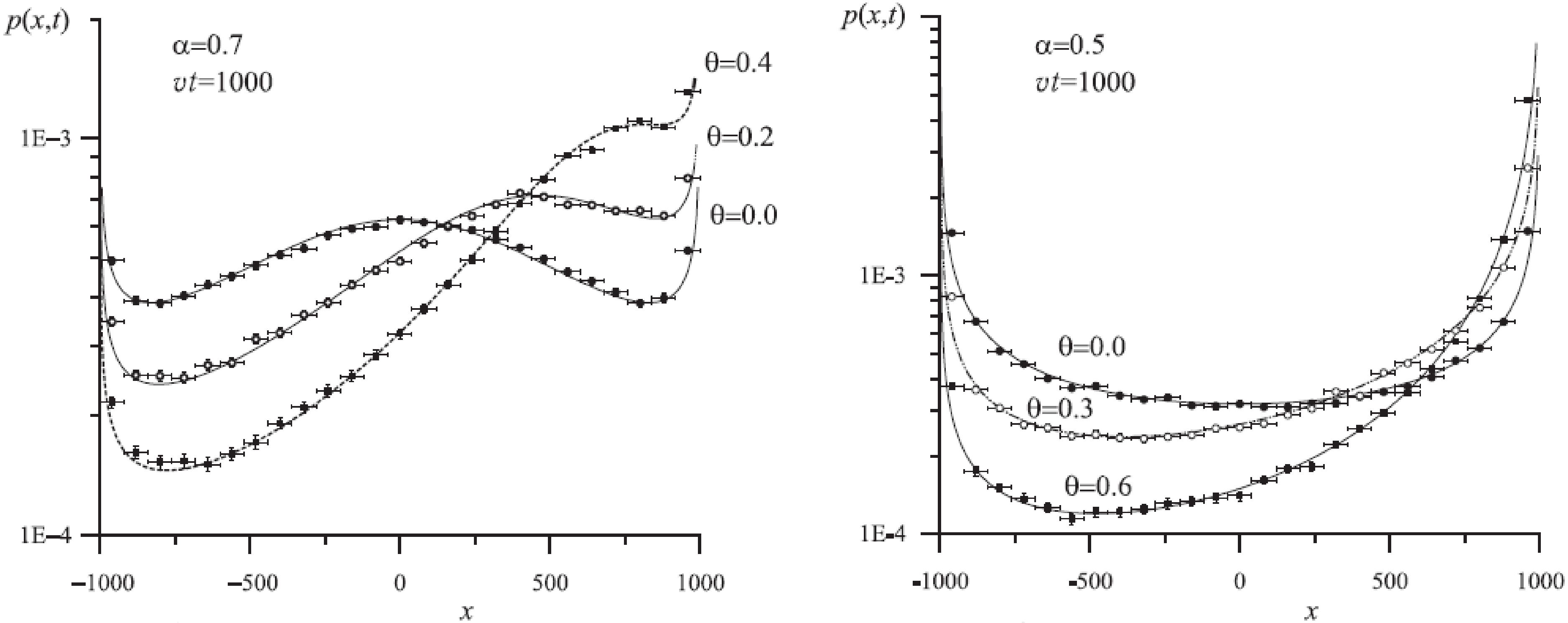
Solution

$$G_{\|}(x,t) = \frac{2\sin\pi\alpha}{\pi v t} \frac{\gamma_1\gamma_2\left(1 - x^2/v^2t^2\right)^{\alpha-1}}{\gamma_1^2(1-x/vt)^{2\alpha} + \gamma_2^2(1+x/vt)^{2\alpha} + 2\gamma_1\gamma_2\left(1 - x^2/v^2t^2\right)^{\alpha}\cos\pi\alpha}, \quad \alpha \in (0,1).$$

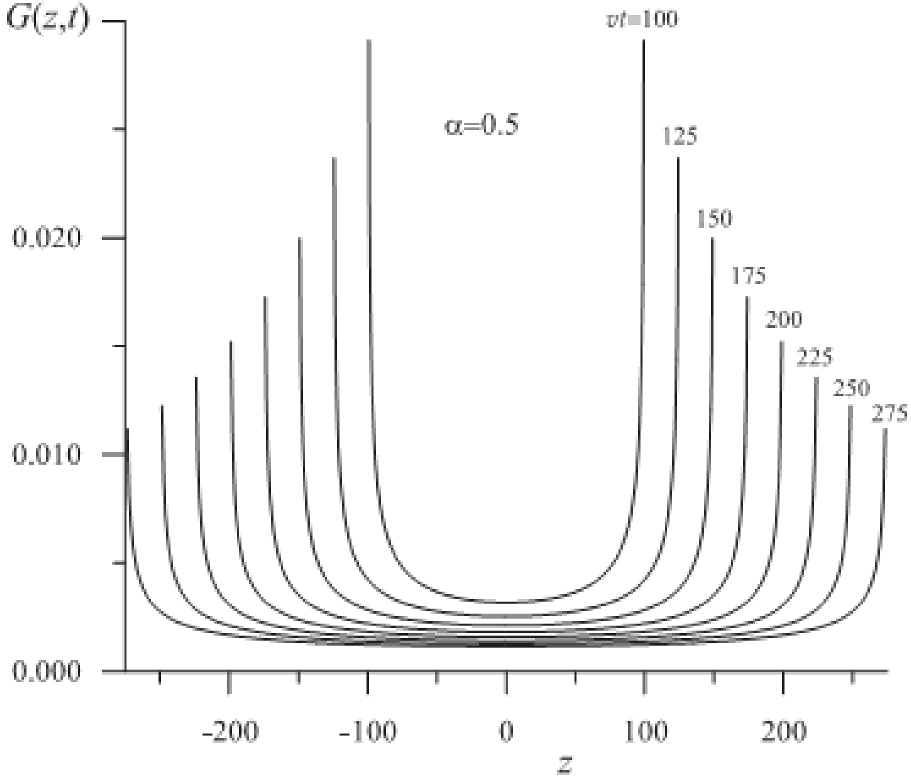
J. Lamperti, Trans. Am. Math. Soc. 88, 380 (1958).
G. Bel and E. Barkai, Phys. Rev. Lett. 94, 240602 (2005).
Rebenshtok, A., & Barkai, E. (2007). *PRL*, *99*(21), 210601.

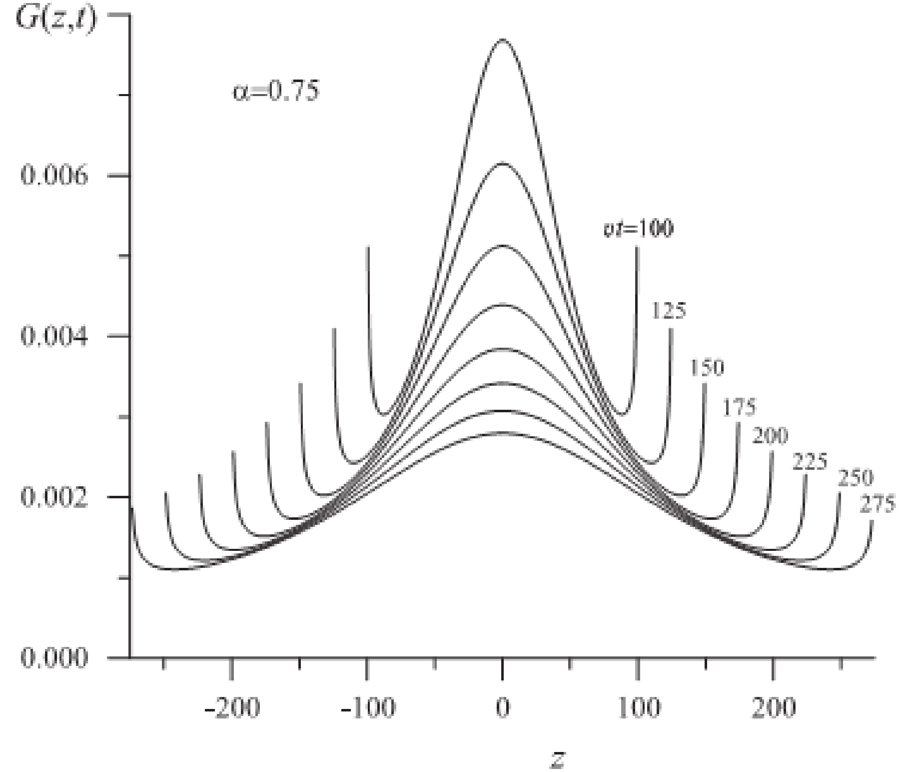
Uchaikin, V. V., & Sibatov, R. T. (2012). *Gravitation and Cosmology*, *18*(2), 122-126.

One-dimensional asymmetric pdf



One-dimensional propagator





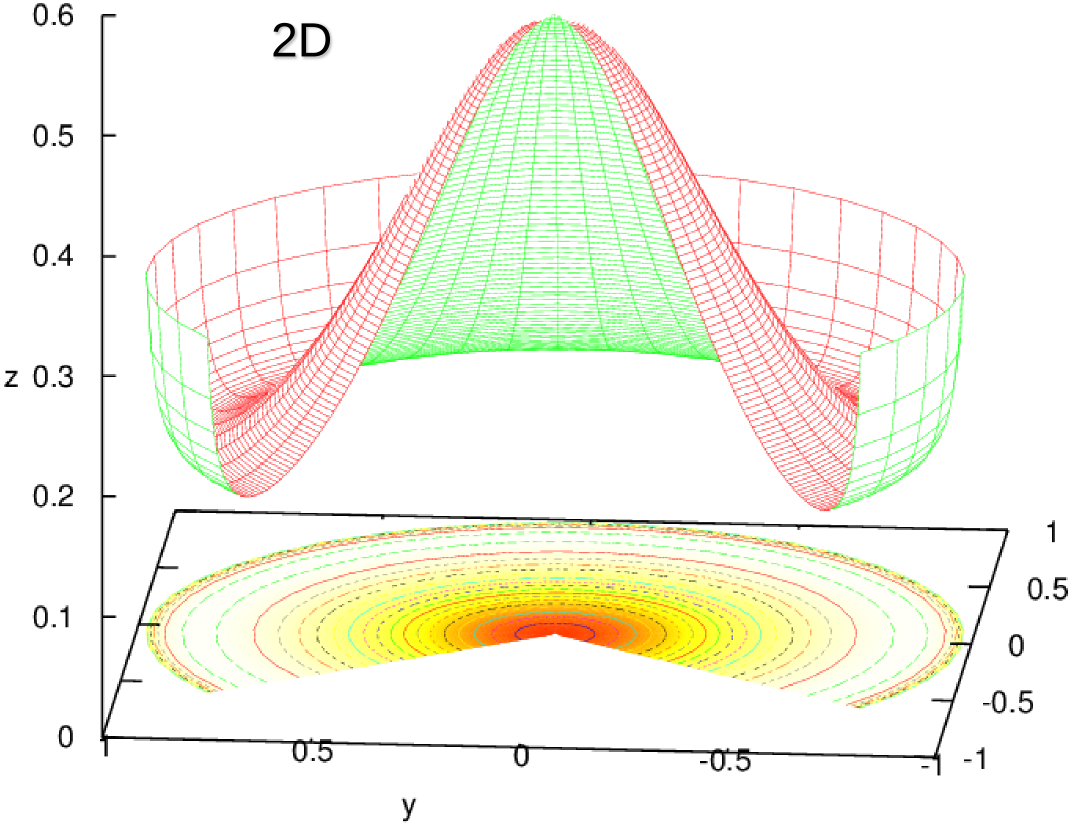
J. Lamperti, Trans. Am. Math. Soc. 88, 380 (1958).
G. Bel and E. Barkai, Phys. Rev. Lett. 94, 240602 (2005).
Rebenshtok, A., & Barkai, E. (2007). *PRL*, *99*(21), 210601.

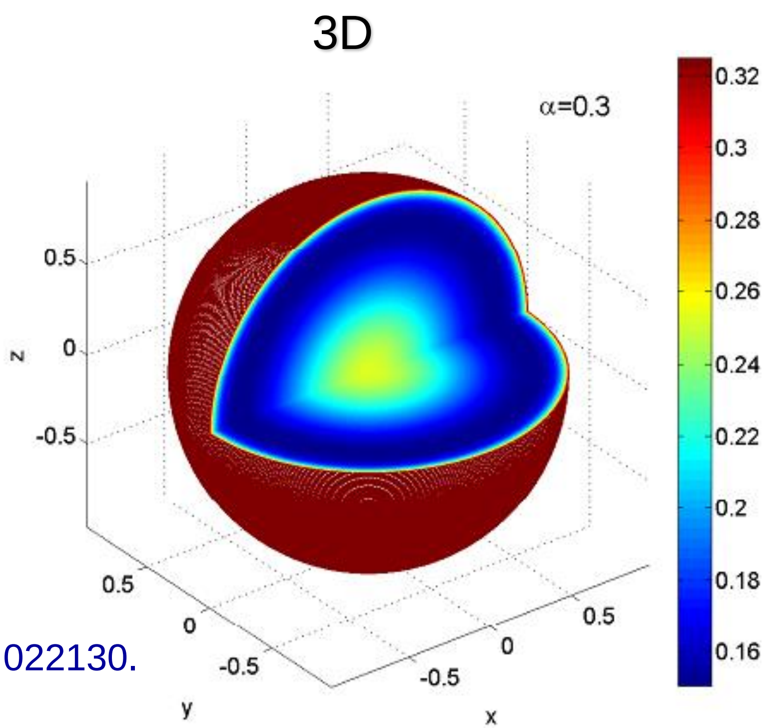
Uchaikin, V. V., & Sibatov, R. T. (2012). *Gravitation and Cosmology*, *18*(2), 122-126.

Uchaikin, V. V., & Sibatov, R. T. (2011). *J Phys A: Math & Theor*, *44*, 145501.

NoRD-propagator

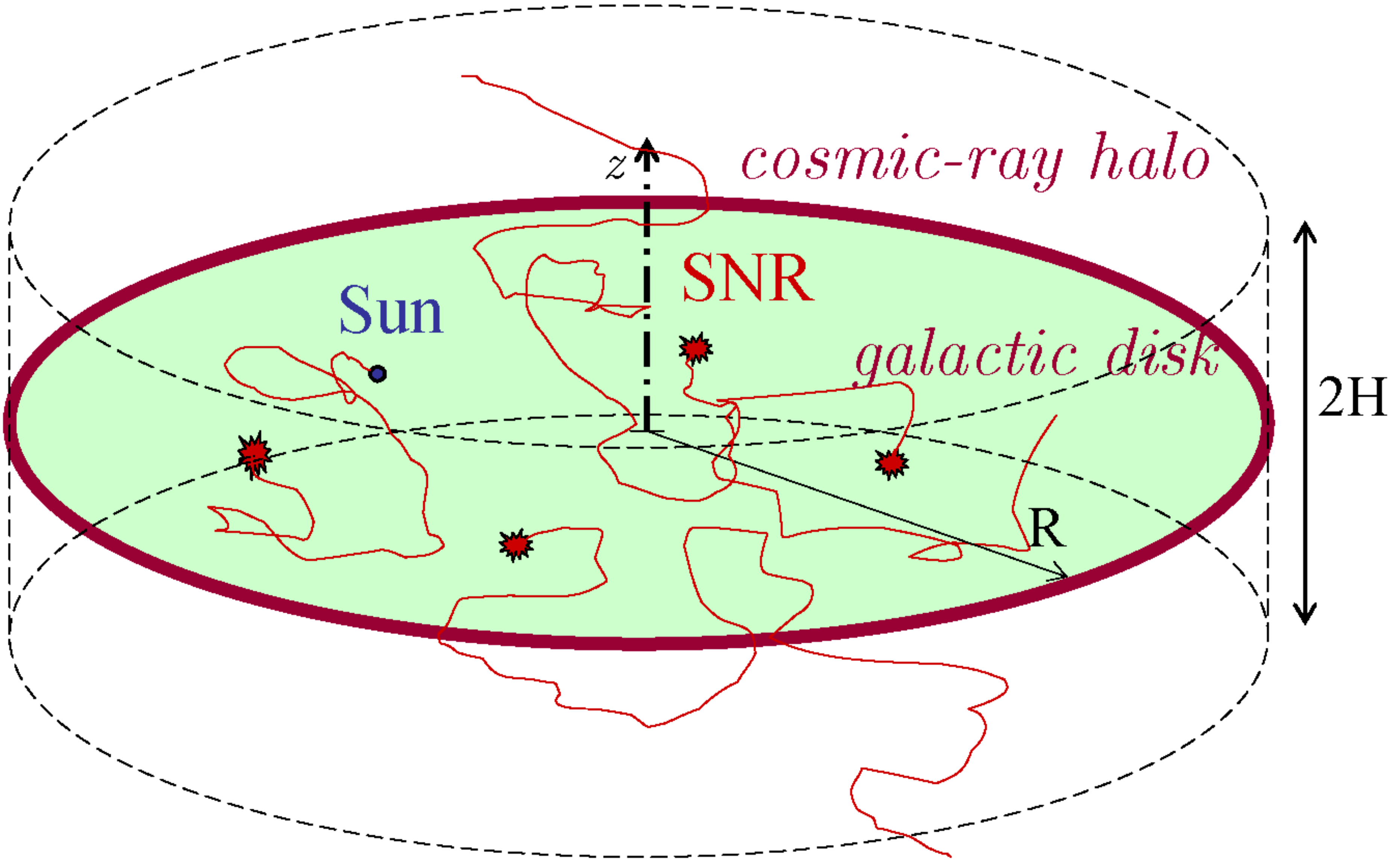
Uchaikin, V. V., & Zolotarev, V. M. (1999). *Chance and Stability.*



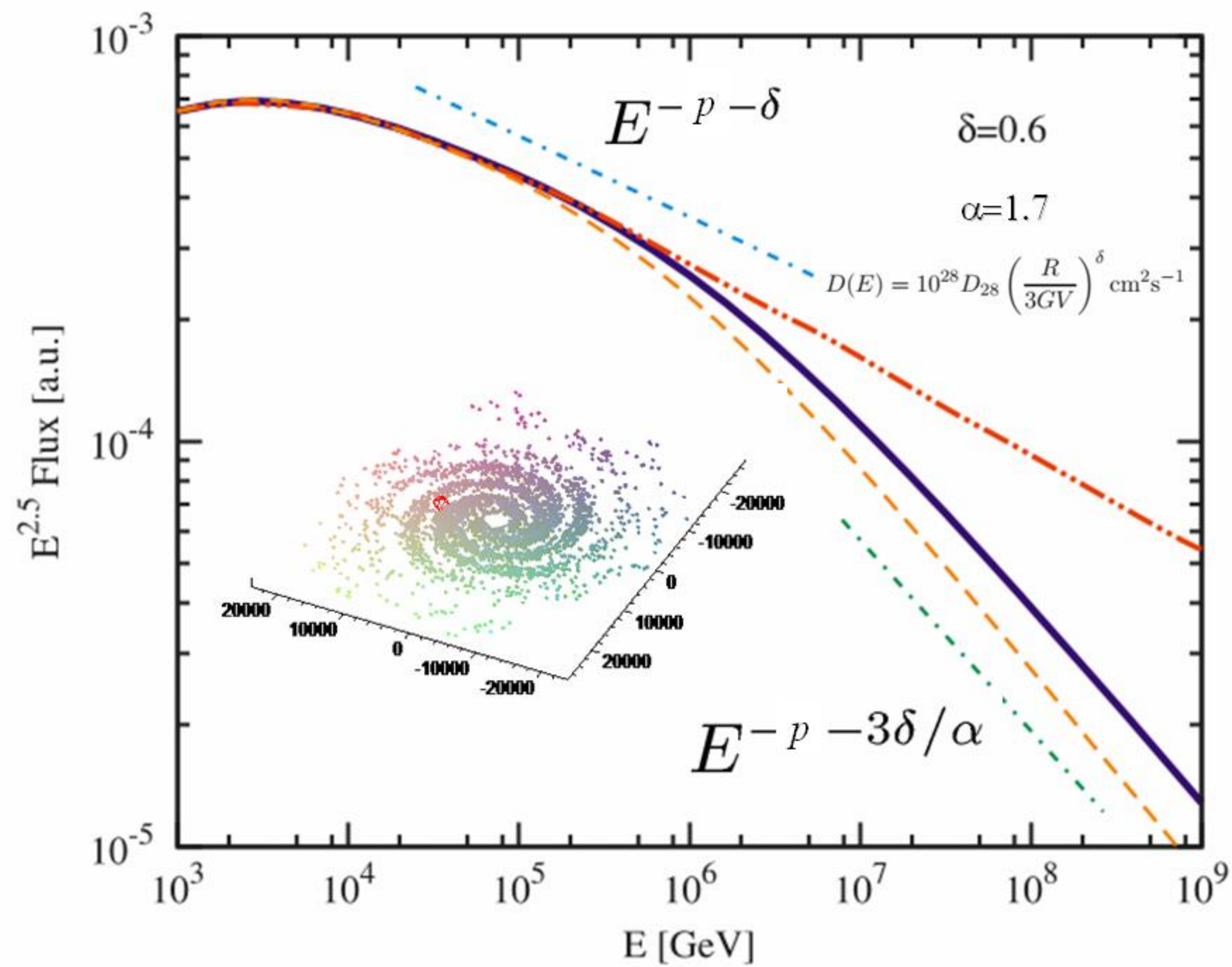


Magdziarz, M., & Zorawik, T. (2016). *Physical Review E*, *94*(2), 022130.

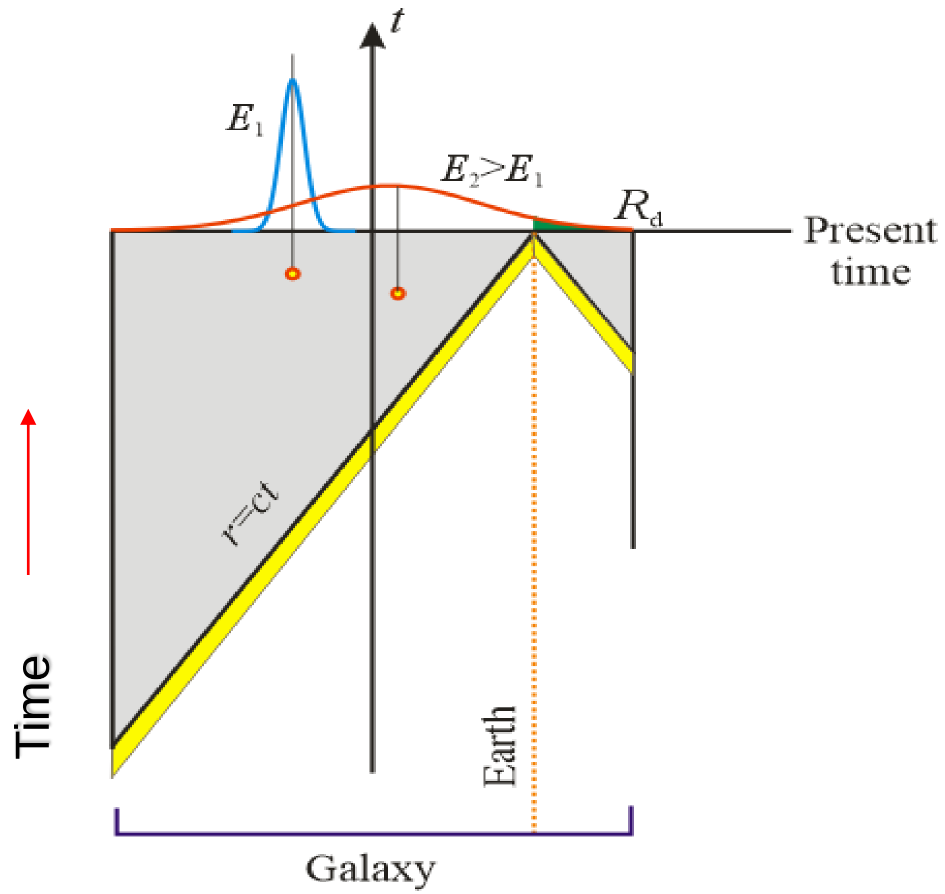
Source distribution and boundary conditions

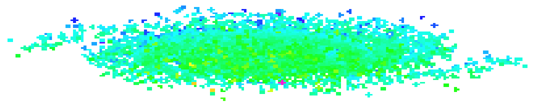


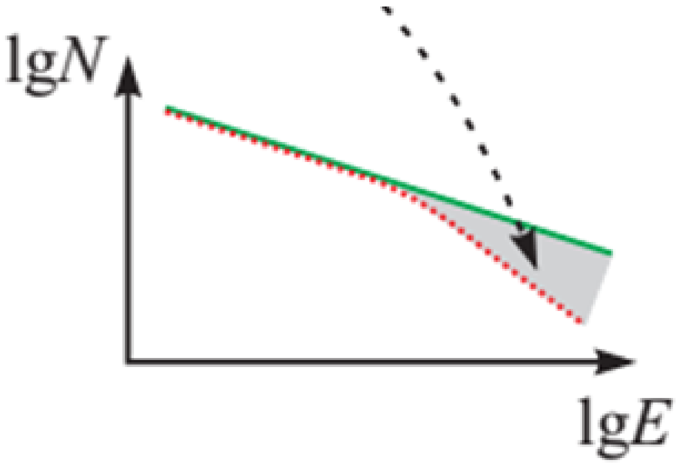
Equilibrium spectrum in the NoRD-model



Explanation of the relativistic steepening



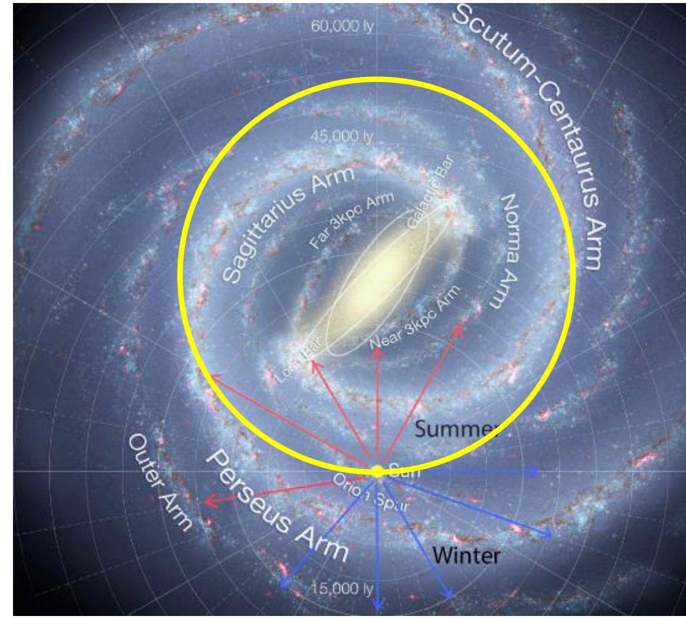




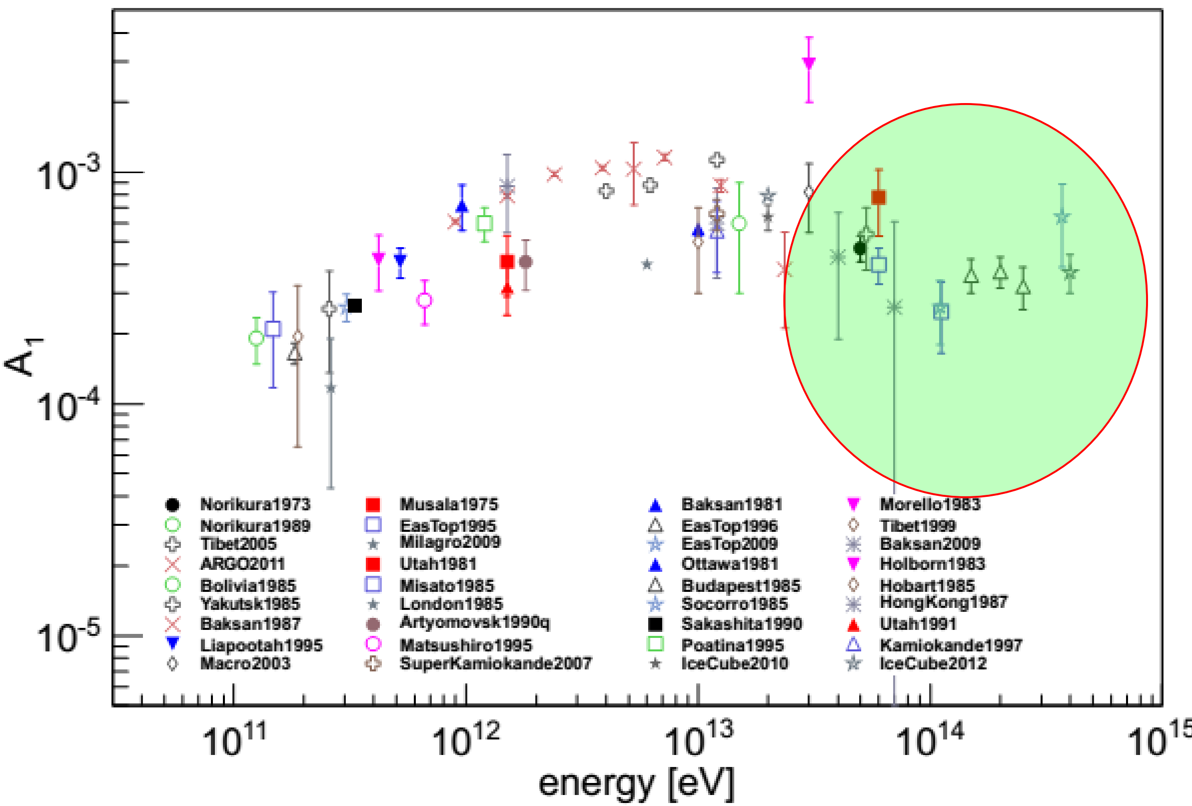
Eliminating of sources out of the ballistic cone affects predominantly on high energy part of CRs

The anisotropy problem

A well known problem is that, although most of the likely sources are in the Inner Galaxy, the direction from which the lowest energy particles (less than about 1 PeV) come is largely from the Outer Galaxy.

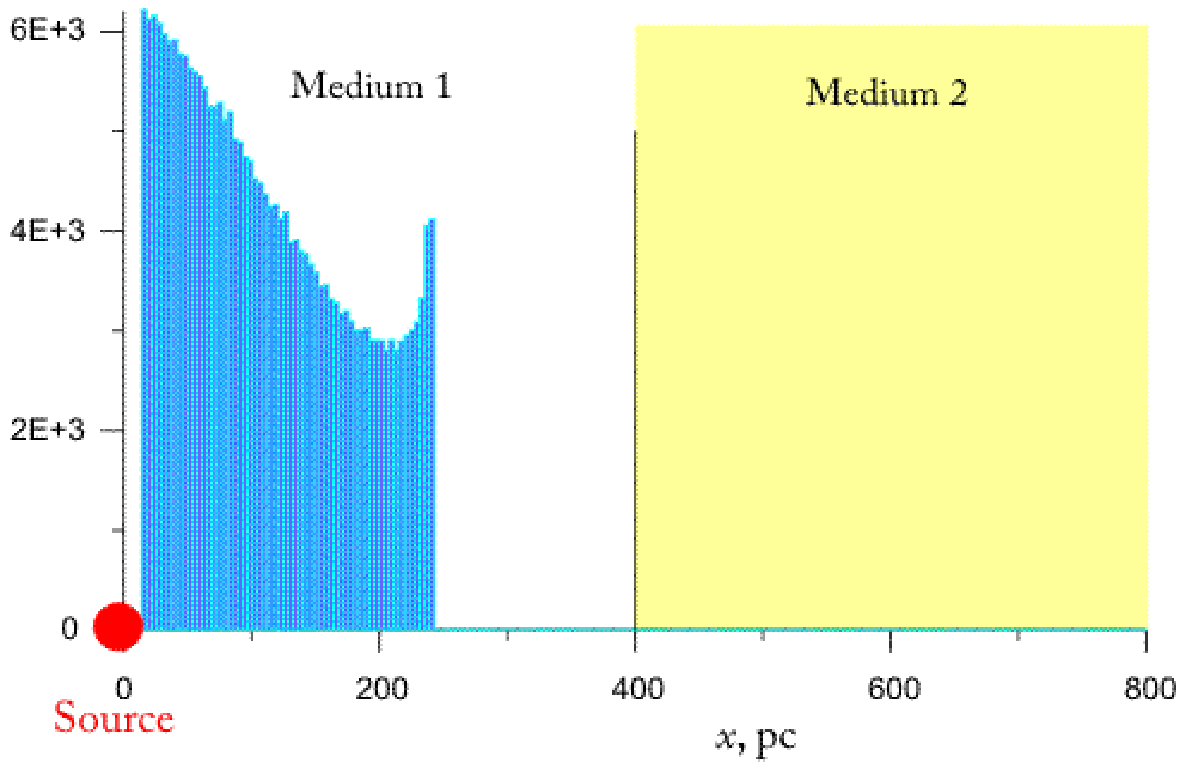


Can particles originated from a source in a particular direction be observed at Earth as coming from the opposite direction?



$$\delta = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

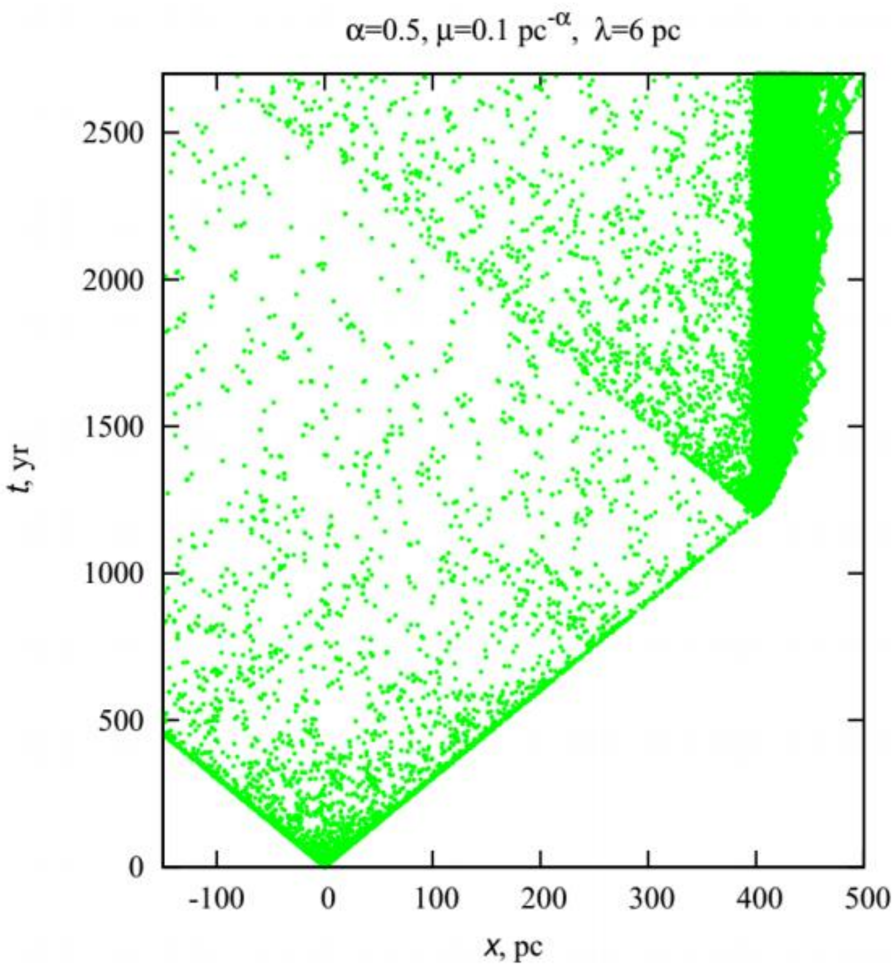
NoRD in a two-layer medium

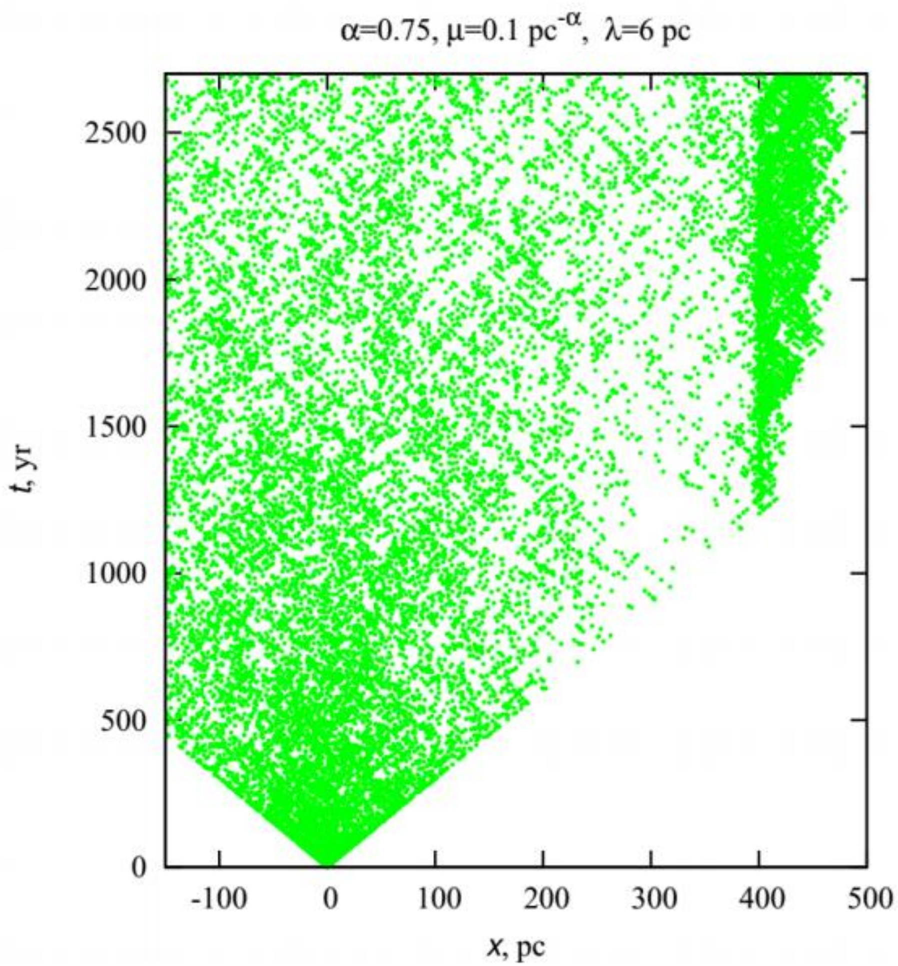


Can particles originated from a source in a particular direction be observed at Earth as coming from the opposite direction?

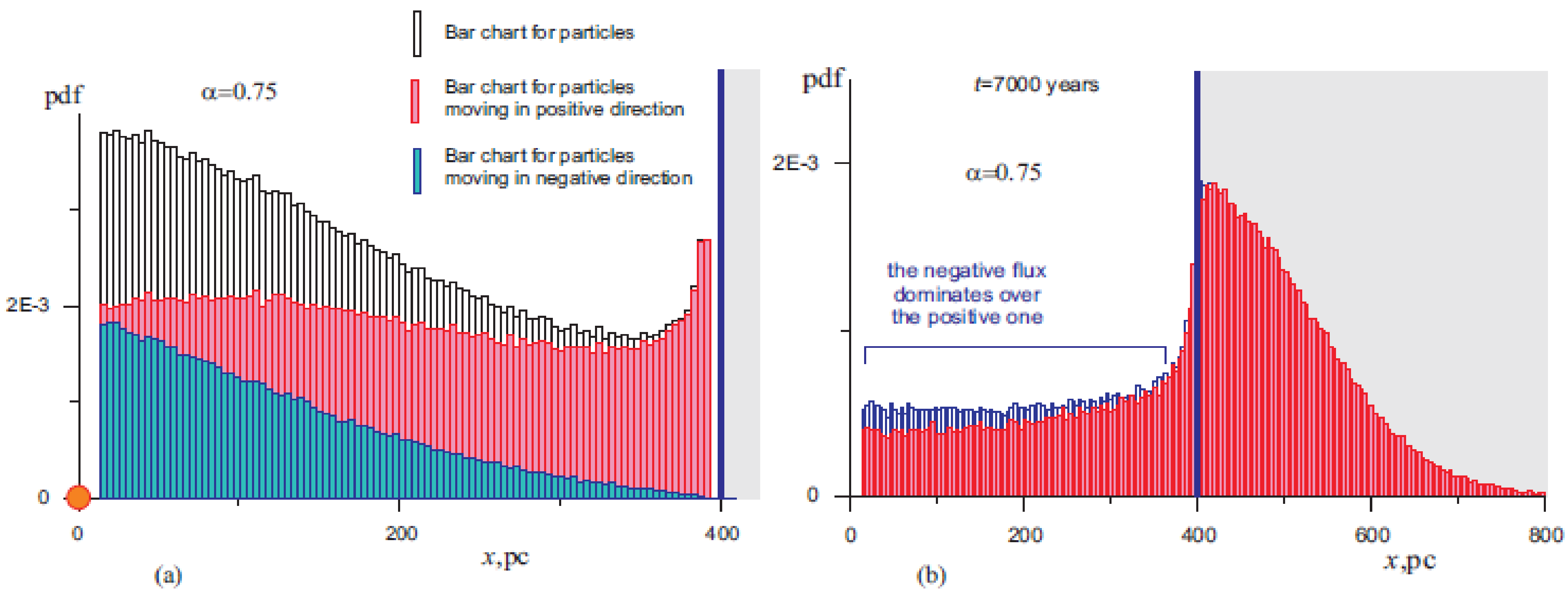
Erlykin AD,
Sibatov RT, Uchaikin VV,
Wolfendale AW, ICRC, 2015

Stochastic reflection of the front





Anisotropy Inversion



Erlykin AD,
Sibatov RT, Uchaikin VV,
Wolfendale AW, ICRC, 2015

‘Negative’ anisotropy

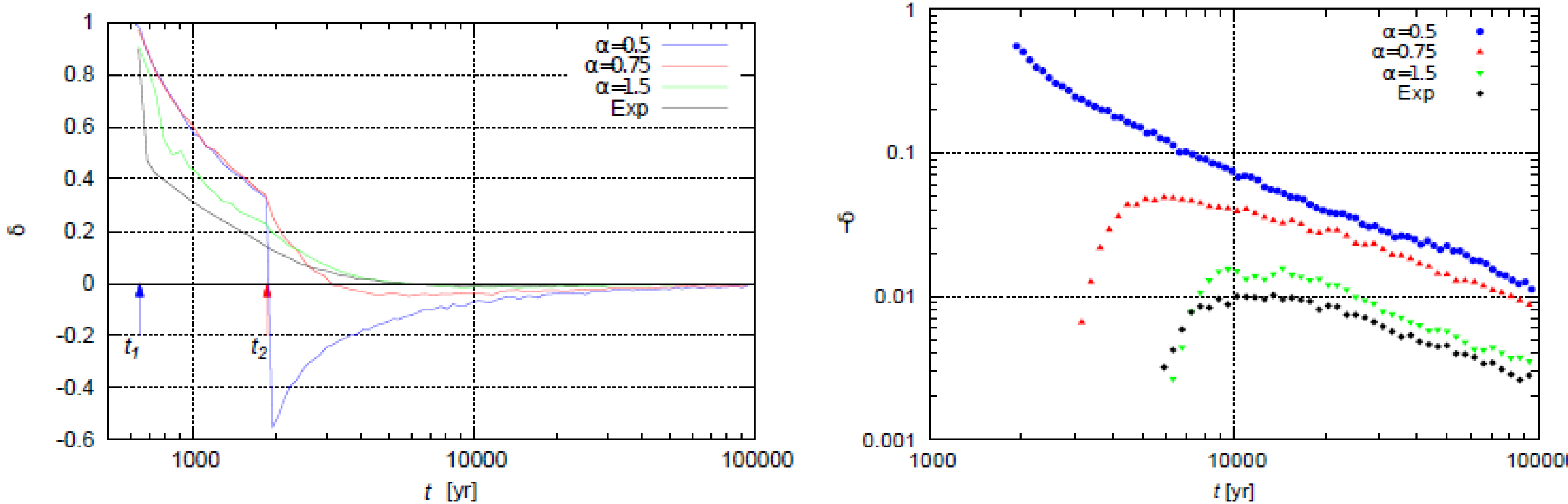
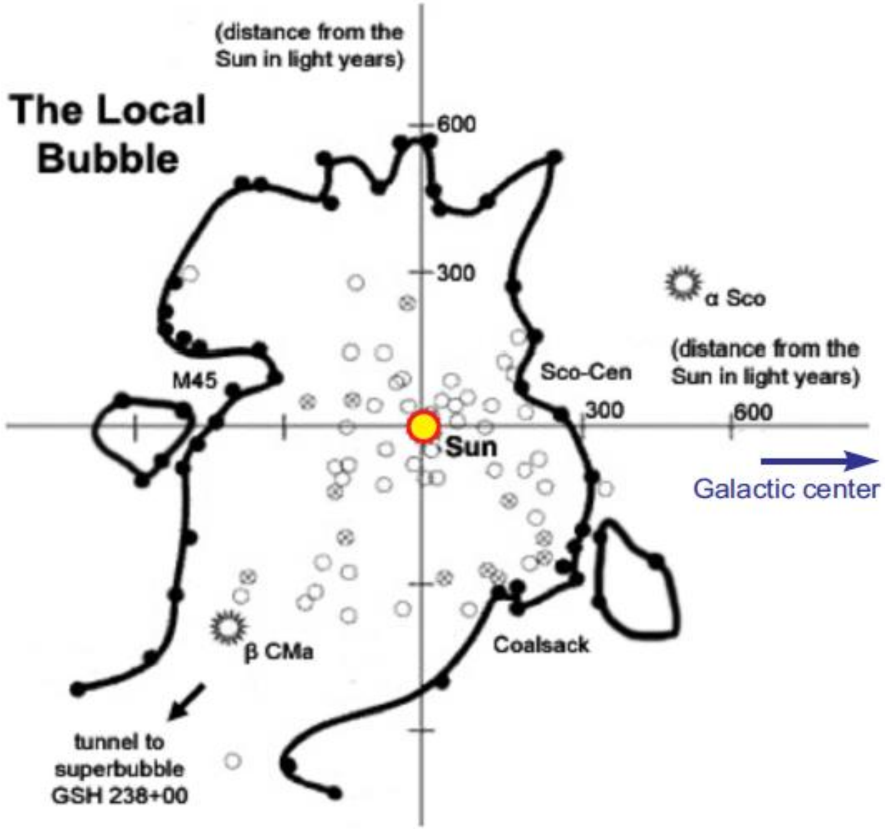


Figure 4: Left panel: The family of anisotropy time-dependence $\delta(t)$ for various values of α. Right panel: Function $-\delta(t)$ in the log-log scale. The observation point is to the left of the boundary (that is, in 1-medium). Distances: source-observer $R = 200$ pc and source-boundary $a = 400$ pc. Random free paths are distributed according to a power law with exponents α in 1-medium and β in 2-medium ($\beta = 1.9$; mean path in 2-medium $\lambda_2 = 0.3$ pc). First-particles front reaches the detector at the time point $t_1 = R/v$ than it reaches boundary, dives into 2-medium, diffuses there, partially returns through the boundary and occurs the observer point after $t_2 = [a + (a - R)]/v$. Label ‘Exp’ corresponds to the case of exponential distribution of free path lengths.

Local Bubble



1. This phenomenon usually does not take place in the case of stationary (time-independent) transport.

2. The ordinary diffusion theory cannot catch this phenomenon because it is not in a position to describe the front splash: a diffusion packet is instantaneously spread around all space, breaking the relativistic principle.

3. The more advanced transport theory reveals the reality of such anisotropy inversion mechanism: the most auspicious conditions for the phenomenon appear after the front splash passed through the boundary of two domains in the direction of the more dense one.

Effects provided by the NoRD-model

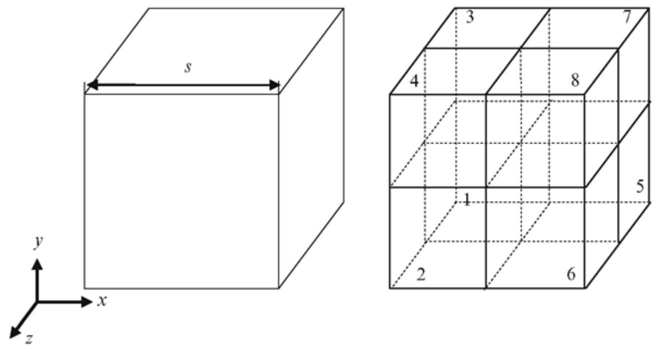
1. Steepening of equilibrium spectra due to the relativistic principle of the speed limit
2. Hardening and softening of single source spectra in different parts of the NoRD-propagator
3. Anisotropy inversion as a stochastic reflection in heterogeneous medium
4. Break in the energy dependence of the escape time (statistical interpretation of the Galactic modulation)

Simulation of charged particle transport in turbulent MF

Pucci, F. et al. (2016). *MNRAS*, *459*(3), 3395-3406.

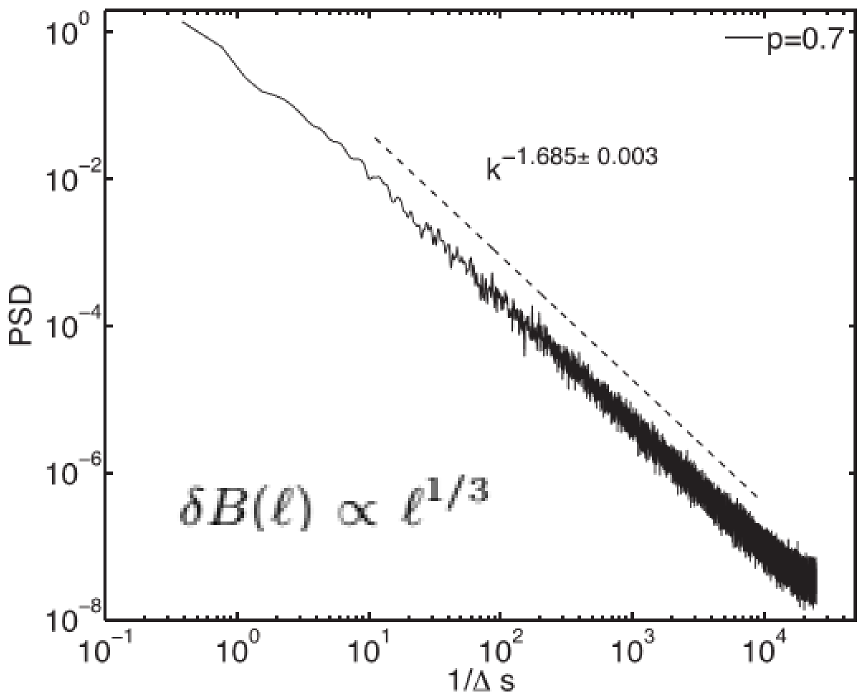
Turbulent magnetic field

$$\boldsymbol{B}(x, y, z) = \boldsymbol{B}_0 + \sum_{m=m_{\min}}^{N_s} \sum_{i,j,k=1}^{2^m} \delta \boldsymbol{B}^{(i,j,k;m)}(x, y, z)$$



$$\delta \mathbf{B}^{(i,j,k;m)} = a^{(i,j,k;m)} \nabla \times \mathbf{A}^{(i,j,k;m)}$$

$$a^{(i,j,k;m)} = \sigma^{(i,j,k;m)} a_0 \left[\frac{\epsilon_m^{(i,j,k;m)}}{\epsilon_0} \frac{\ell_m}{\ell_0} \right]^{1/3}$$



The model builds up a turbulent magnetic field as a superposition of space-localized fluctuations at different spatial scales. The resulting spectrum is isotropic with an adjustable spectral index. The model allows them to reproduce a spectrum broader than four decades, and to regulate the level of intermittency through a technique based on the p-model.

Simulation of charged particle transport in turbulent MF

Pucci, F. et al. (2016). *MNRAS*, *459*(3), 3395-3406.

$$B_0 = 10^{-4}G$$

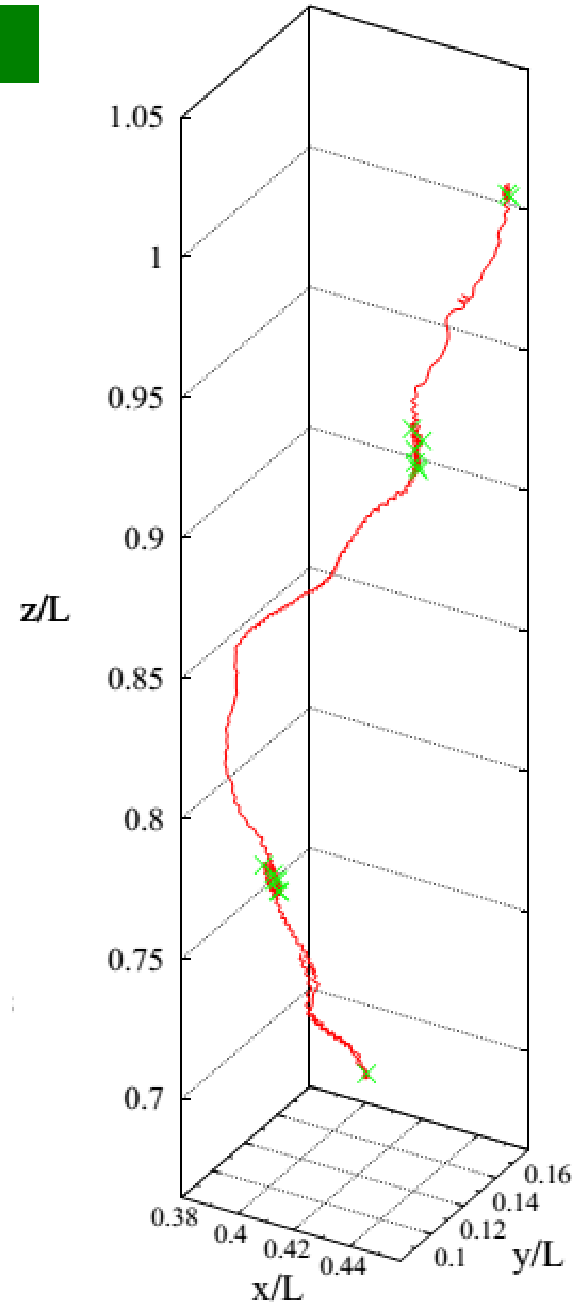
Equations of motion

$$\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{\Delta t} = \boldsymbol{v}_{i+1}$$

$$\frac{\boldsymbol{v}_{i+1} - \boldsymbol{v}_i}{\Delta t} = \frac{q}{m}\left[\boldsymbol{E}_i + \frac{(\boldsymbol{v}_{i+1} + \boldsymbol{v}_i) \times \boldsymbol{B}_i}{2c}\right]$$

The guiding center motion

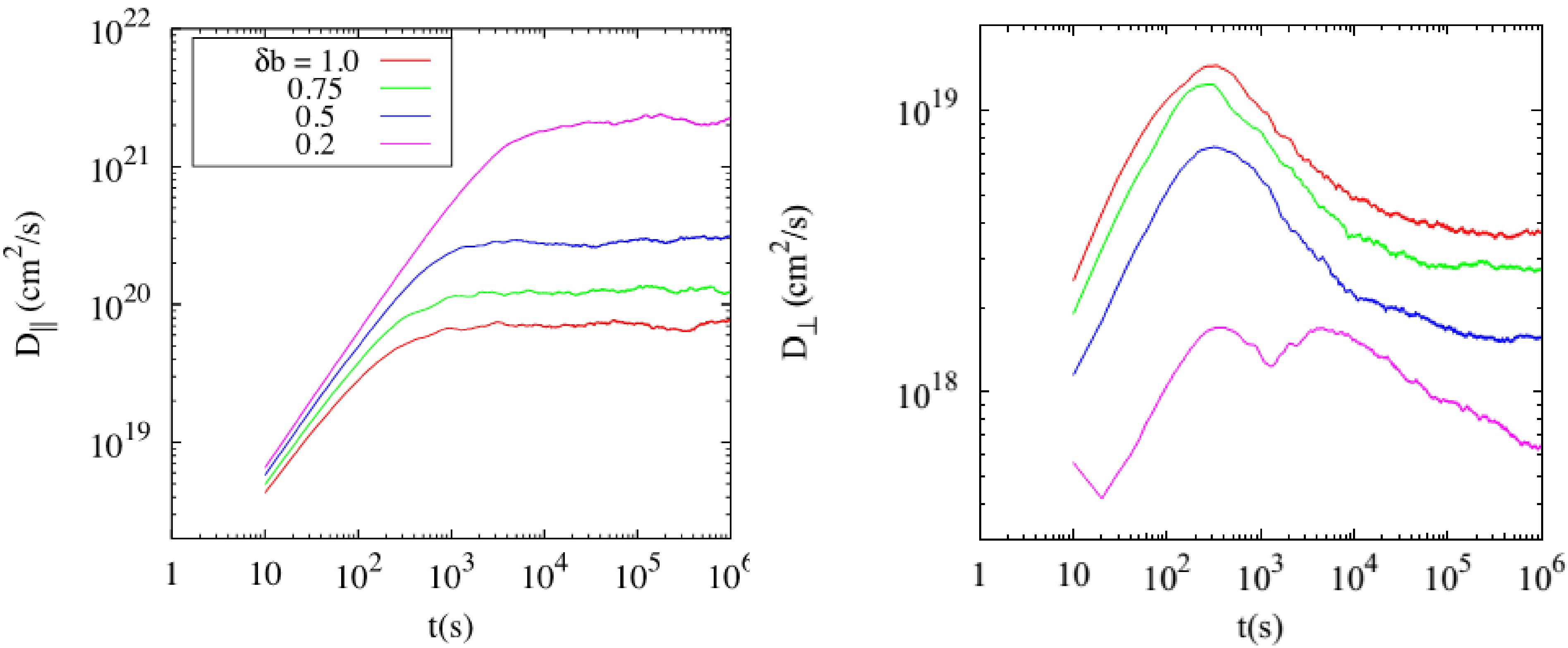
Resonant interactions and magnetic mirroring



Parallel and perpendicular diffusion coefficients

Pucci, F. et al. (2016). *MNRAS*, *459*(3), 3395-3406.

$$D_{\parallel}(t) = \frac{1}{2N_{\mathrm{p}}t}\sum_{i=1}^{N_{\mathrm{p}}}[z_i(t) - z_i(0)]^2 \qquad D_{\perp}(t) = \frac{1}{2N_{\mathrm{p}}t}\sum_{i=1}^{N_{\mathrm{p}}}\left\{[x_i(t) - x_i(0)]^2 + [y_i(t) - y_i(0)]^2\right\}$$



Transient superdiffusion

Baeumer, B., & Meerschaert, M. M. (2010). *Journal of Computational and Applied Mathematics*, *233*(10), 2438.

Inverse problem

Second moment of the random walk with finite velocity

$$m_2(t) = 2\int_0^{vt} xP(x)dx + \int_0^{vt} m_2(t - x/v)p(x)dx$$

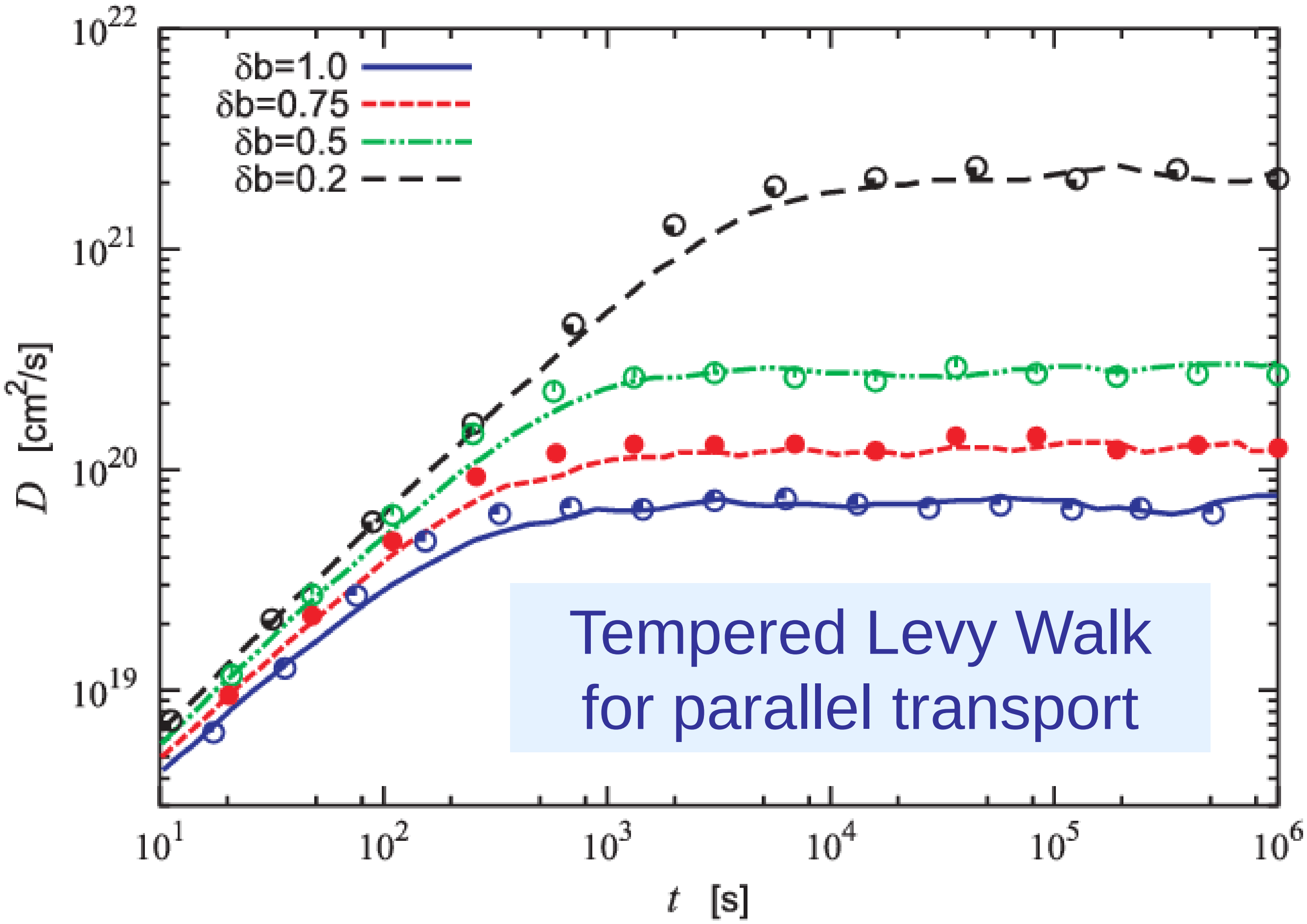
$$D_{\|}(t) = \frac{m_2(t)}{2t} \sim \begin{cases} v^2 t(1-\alpha)/2, & vt \ll \gamma^{-1}; \\ v(1-\alpha)(2\gamma)^{-1}, & vt \gg \gamma^{-1}. \end{cases} \qquad 0 < \alpha < 1$$

$$D_{\|}(t) \sim \begin{cases} (\alpha-1)v(c_\alpha/c_1)(vt)^{2-\alpha}/\Gamma(4-\alpha), & vt \ll \gamma^{-1}; \\ \alpha(\alpha-1)v(c_\alpha/2l)\gamma^{\alpha-2}, & vt \gg \gamma^{-1}. \end{cases} \qquad 1 < \alpha < 2.$$

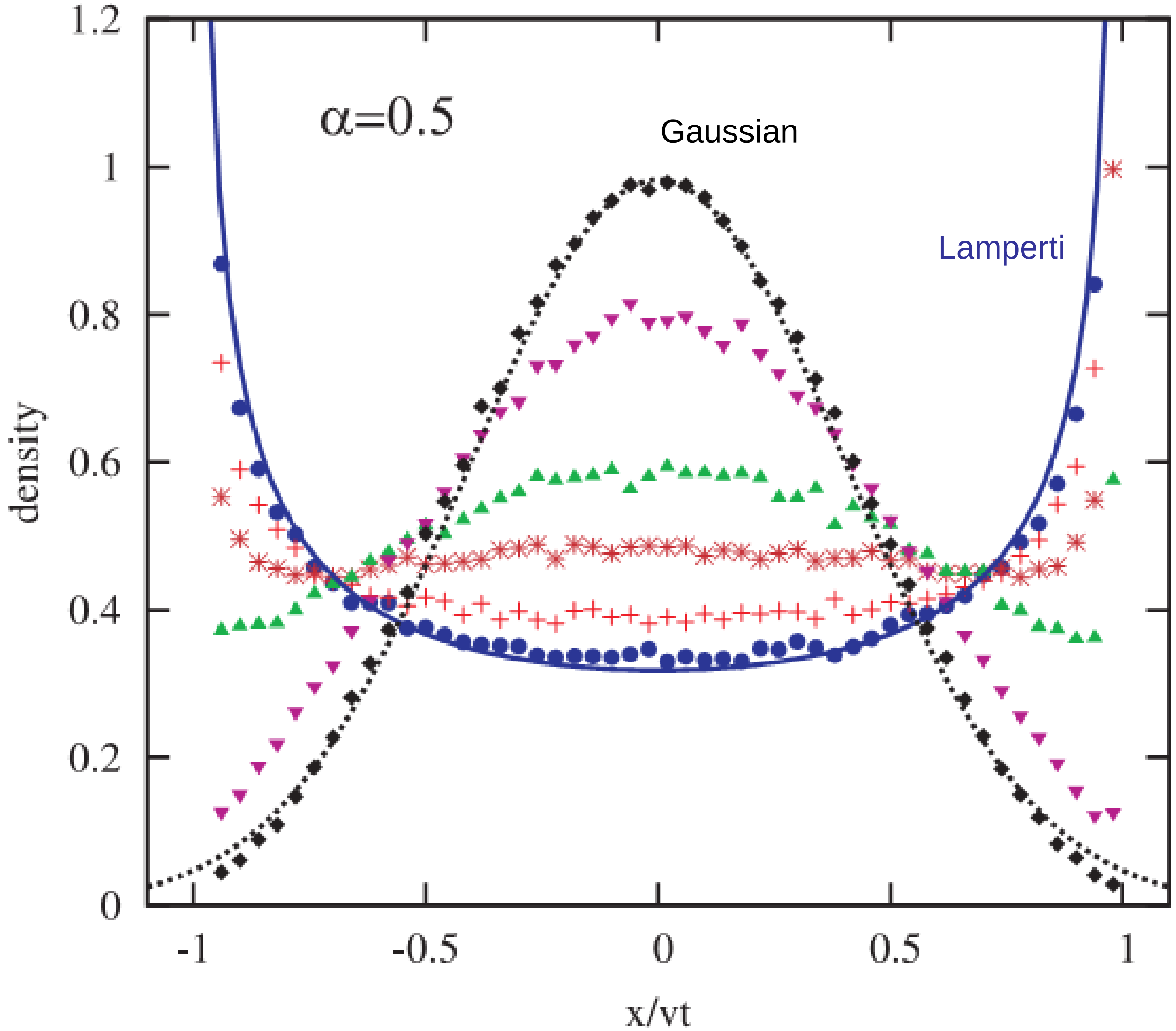
Table 1. *Parameters of tempered superdiffusion dependent on turbulence level* $\delta b = \delta B/B_0$ *for 1 MeV proton,* $p = 0.5$, $\xi = 1024$, $v = 1.41 \cdot 10^9$ *cm/s.*

δb	0.2	0.5	0.75	1.0
α	0.37	0.44	0.55	0.63
γ^{-1}, 10^{11} cm	48	7.1	3.8	2.8

Comparison of simulations

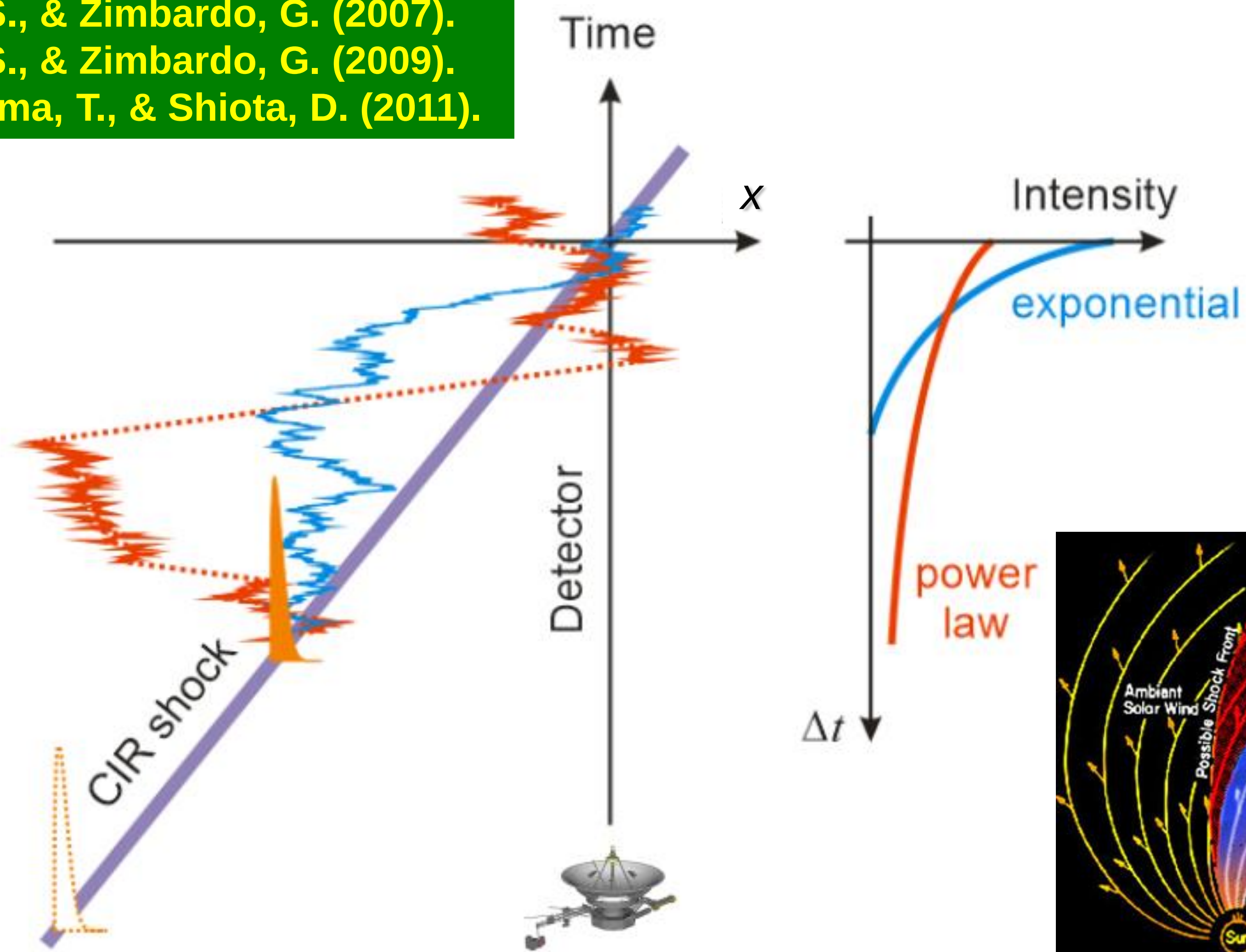


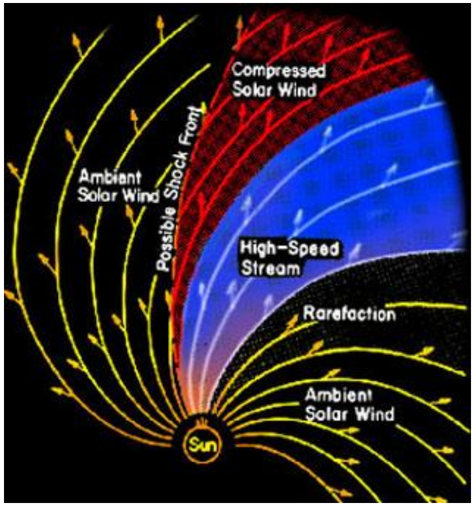
Density



Protons and electrons accelerated by corotating interaction region

1. **Perri, S., & Zimbardo, G. (2007).**
2. **Perri, S., & Zimbardo, G. (2009).**
3. **Sugiyama, T., & Shiota, D. (2011).**





$$f(x,E,t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' G(x-x', t-t') f_{sh}(x', E, t')$$

Superdiffusion from analysis of energetic particle profiles measured by spacecraft

$$f(x,E,t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' G(x-x', t-t') f_{sh}(x',E,t')$$

$$f_{sh}(x,E,t) = f_0(E)\delta(x - V_{sh}t)$$

Normal diffusion – exponential decay of intensity

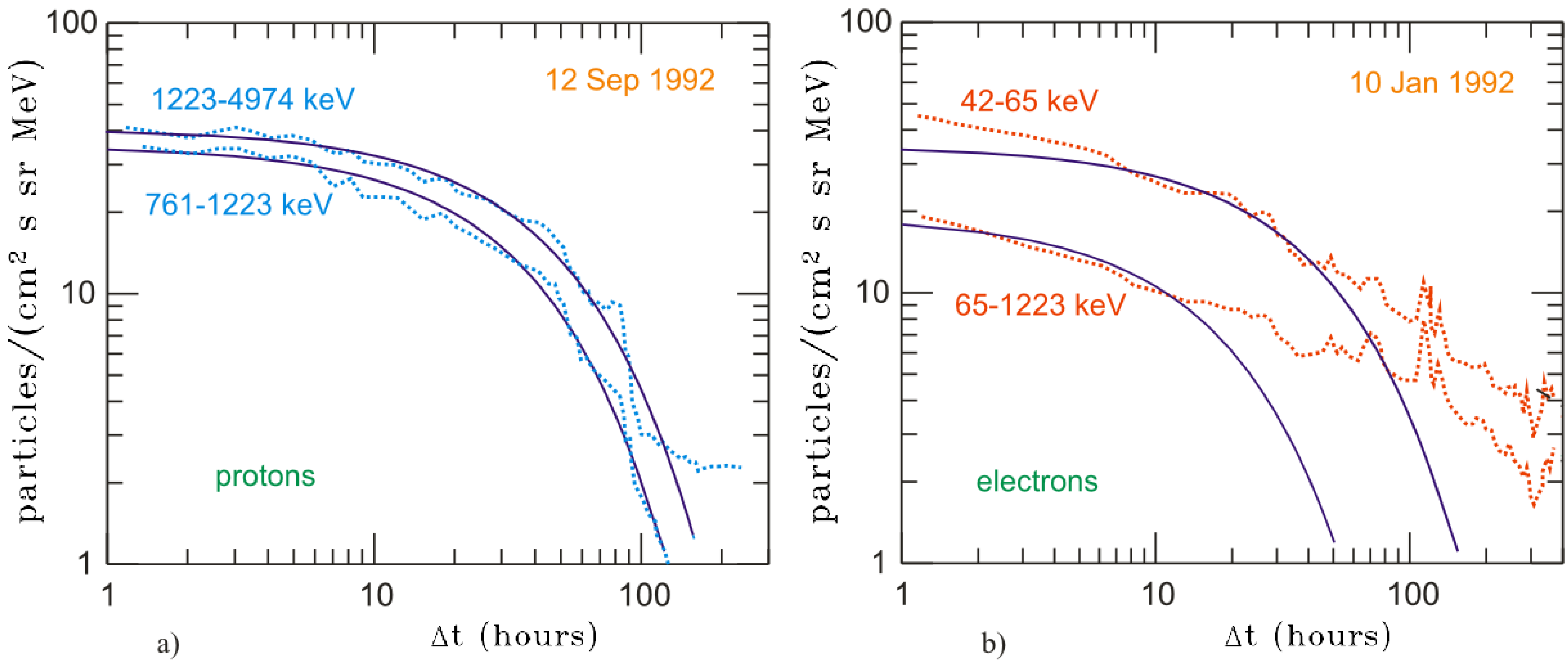
$$G(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right) \qquad f(0,E,t) = \frac{f_0(E)}{V_{sh}} \exp\left(-\frac{V_{sh}^2|t|}{D}\right)$$

Anomalous diffusion – power law decay of intensity

$$1 < \alpha < 2$$

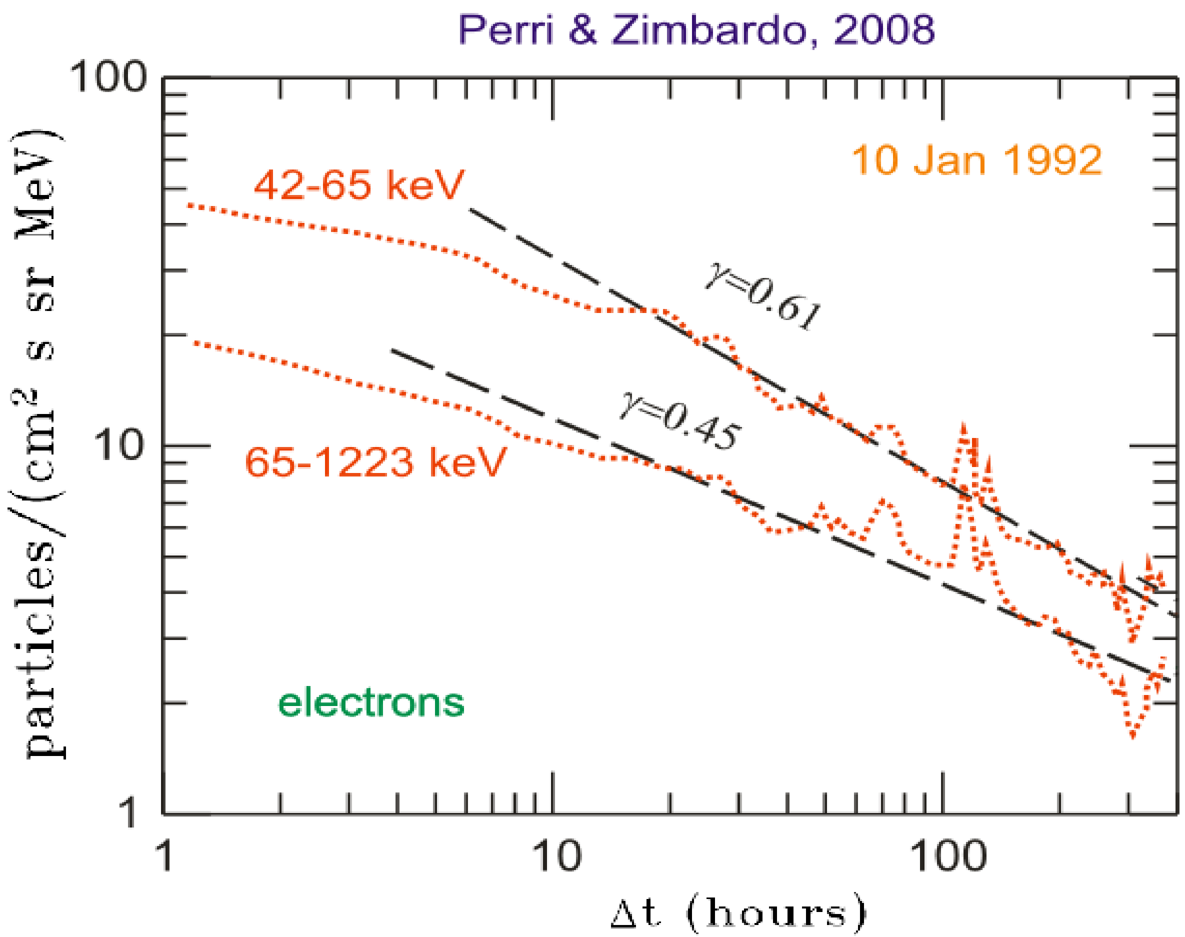
$$f(0,E,t) = \int G(0-x', t-t') f_{sh}(x',E,t') dx' dt' \propto (\Delta t)^{1-\alpha}$$

Superdiffusion from analysis of energetic particle profiles measured by spacecraft



Electron transport is superdiffusive.
Proton transport is normal diffusive.
(Perri and Zimbardo, Adv. Spa. Res. 2009)

Superdiffusion of electrons



$\Delta t = |t - t_{sh}|$

Power law $J = A(\Delta t)^{-\gamma}$

Exponential

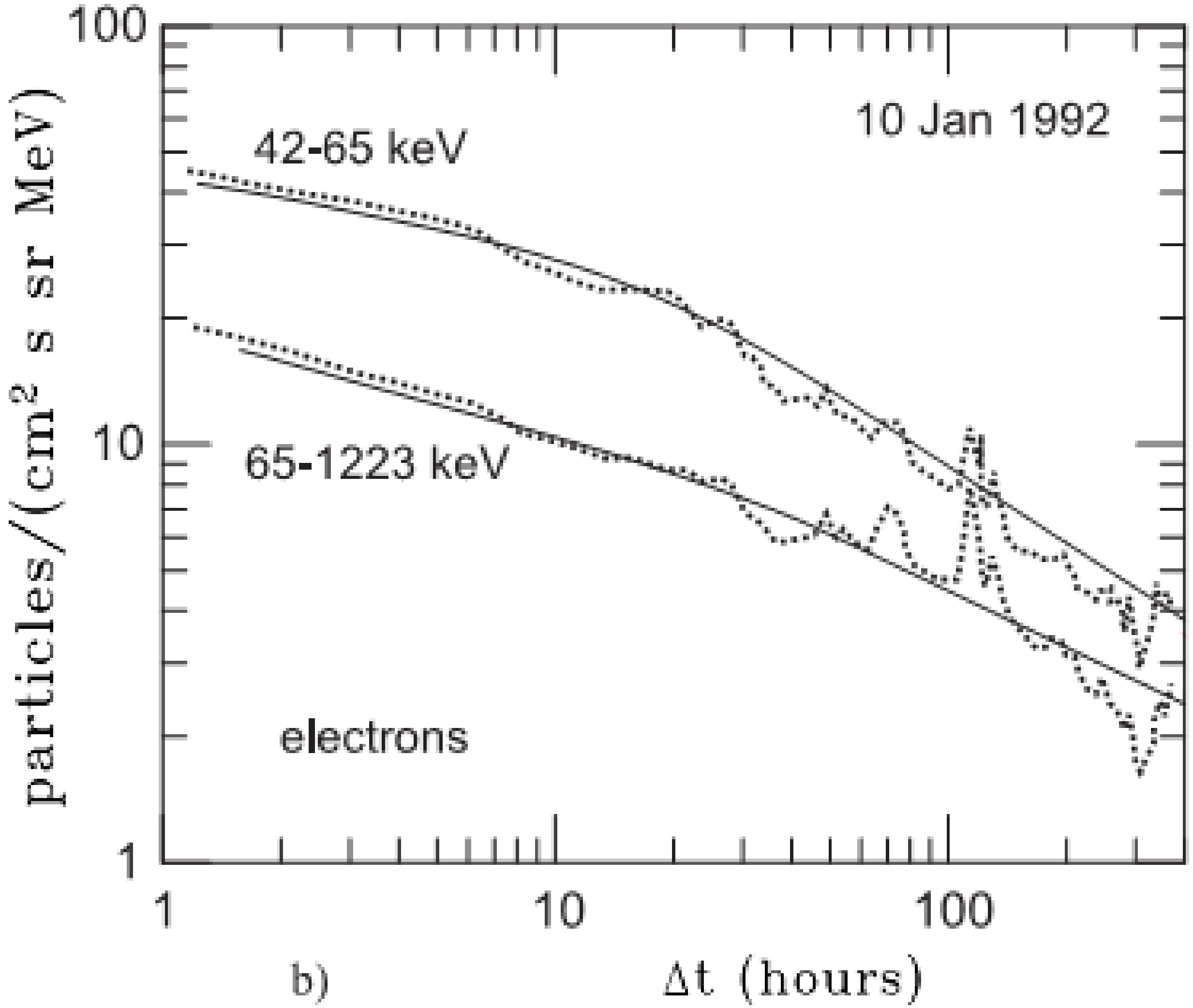
$J = K \exp(-G\Delta t)$

Statistical parameters for the fits of the electron time profiles.

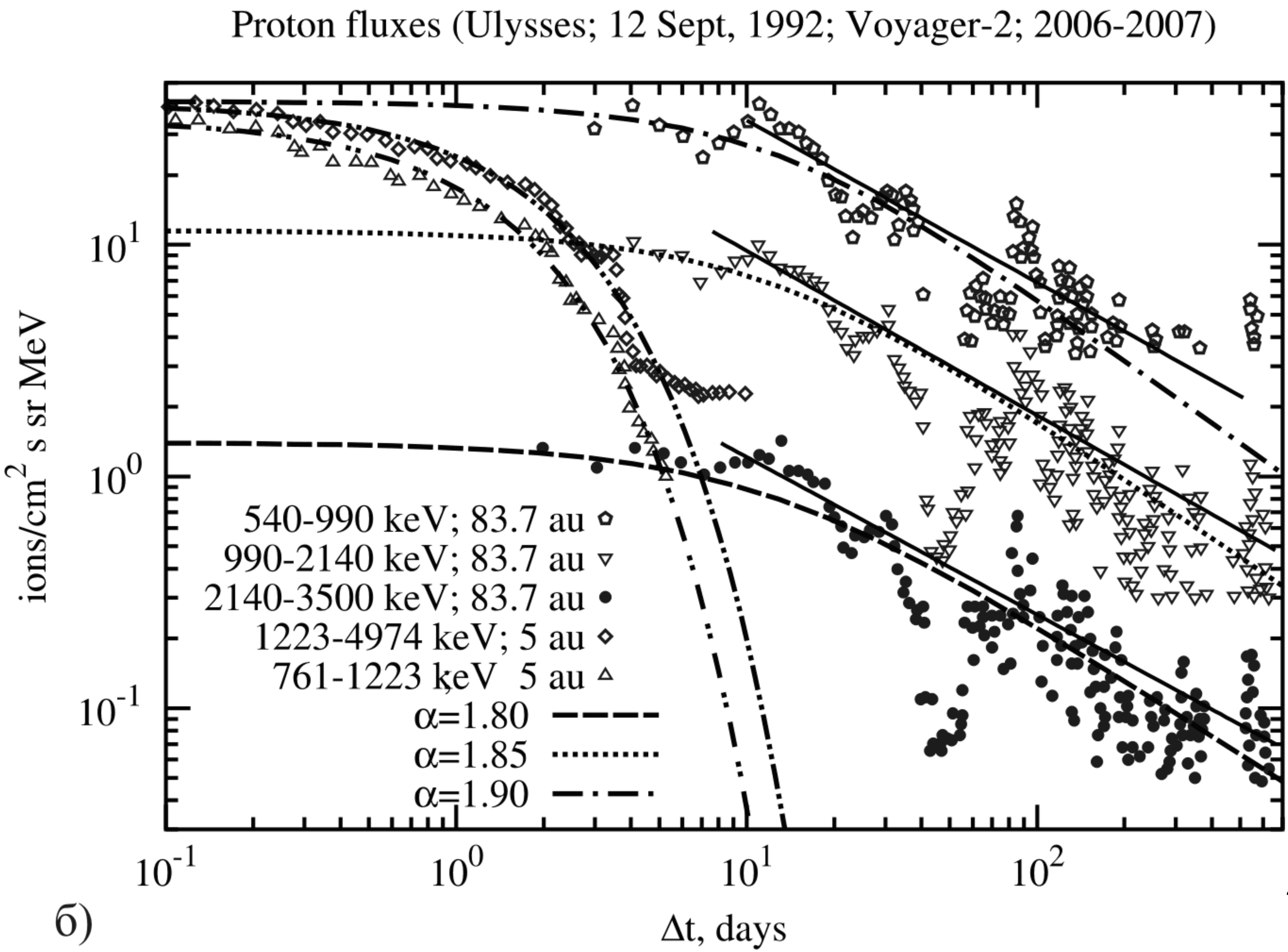
S/C	DD/MM/YYYY	Energy (keV)	γ	α
Ulysses	11/10/1991	42–65	0.56 ± 0.08	1.44
		65–112	0.44 ± 0.09	1.56
		112–178	0.3 ± 0.1	1.70
		178–290	0.4 ± 0.2	1.60

Specification of the Green function

Uchaikin, V., Sibatov, R., & Byzykchi, A. (2014). *Commun in Appl and Ind Math*, *6*(1).

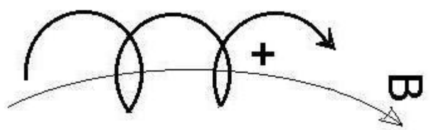


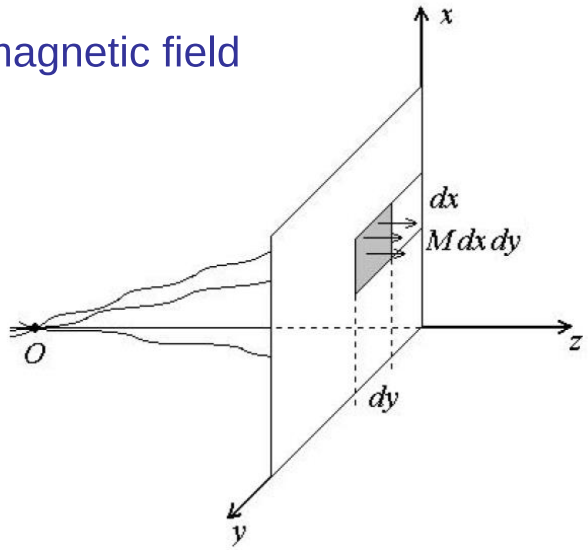
Normal and anomalous diffusion of protons



Perpendicular subdiffusion as a result of compound diffusion

Dominant background magnetic field





Total magnetic field is a superposition of the background field and a turbulent component:

$$B = B_0 + \delta B, \qquad \langle B \rangle = B_0, \quad \langle \delta B \rangle = 0$$

Getmantcev 1963,
Jokipii 1966, Jokipii & Parker 1969,
Zybin & Istomin,1985
Chuvilgin & Ptuskin, 1993

$$\frac{\partial N_\perp}{\partial t} = D_\perp \frac{\partial^{1/2}}{\partial t^{1/2}} \triangle_\perp N_\perp(\mathbf{r}_\perp, t) + N_\perp(\mathbf{r}_\perp, 0)\delta(t)$$

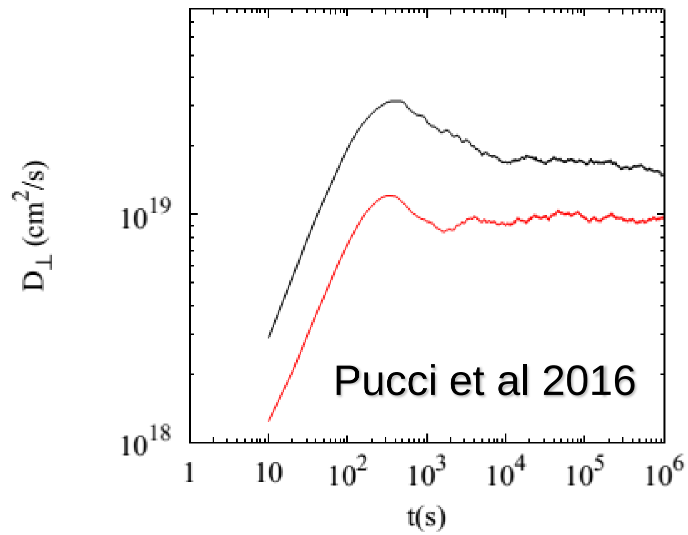
Magnetic field line random walk

Mean square displacement

$$\left\langle \left[X^H(z)\right]^2 \right\rangle = C^H(z,z) = \sigma^2 |z|^{2H}$$

Fractional Brownian motion

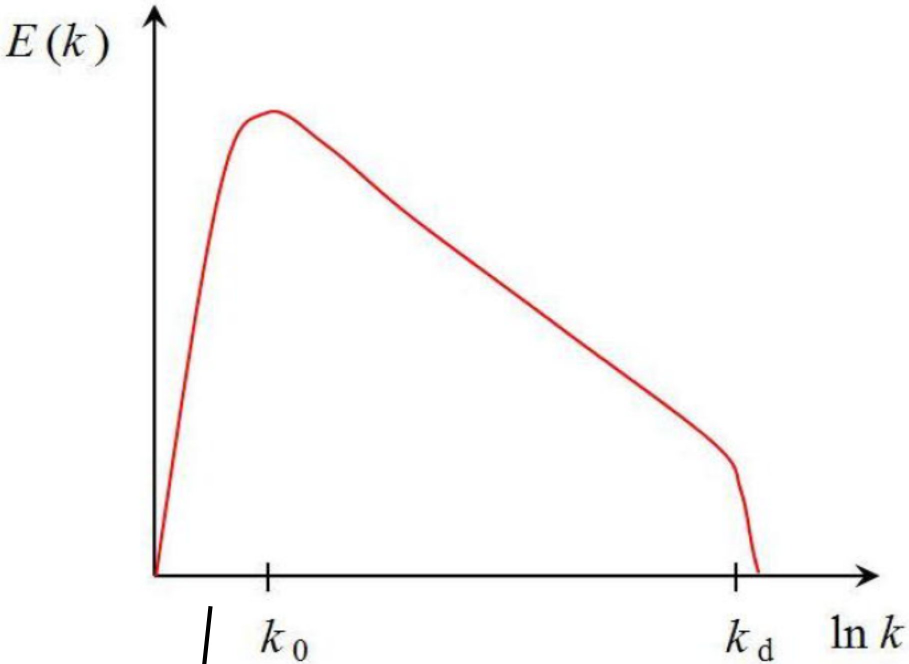
$$X^H(z) = X_+^H(z) + \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} \left[(z-z')^{H-1/2} - (-z)^{H-1/2}\right] W(z') dz'$$



Perpendicular diffusion coefficient

$$D_\perp(t) = \frac{1}{2t} \int_{-\infty}^{\infty} 2K_\perp |z|^{2H} f_\parallel(z,t) dz = \frac{K_\perp}{t} \langle |\zeta|^{2H} \rangle$$

From turbulence spectrum to tempered fractional model



k_0 = energy injection scale
(= integral scale ≈ correlation length),

k_d = dissipation scale,

$k_0 < k < k_d$, the inertial range
(where the flow is controlled by inertia forces, and where kinetic energy does not dissipate).

The Heisenberg-Kolmogorov-Weizsacker formalism leads to

$$\frac{\partial f}{\partial t} = \mathrm{R}_L^\alpha f(\mathbf{x}, t) + \delta(\mathbf{x})\delta(t)$$

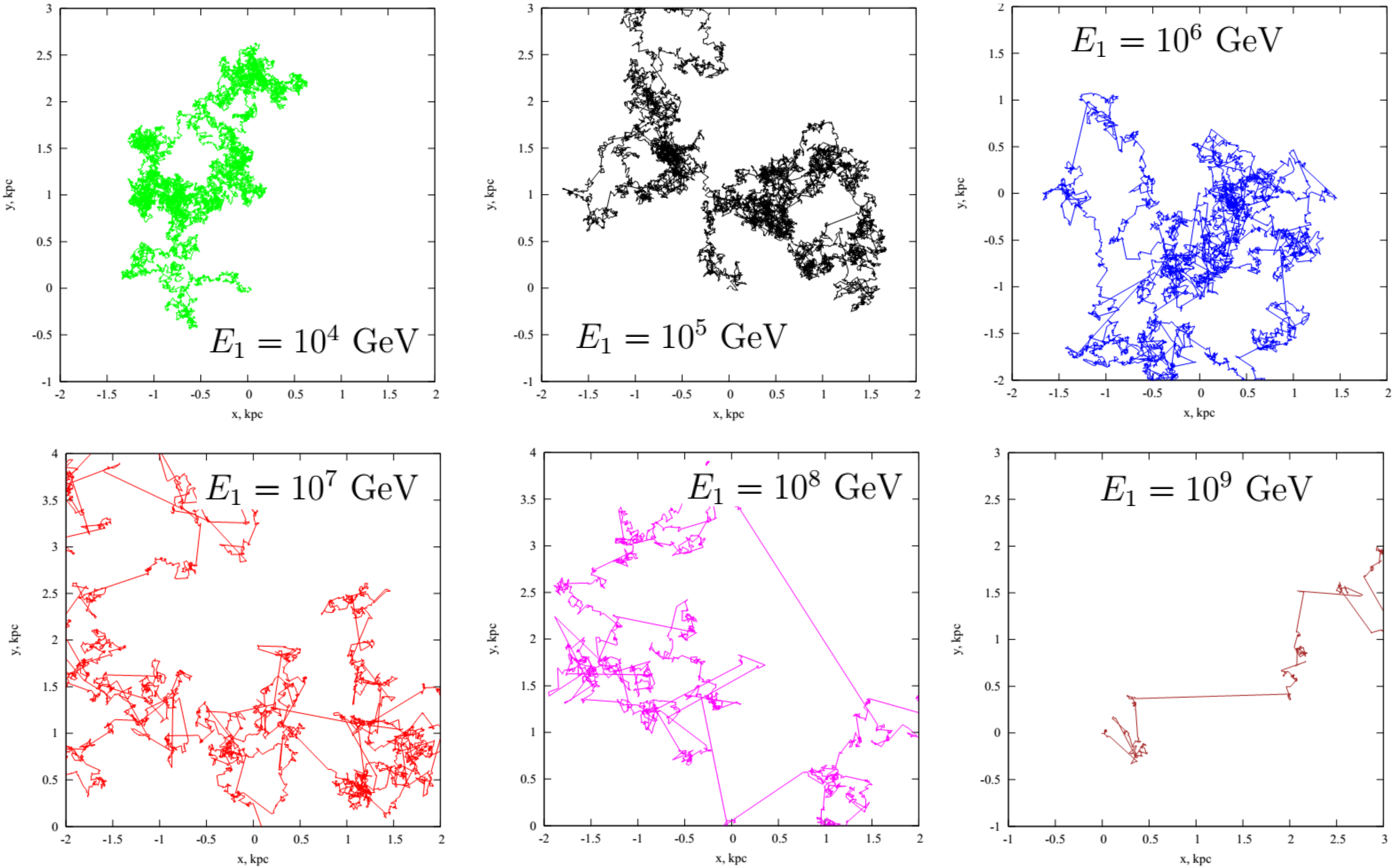
Tempered fractional Laplacian

Among mathematical models of the turbulent diffusion, the nonlocal model seems to be the most promising, especially in its fractional version.

$$\mathrm{R}_L^\alpha \sim \begin{cases} \Delta, & \text{on scales much larger than } L; \\ -(-\Delta)^{\alpha/2}, & \text{on scales much smaller than } L. \end{cases}$$

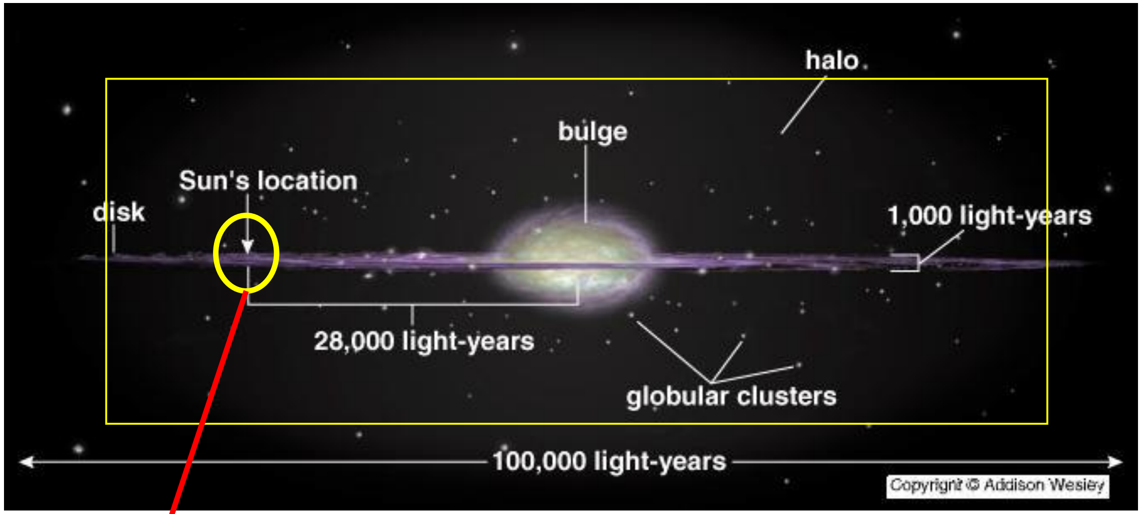
Cartea, Á., & del-Castillo-Negrete, D. (2007). *Physical Review E*,*76*(4), 041105.

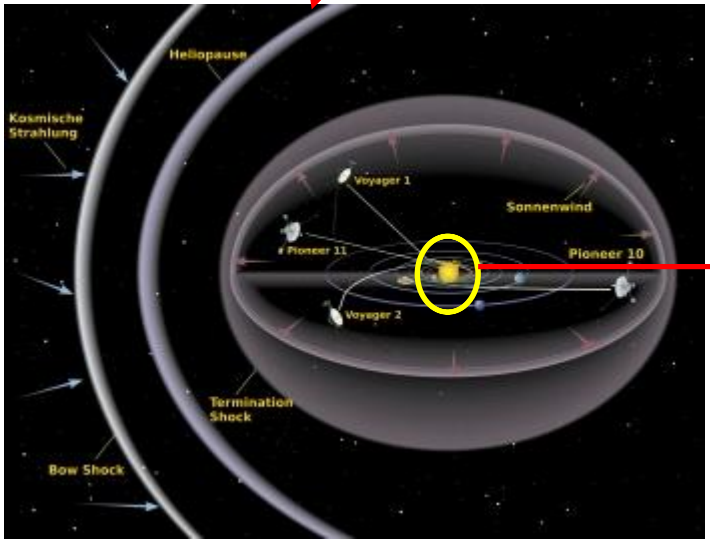
NoRD+ trajectories for different energies

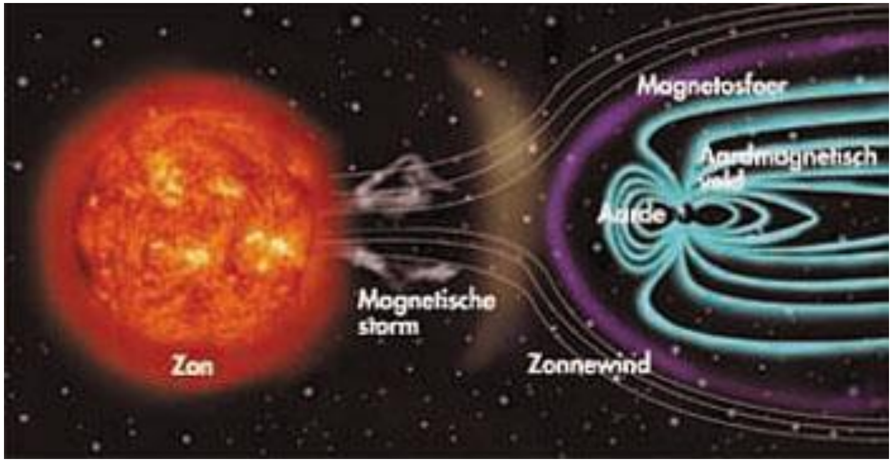


δ=0.3

$$p_R(r) \propto e^{-r/R_{\rm tr}(E)} r^{-1-\alpha} \qquad \alpha = 1.3 \qquad R_{\rm tr}(E) \propto E^{\delta}$$

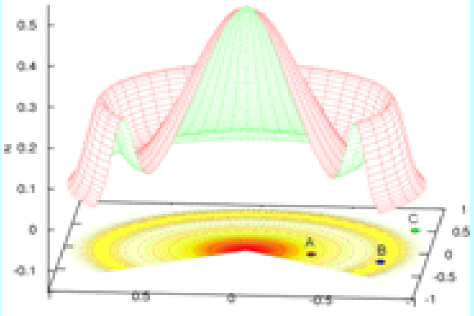

halo
bulge
Sun's location
disk
1,000 light-years
28,000 light-years
globular clusters
100,000 light-years
Copyright © Addison Wesley

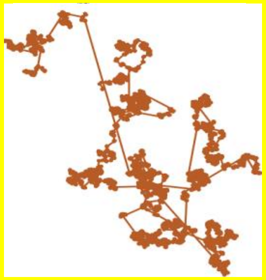

Heliopause
Kosmische Strahlung
Voyager 1
Sonnenwind
Pioneer 11
Pioneer 10
Voyager 2
Termination Shock
Bow Shock


Magnetosfeer
Aardmagnetisch
Aarde
Magnetische storm
Zon
Zonnewind

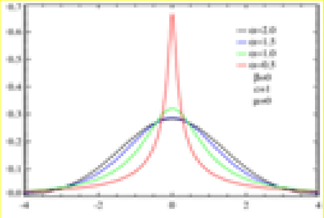
Thank you for your attention!

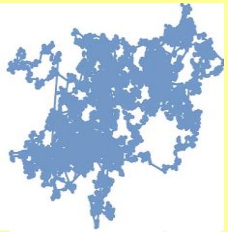
RELATIVISTIC PRINCIPLE
(finite speed requirement)



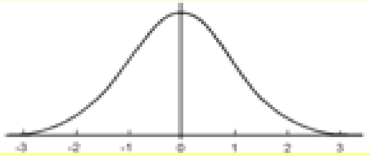


NONLOCAL DIFFUSION (Non-Fickian)
Statistical ground: GENERALIZED LIMIT THEOREM
Trajectory: LEVY FLIGHT





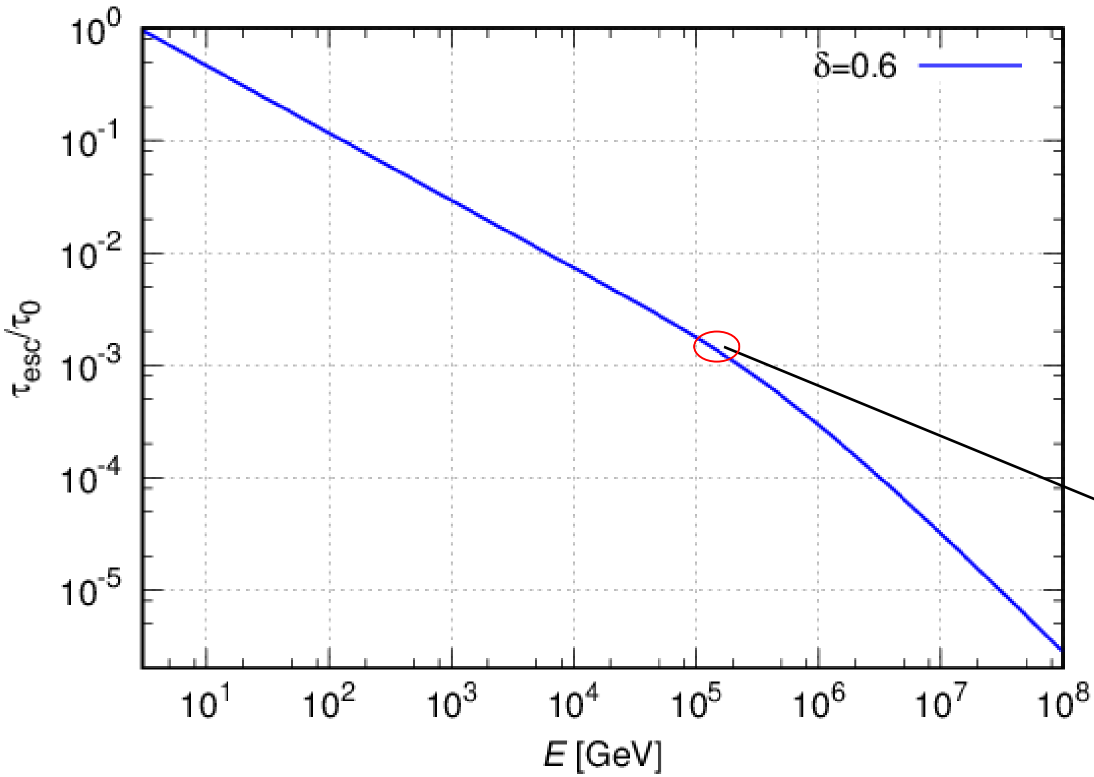
NORMAL DIFFUSION
Statistical ground: CENTRAL LIMIT THEOREM
Trajectory: BROWNIAN MOTION

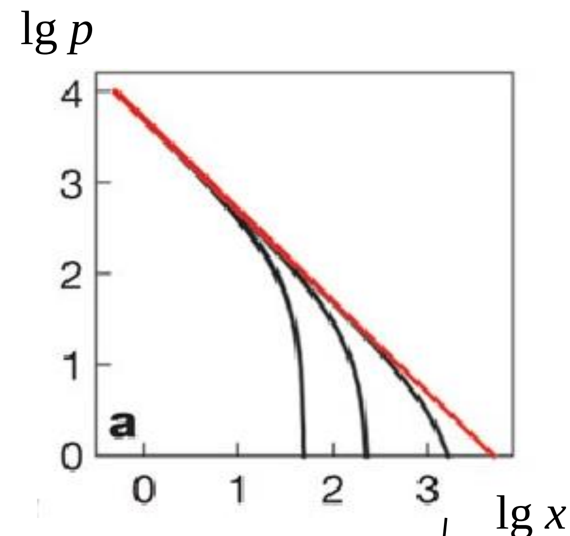


NoRD+ model and leaky-box approximation

$$N(E) \propto S(E)\tau_{\rm esc}(E)$$

$$p_R(r) \propto e^{-r/R_{\rm tr}(E)} r^{-1-\alpha}$$

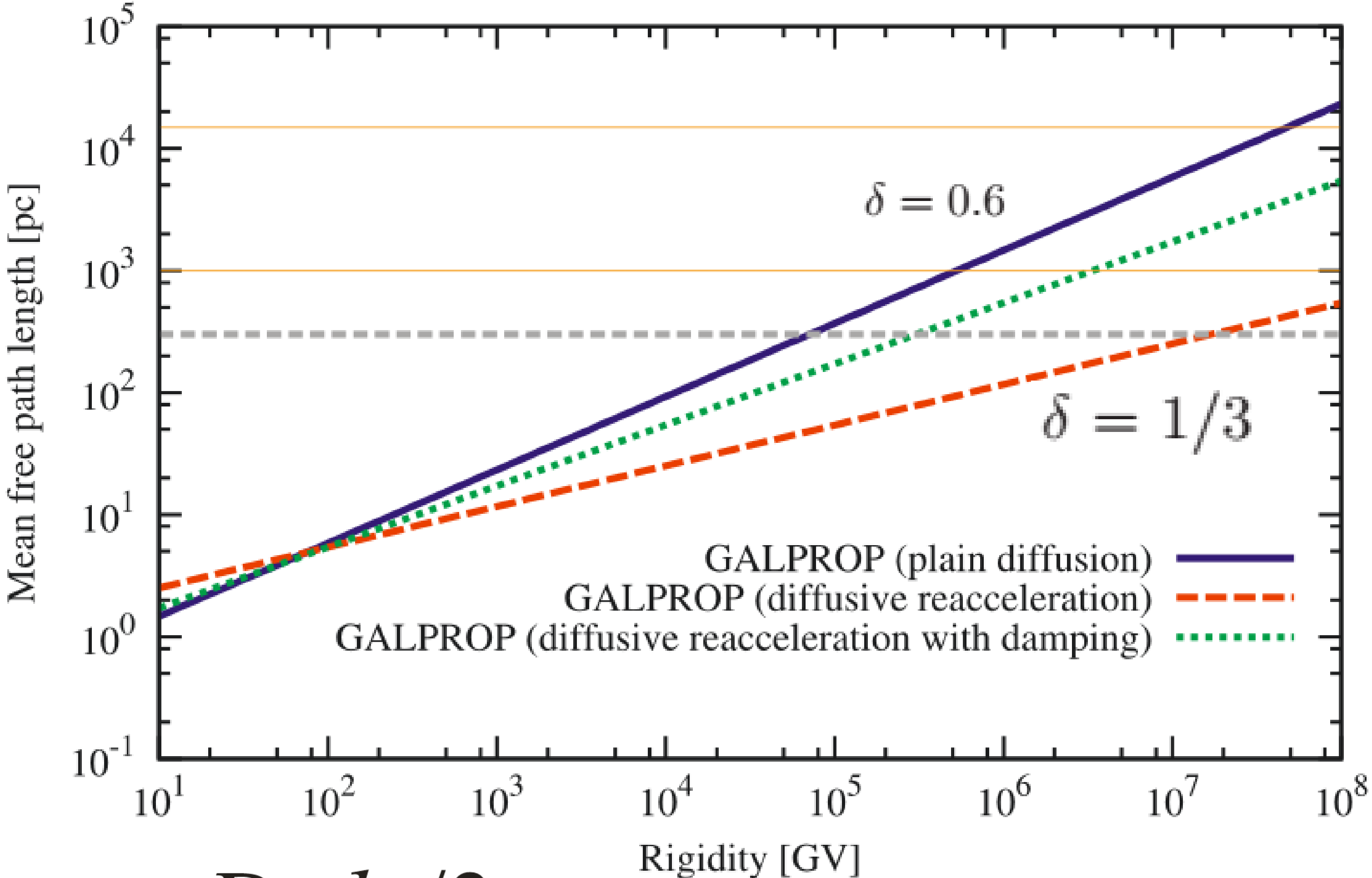




$$R_{\rm tr}(E) \approx H$$

Giacinti, G., Kachelrieß, M., & Semikoz, D. V. (2014). *Physical Review D*, *90*(4), 041302.

Energy dependence of the diffusion coefficient



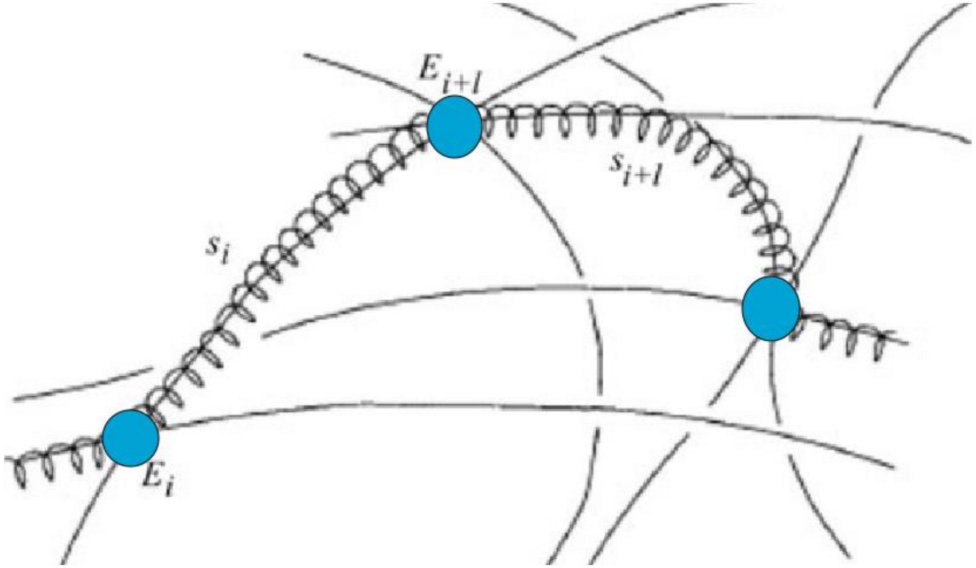
$$D = lv/3$$

$$D(E) = 10^{28} D_{28} \left(\frac{R}{3GV}\right)^{\delta} \text{cm}^2\text{s}^{-1}$$

$$D_{28}/H_{\text{kpc}} = 1.33 \text{ for } \delta = 1/3$$

$$D_{28}/H_{\text{kpc}} = 0.55 \text{ for } \delta = 0.6$$

Second order Fermi acceleration and turbulence



A **particle** (*spiraling line*) basically follows the **magnetic field lines** (*solid lines*), although also undergoing drifts and travel freely a distance s until it enters the "**scattering center or acceleration node**" (*filled circle*), where it is accelerated by the local "electric field *Ei* ". After spending a time *τi* inside the acceleration node it move freely again till it meets the new "acceleration node".

Second order Fermi acceleration

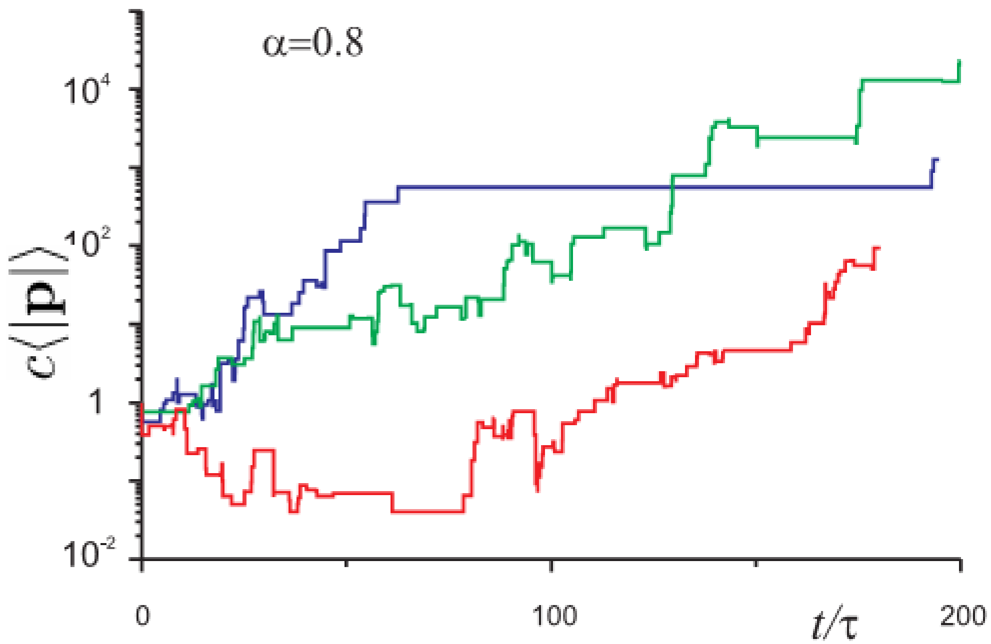
$$\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}_1 + \Delta\mathbf{p}_2 + \Delta\mathbf{p}_3 + \dots$$

$$\Delta\mathbf{p} = p'\mathbf{q},$$

$$\int\limits_{|\Delta\mathbf{p}|>p} w(\Delta\mathbf{p};\mathbf{p}')d\Delta\mathbf{p} \propto (p/p')^{-\gamma}, \ p \to \infty$$

$$\boxed{{}_0\mathsf{D}_t^{\alpha} f(\mathbf{p},t) = \mu \mathsf{A} f(\mathbf{p},t) + f_0(\mathbf{p})\delta_\alpha(t)}$$

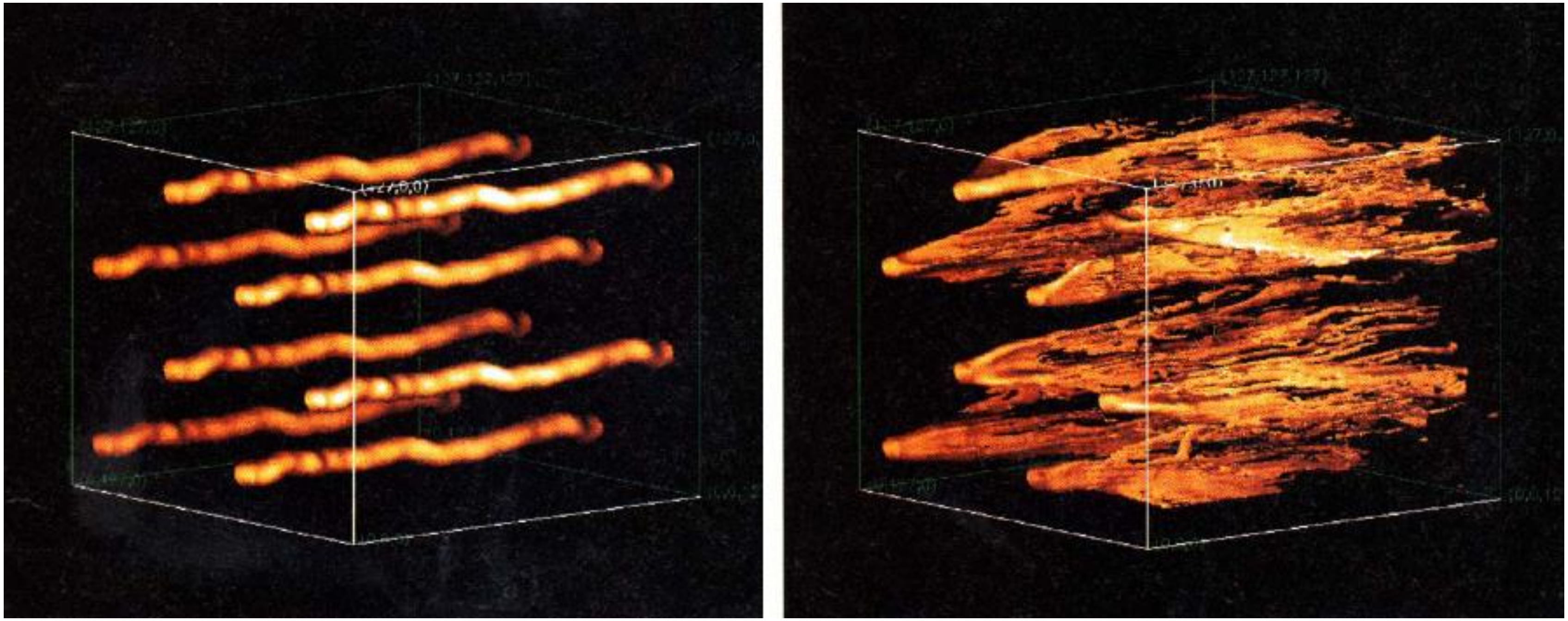
$$\int\limits_{-1}^{1}\frac{d\xi}{2}\int\limits_{0}^{\infty}\frac{f\left(p\Big/\sqrt{1+2\xi q+q^2},t\right)}{\left(\sqrt{1+2\xi q+q^2}\right)^3}V(q)dq - f(p,t)$$



Diverging Field Lines

Jokipii, 1966

Bieber et al. 1996.



The 2D component (right) leads to perpendicular spatial variation so that field lines and the energetic particles on them diverge.

Zybin & Istomin,1985
Chuvilgin & Ptuskin, 1993

Compound diffusion

Getmantsev (1963)
Lingenfelter et al. (1971) and Fisk et al. (1973)
Rechester & Rosenbluth 1978; Kadomtsev & Pogutse 1979; Krommes et al. 1983
Chuvilgin & Ptuskin (1993)
Webb et al. (2006)

$$p_\perp(\mathbf{r}, t \leftarrow \mathbf{r}_0, t_0) = \int_{-\infty}^{\infty} q(\mathbf{r} - \mathbf{r}_0 | z - z_0)\ G_\parallel(z, t \leftarrow z_0, t_0) dz.$$

$$\frac{\partial G_\parallel(z, t \leftarrow z_0, t_0)}{\partial t} = D_\parallel \frac{\partial^2 G_\parallel(z, t \leftarrow z_0, t_0)}{\partial z^2} + \delta(z - z_0)\delta(t - t_0),$$

$$\frac{\partial q(\mathbf{r} - \mathbf{r}_0, z - z_0)}{\partial z} = D_L \triangle_{\mathbf{r}} q(\mathbf{r} - \mathbf{r}_0, z - z_0) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(z - z_0).$$

G. M. Webb, G. P. Zank, E. Kh. Kaghashvili,1 and J. A. le Roux, 2006

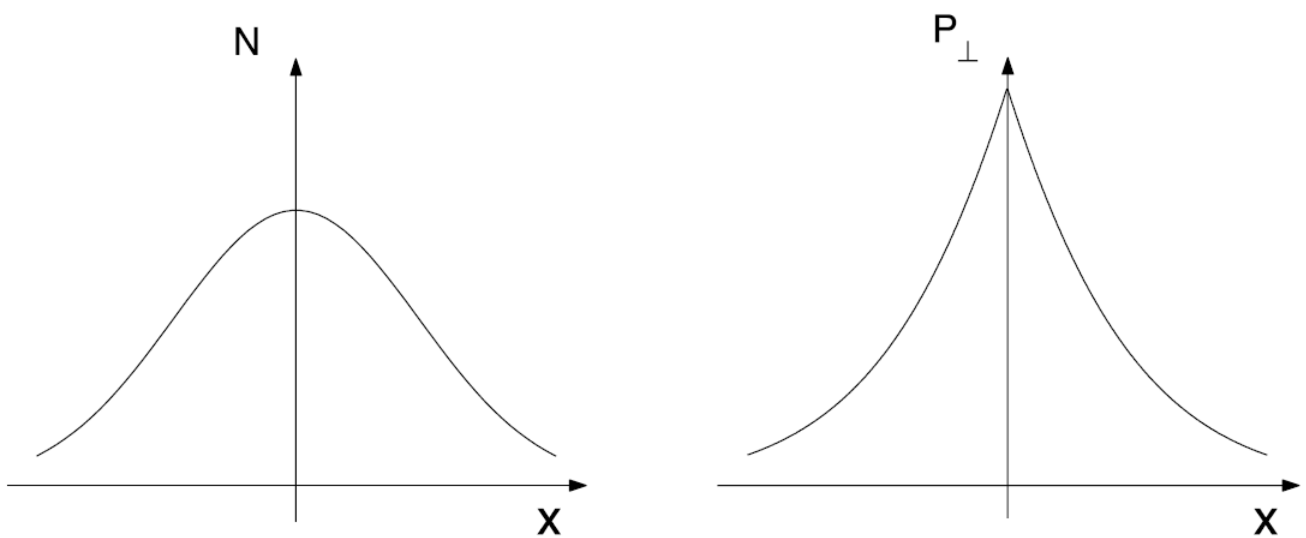


FIG. 3.—*Left:* pdf $N \equiv P_\perp^{(n)}$ for normal perpendicular diffusion (2.8); *right:* the compound diffusion pdf $P_\perp$ (2.16) vs. x at a fixed time t. Note the cusp in the compound diffusion case at $x = 0$.

Aloisio, R., Berezinsky, V., Gazizov, A. (2009). Astrophys. J, 693(2), 1275

"Diffusion equations are intrinsically non-relativistic and superluminal velocities appear naturally there. The cardinal solution of this problem – the relativistic generalization of the diffusion equation – still expects to be found after more than 70 years of unsuccessful attempts."

Litvinenko, Y. E., Effenberger, F., Schlickeiser, R. (2015). Astrophys J, 806(2), 217

"A general shortcoming of the diffusion approximation is that the diffusion equation implies an infinite speed of signal propagation, whereas particle speeds are finite, of course. A more accurate description may be provided by the telegraph equation."

Prosekin, A. Y., Kelner, S. R., Aharonian, F. A. (2015). *Phys Rev D*, *92*(8), 083003

"While the diffusion of cosmic rays has been comprehensively studies in the literature, the description of propagation in the intermediate stage, i.e. at the transition from the ballistic to the diffusive regime, is a problem of greater complexity regarding the exact analytical solutions."