Anomalous Diffusions and Fractional Order Differential Equations

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Workshop on Future Directions in Fractional Calculus Research and Applications

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Brownian Motion

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• Louis Bachelle 1900 Ph.D thesis: Theory of Speculations (advisor: Henri Poincaré).

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BM and Heat Equation

• Albert Einstein (1905): derive equation for BM based on the kinetic-molecular conception of matter

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Probabilistic representation

Mathematically, a Brownian motion B_t is a continuous process that has independent stationary increments. The increment is of Gaussian distribution.

Give initial data (temperature) f, $u(t,x) = \mathbb{E}_x[f(B_t)]$ solves the heat equation $\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x)$ with $u(0,x) = f(x)$.

Brownian motion is the scaling limit of random walks whose step size has finite second moment.

Random walk model: $S_n = \sum_{k=1}^n \xi_k$, $T_n = \sum_{j=1}^n \eta_j$, where ξ_k is the kth displacement and η_j is the jth waiting or holding time. Let $N_t = \max\{n : T_n \leq t\}$. Then $X_t = S_{N_t}$.

Key assumption for Brownian approximation: finite second moment displacement for each ξ_k and finite mean time for each η_j.

An increasing number of natural phenomena do not fit into the standard diffusion model. That is, either $|\xi_k|^2$, or η_j, or both has infinite mean, such as stable diffusion. (Pareto-Lévy or power-law distribution.)

Subdiffusion

Particle moves slower than Brownian motion, for example, due to particle sticking and trapping.

Example: (i) xerox machine, electrons in amorphous media tend to get trapped by local imperfections and then released due to thermal fluctuations.

(ii) hydrology: travel times of contaminants in groudwater are much longer than that of diffusion.

(iii) biology: proteins diffuse across cell membranes.

It can be described by an equation involving fractional time derivative: $\dfrac{\partial^\beta p}{\partial t^\beta} = a\dfrac{\partial^2 p}{\partial x^2}$.

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Caputo fractional derivative

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \left(f(s) - f(0)\right) ds,$$

where Γ is the Gamma function defined by
$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t} dt$.

A little physics: Let $u(t,x)$, $e(t,x)$ and $\vec{F}(t,x)$ denote the body temperature, internal energy and flux density, respectively. Then the relations

$$e(t,x) = \beta u(t,x), \quad \vec{F}(t,x) = -\lambda \nabla u(t,x), \qquad \beta, \lambda > 0,$$

$$\frac{\partial e}{\partial t}(t,x) = -\mathrm{div}\, \vec{F} \quad \text{(conservation law)}$$

yield the classical heat equation $\beta \frac{\partial u}{\partial t} = \lambda \Delta u$.

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Material of thermal memory

However in real modeling, heat flow can be disrupted by the response of the material. It has been shown (e.g. in Lunardi and Sinestrari (1988), von Wolfersdorf (1994)) that in a material with thermal memory, the internal energy

$$e(t,x) = \overline{\beta} u(t,x) + \int_0^t n(t-s)u(s,x)ds.$$

Typically, $n(t)$ is a positive decreasing function that blows up near $t = 0$, indicating the nearer past affects the present more. When $n(t) = t^{-\alpha}$ for $\alpha \in (0,1)$ and $u(0,x) = 0$, the heat equation becomes

$$\overline{\beta}\frac{\partial u}{\partial t} + \frac{1}{\Gamma(1-\alpha)}\frac{\partial^\alpha u}{\partial t^\alpha} = -\mathrm{div}\vec{F} = \lambda \frac{\partial^2 u}{\partial x^2}.$$

Fractional time SPDE

If in addition the internal energy $e(t,x)$ depends also on past random effects, then the internal energy is given by

$$e(t,x) = \overline{\beta} u(t,x) + \int_0^t n(t-s)u(s,x)ds + \int_0^t \ell(t-s)h(s,u(s,x))dW_s,$$

where W is a random process, such as Brownian motion, modeling the random effects. If $u(0,x)=0$, $\overline{\beta}=0$, $n(t)=\Gamma(1-\beta_1)^{-1}t^{-\beta_1}$ and $\ell(t)=\Gamma(2-\beta_2)^{-1}t^{1-\beta_2}$, we get fractional time stochastic partial differential equation

$$\partial_t^{\beta_1} u = \lambda \Delta u + \partial_t^{\beta_2} \int_0^t h(s,u(s,x))dW_s.$$

This type SPDE has recently been introduced and studied in C.-Kim-Kim, *SPA* **125**, 2015.

Superdiffusion

Another possibility for anomalous diffusion is that the random walker remains in motion without changing direction for a time that follows a Pareto-Lévy distribution.

Bird search: more effective

UCLA burglary hotspot model: study and predict burglary location

• S. Chaturapruek et al, Crime modeling with Lévy flights. *SIAM J. Appl. Math.* **73** (2013), 1703-1720.

Measured in terms of number of stretches, this corresponds to ξ_j of Lévy distribution and $\eta_j = 1$. The limiting process is a Lévy process. It can be described by an equation with fractional derivative in space: $\frac{\partial p}{\partial t} = a\Delta^{\alpha/2}p$.

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Central Limit Theorem

Recall random walk $S_t = \sum_{j=1}^{[t]} \xi_j$. Assume ξ_1 is spherically symmetric.

• If $\sigma^2 := \mathbb{E}[\xi_1^2] < \infty$, then $\lambda^{-1/2} S_{\lambda t}$ converges weakly to Brownian motion σB_t.

• If $\mathbb{P}(|\xi_1| \geq \lambda) \sim C\lambda^{-\alpha}$ for some $C > 0$ and $0 < \alpha < 2$ as $\lambda \to \infty$, the (extended) central limit theorem tells us that $\{\lambda^{-1/\alpha} S_{\lambda t}, t \geq 0\}$ converges weakly to a rotationally symmetric α-stable Lévy motion $\{Y_t, t \geq 0\}$ with

$$\mathbb{E}[e^{i\xi \cdot Y_t}] = e^{-C_0 |\xi|^\alpha t} \qquad \text{for every } \xi \in \mathbb{R}^d \text{ and } t \geq 0,$$

where the constant C_0 depends only on C and the dimension d.

The α-stable process Y has the following scaling property: $\{\lambda^{1/\alpha} Y_t, t \geq 0\}$ has the same distribution as $\{Y_{\lambda t}, t \geq 0\}$, it represents a model for anomalous super-diffusion, where particles spread faster than Brownian particles.

The infinitesimal generator of Y is $\Delta^{\alpha/2}$: $\widehat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \widehat{f}(\xi)$. Alternatively,

$$\Delta^{\alpha/2} u(x) = \int_{d} (u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) J(z) dz$$

where

$$J(z) = \frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} = J(x, x+z)$$

with $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1} \asymp \alpha(2-\alpha)$.

Fractional Laplacian

It can be shown that $u(t,x) = \mathbb{E}_x[f(Y_t)]$ solves the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta^{\alpha/2} u(t,x) \qquad \text{with } u(0,x) = f(x).$$

Fundamental solution (or transition density function of Y) $p(t,x,y)$:

$$u(t,x) = \mathbb{E}_x[f(Y_t)] = \int_{R^d} p(t,x,y) f(y) dy.$$

Heat kernel

Transition density function $p(t, x, y)$ of X encodes all the information of the process.

Unlike Brownian motion, typically it is impossible to get its explicit exact formula except for a very few special cases such as Cauchy process. Estimates

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \asymp \frac{t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}}.$$

Waiting time and Subordinator

Recall that the n-th jumping time is given by $T_n = \sum_{k=1}^n \eta_k$. The number of jumps by time $t > 0$ is $N_t = \max\{n : T_n \le t\}$, so the position of the particle at time $t > 0$ is S_{N_t}.

If $\mathbb{P}(\eta_1 > t) \sim Ct^{-\beta}$ as $t \to \infty$ for some $0 < \beta < 1$, then the scaling limit of $c^{-1/\beta} T_{[ct]} \Rightarrow Z_t$ as $c \to \infty$ is a strictly increasing stable Lévy motion with index β, called β-stable subordinator.

Meerschaert and Scheffler (2004) showed that $\{c^{-\beta} N_{ct}, t \ge 0\}$ converges weakly to the process $\{E_t, t \ge 0\}$, where $E_t = \inf\{s : Z_s > t\}$ and so the scaling limit of the particle location $\{c^{-\beta/\alpha} S_{N_{[ct]}}, t \ge 0\}$ is $\{B_{E_t}, t \ge 0\}$, Brownian motion time-changed by an inverse β-stable subordinator.

Thus B_{E_t} provides a model for anomalous sub-diffusion, where particles spread slower than Brownianian particles.

Inverse subordinator

In general, given a Markov process Y_t and an independent β-subordinator Z, one can do time change to get a new process $X_t = Y_{E_t}$, where $E_t = \inf\{x : Z_x > t\}$.

Question: What is the marginal distribution of X_t?

Denote by $g_\beta(u)$ the density of Z_1. Then by scaling, Z_s has density $s^{-1/\beta} g_\beta(s^{-1/\beta} u)$ for any $s > 0$. Using the inverse relation $\mathbb{P}(E_t \le s) = \mathbb{P}(Z_s \ge t)$ and taking derivatives, it follows that E_t has the density

$$f_t(s) = \frac{d}{ds}\mathbb{P}(Z_s \ge t) = t\beta^{-1} s^{-1-1/\beta} g_\beta(ts^{-1/\beta}).$$

Marginal distribution

For $\phi \geq 0$,

$$\begin{aligned} u(t,x) &:= \mathbb{E}_x\left[\phi(X_t)\right] = \mathbb{E}_x\left[\phi(Y_{E_t})\right] \\ &= \int_0^\infty \mathbb{E}_x[\phi(Y_s)]\mathbb{P}(E_t \in ds) = \int_0^\infty P_s\phi(x) f_t(s)ds \\ &= \int_0^\infty P_{(t/s)^\beta}\phi(x) g_\beta(s)ds. \end{aligned}$$

Theorem (Baeumer and Meerschaert, 2001):

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \mathcal{L}_x u(t,x).$$

C.-Meerschaert-Nane (*J. Math. Anal. Appl.* 2012): Space-time fractional diï¬*usiononboundeddomains*

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Symmetric non-local operator

Since the media can have impurities, of composite media, we need to consider stochastic processes whose movements are state-dependent.

An effective way to model a Markov process is through its infinitesimal generator.

There are two versions of state-dependent non-local operators: symmetric and non-symmetric.

• Symmetric operator on $\mathbb{R}^d$:

$$\mathcal{L}f(x) = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |y-x| > \varepsilon\}} (f(y) - f(x)) J(x, y) dy.$$

Probabilistic meaning of $J(x, y)$: jumping intensity from x to y.

It can be viewed as a counterpoart of divergence form operator.

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Non-symmetric non-local operator

• Non-symmetric operator (counterpart of non-divergence form elliptic operators)

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Symmetric stable-like operator

Symmetric operator on $\mathbb{R}^d$:

$$\mathcal{L}f(x) = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |y-x| > \varepsilon\}} (f(y) - f(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}} dy,$$

where $c(x,y)$ is a symmetric function bounded between two positive constants and $0 < \alpha < 2$.

Heat kernel estimates

Theorem (C.-Kumagai, *SPA* 2003)

$\mathcal{L}$ admits a jointly Hölder continuous heat kernel $p(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$, which satisfies

$$C^{-1}\frac{t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}} \le p(t,x,y) \le C\,\frac{t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}}$$

for every $t>0$ and $x,y\in\mathbb{R}^d$. The constant C depends only on the "ellipticity" of $c(x,y)$ and (d,α).

• Approach: Dirichlet form, Nash's inequality, Davies method, probabilistic approach by estimating various hitting probabilities.

• stability result. Holds on Ahlfors *d*-sets as well.

• Can be viewed as DeGiorgi-Nash-Moser-Aronson type theory for symmetric non-local operators.

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• Bass-Levin (2002): Two-sided HK estimate for random walk on $\mathbb{Z}^n$ with stable-like one-step transition distribution.

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$$\mathcal{L}^\kappa f(x) = \lim_{\varepsilon \to 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} dz.$$

is the counterpart of non-divergence form second order differential operator.

Here $d \geq 1$, $0 < \alpha < 2$, and $\kappa(x,z)$ a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $0 < \kappa_0 \leq \kappa(x,z) \leq \kappa_1$, $\kappa(x,z) = \kappa(x,-z)$. In addition, $\kappa(x,z)$ is uniformly Hölder continuous in x; that is,

$$|\kappa(x,z) - \kappa(y,z)| \leq \kappa_2 |x-y|^\beta.$$

Such operators have also been studied extensively in analysis, e.g., by Caffarelli, Silvestre, etc.

While heat kernels of differential operators have been studied extensively and there are quite many progress recently for symmetric non-local operators, there are very limited results on heat kernels for non-symmetric non-local operators.

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HK for non-local operators

Theorem (C.-Zhang, *PTRF* 2016)

There exists a unique nonnegative continuous function $p^\kappa(t,x,y)$ on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving

$$\partial_t p^\kappa(t,x,y) = \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x),$$

and satisfying the following four properties:
(i) *There is a constant $c_1 > 0$ so that for all $t \in (0,1]$ and $x,y \in \mathbb{R}^d$,*

$$p^\kappa(t,x,y) \le \frac{c_1\, t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}}.$$

(ii) *For every $\gamma \in (0, \alpha \wedge 1)$, there is a constant $c_2 > 0$ so that for every $t \in (0,1]$ and $x, x', y \in \mathbb{R}^d$,*

HKE

Theorem (C.-Zhang, *PTRF* 2016)

$$\begin{aligned} &|p^\kappa(t,x,y) - p^\kappa(t,x',y)| \\ \le\ & c_2|x-x'|^\gamma t^{1-\frac{\gamma}{\alpha}} \left(t^{1/\alpha} + |x-y| \wedge |x'-y|\right)^{-d-\alpha}. \end{aligned}$$

(iii) *For all* $x, y \in \mathbb{R}^d$, *the mapping* $t \mapsto \mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)$ *is continuous on* $(0,1]$, *and*

$$|\mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)| \le c_3(t^{1/\alpha} + |x-y|)^{-d-\alpha}.$$

(iv) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\limsup_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t,x,y) f(y)dy - f(x) \right| = 0.$$

Theorem (C.-Zhang, *PTRF* 2016)

Moreover, we have the following conclusions.

- *The constants* c_1, c_2 *and* c_3 *in (i)-(iii) above can be chosen so that they depend only on* $(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2, \gamma)$.
- $p^\kappa(t,x,y) \geq 0$ *and* $\int_{\mathbb{R}^d} p^\kappa(t,x,y)dy = 1$.
- $p^\kappa(t,x,y)$ *satisfies the Chapman-Kolmogorov's equation.*
- *For all* $t \in (0,1]$ *and* $x, y \in \mathbb{R}^d$,

$$p^\kappa(t,x,y) \geq \frac{c_4 t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}}.$$

Theorem (C.-Zhang, *PTRF* 2016)

- *If* $\alpha \in [1, 2)$*, for all* $x, y \in \mathbb{R}^d$ *and* $t \in (0, 1]$*,*

$$|\nabla_x \log p^\kappa(t, x, y)| \le c_5 t^{-1/\alpha}.$$

- *For any* $f \in C_b^2(\mathbb{R}^d)$*,*

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} p^\kappa(t, x, y)(f(y) - f(x))dy = \mathcal{L}^\kappa f(x),$$

and the convergence is uniform.

Some history

• Kochubei (1998) studied the existence of heat kernel of $\mathcal{L}^\kappa$ under strong smoothness assumption on $\kappa(x, y)$ in y and requires $\alpha \in [1, 2)$.

• C.-Jieming Wang (2014) studied the sharp HKE for $\Delta^{\alpha/2}$ perturbed by lower order non-local operators, corresponds to the case where $\kappa(x, z) = a + b(x, z)|z|^{\alpha-\delta}$ for some constant $a > 0$ and a bounded measurable $b(x, z)$ with $b(x, z) = b(x, -z)$. It includes the following as a special case.

$$dX_t = dY_t + b(X_{t-})dZ_t.$$

• C.-Ting Yang (2015): Dirichlet heat kernel estimates for above operator with non-local perturbation in smooth domains.

Suppose that $A(x) = (a_{ij}(x))_{1\le i,j\le d}$ is a bounded continuous $d \times d$-matrix-valued function on $\mathbb{R}^d$ that is non-degenerate at every $x \in \mathbb{R}^d$, and Y is a (rotationally) symmetric α-stable process on $\mathbb{R}^d$ for some $0 < \alpha < 2$.

Theorem (Bass-C. *PTRF* 2006)

For every $x \in \mathbb{R}^d$, SDE

$$dX_t = A(X_{t-})dY_t, \qquad X_0 = x,$$

has a unique weak solution.

These weak solutions form a strong Markov process X. Does X have a transition density function? What is its estimates?

Infinitesimal generator

of X is

$$\begin{aligned}\mathcal{L}f(x) &= \text{p.v.}\int_{\mathbb{R}^d} (f(x+A(x)y)-f(x))\,\frac{\mathcal{A}(d,-\alpha)}{|y|^{d+\alpha}}dy \\ &= \text{p.v.}\int_{\mathbb{R}^d} (f(x+z)-f(x))\,\frac{\kappa(x,z)}{|z|^{d+\alpha}}dz,\end{aligned}$$

where

$$\kappa(x,z)=\frac{\mathcal{A}(d,-\alpha)}{|\mathrm{det}A(x)|}\left(\frac{|z|}{|A(x)^{-1}z|}\right)^{d+\alpha}.$$

Theorem (C.-Zhang, *PTRF* 2016)

Suppose that $A(x) = (a_{ij}(x))$ *is uniformly bounded and elliptic and there are constants* $\beta \in (0,1)$ *and* $\lambda_2 > 0$ *so that*

$$|a_{ij}(x) - a_{ij}(y)| \leq \lambda_2 |x-y|^\beta \quad \text{for } 1 \leq i,j \leq d.$$

Then the strong Markov process X *formed by the unique weak solution to SDE has a jointly continuous transition density function* $p(t,x,y)$ *with respect to the Lebesgue measure on* $\mathbb{R}^d$*, and there is a constant* $C > 0$ *that depends only on* $(d, \alpha, \beta, \lambda_0, \lambda_1)$ *so that*

$$C^{-1} \frac{t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}} \leq p(t,x,y) \leq C \frac{t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}}$$

for every $t \in (0,1]$ *and* $x, y \in \mathbb{R}^d$*. Moreover,* $p(t,x,y)$ *enjoys all the properties stated in the conclusion of the previous Theorem.*

HK Stability

Any mathematical model is an approximation and simplification of the real model. How stable are the results in response to the approximation error?

For two uniformly elliptic divergence form operators $\mathcal{L} = \nabla(A\nabla)$ and $\widetilde{\mathcal{L}} = \nabla(\widetilde{A}\nabla)$, C.-Hu-Qian-Zheng (*JFA*, 1998) showed that

$$|p(t,x,y) - \widetilde{p}(t,x,y)| \leq t^{-d/2} \exp(-c|x-y|^2/t) F(t, \|A - \widetilde{A}\|_{L^2_{loc}}),$$

where $F(t,r)$ is an explicit bounded continuous function such that $\lim_{r\to 0} F(t,r) = 0$ for each $t > 0$.

Recently, Bass and H. Ren (*JFA*, 2013) extended the above result to symmetric stable-like operators.

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Theorem (C.-Zhang, 2016+)

Suppose $\beta \in (0, \alpha/4]$*, and* $\kappa(x,z)$ *and* $\widetilde{\kappa}(x,z)$ *are two functions that are* β*-Hölder continuous and bounded between two positive constants* κ_1 *and* κ_2*. Then for every* $\gamma \in (0, \beta)$ *and* $\eta \in (0,1)$*, there exists a constant* $C = C((d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2, \gamma, \eta) > 0$ *so that for all* $t \in (0,1]$ *and* $x, y \in \mathbb{R}^d$*,*

$$\left| \frac{p^{\kappa}(t,x,y)}{p^{\widetilde{\kappa}}(t,x,y)} - 1 \right| \leq C \|\kappa - \widetilde{\kappa}\|_{\infty}^{1-\eta} \left(1 + t^{-\gamma/\alpha} (|x-y| \wedge 1)^{\gamma} \right).$$

Approach

We use an analytic method based on Levi's freezing coefficients argument to construct heat kernel of $\mathcal{L}^\kappa$ and obtain its upper heat kernel estimates. From it we construct its associated jump process X and identify its Lévy system of X. We then using the Lévy system to derive the lower bound estimate of the heat kernel through a probabilistic argument.

For fixed $y \in \mathbb{R}^d$, let $\mathcal{L}^{\kappa(y)}$ be the freezing operator

$$\mathcal{L}^{\kappa(y)} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{\kappa(y,z)}{|z|^{d+\alpha}} dz.$$

It determines a Lévy process. Let $p_y(t,x) := p^{\kappa(y)}(t,x)$ be its heat kernel, its existence and heat kernel estimates is known from C.-Kumagai (2003).

Levi's method

We search for heat kernel $p^\kappa(t,x,y)$ of $\mathcal{L}^\kappa$ with the form:

$$p^\kappa(t,x,y) = p_y(t,x-y) + \int_0^t\int_{\mathbb{R}^d} p_z(t-s,x-z)q(s,z,y)dzds.$$

We want $\partial_t p^\kappa(t,x,y) = \mathcal{L}^{\kappa(x)} p^\kappa(t,x,y)$. We deduce from it

$$q(t,x,y) = q_0(t,x,y) + \int_0^t\int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)dzds,$$

where

$$q_0(t,x,y) = (\mathcal{L}^{\kappa(x)} - \mathcal{L}^{\kappa(y)})p_y(t,x-y).$$

Thus for the construction and upper bound HK estimates, the main task is to solve $q(t, x, y)$ recursively, and to make the above argument rigorous. This relies on the fractional derivative estimates on $p_y(t, x - y)$ and strong continuity results on stability results on $p^\kappa(t, x, y)$ and their derivatives for symmetric $\kappa(z)$.

Hard analysis.

For the lower bound heat kernel estimates, we use probabilistic analysis via Lévy system that describes how the process jumps and certain hitting probability estimates.

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The heat kernel stability result can be derived from the construction of $p^\kappa(t, x, y)$ by carefully analyzing its dependence first on $\kappa(x_0, z)$ when x_0 is fixed and then on variable $\kappa(x, z)$.

It in particular implies stability of heat semigroup in terms of operator norms such as $\|P_t^\kappa - P_t^{\widetilde{\kappa}}\|_{2\to 2}$

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Thank you!