A Petrov-Galerkin Spectral Element Method for Fractional Elliptic Problems

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Problem Definition

Fractional **Helmholtz** equation, subjected to **Dirichlet** boundary conditions:

$$ {}^{RL}_{0}\mathcal{D}^{\alpha}_{x} u(x) - \lambda u(x) = f(x), \quad \forall x \in \Omega \qquad \alpha = 1+\mu$$

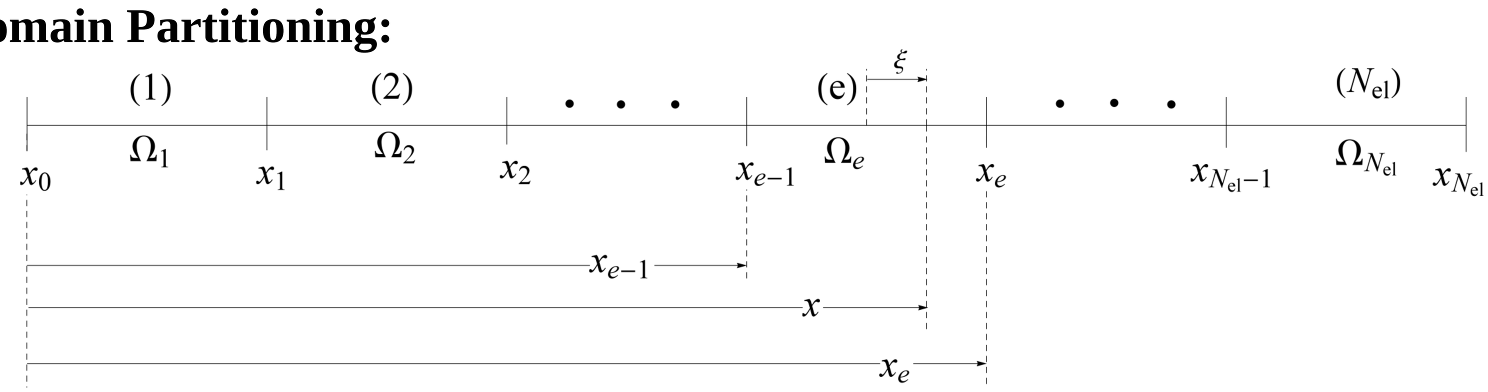
$$u(0) = u(L) = 0, \quad \forall x \in \partial\Omega \qquad \mu \in (0,1]$$

Weak Formulation: multiplying by proper test function and integrating by parts

$$a(u,v) = l(v) \qquad \begin{cases} a(u,v) = \left(\frac{du}{dx}, {}^{RL}_{x}\mathcal{D}^{\mu}_{L} v\right)_{\Omega} - \lambda\left(u,v\right)_{\Omega} \\ l(v) = \left(f, v\right)_{\Omega}, \end{cases}$$

Basis/Test Functions

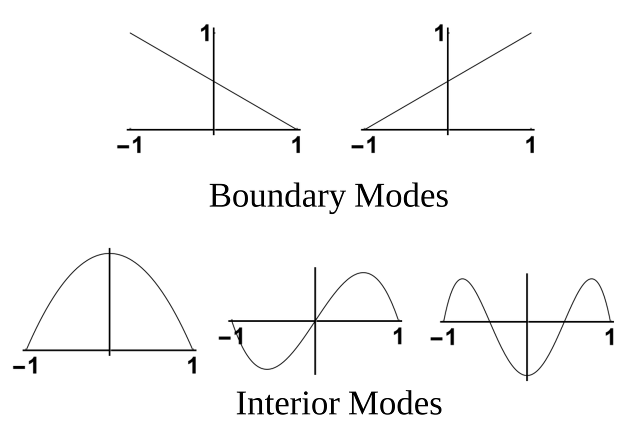
Domain Partitioning:



$$u \approx u^{\delta}(x) = \sum_{e=1}^{N_{el}} \sum_{p=0}^{P} \hat{u}^{(e)}_{p} \psi_p(x)$$

Standard local basis functions [1]: in standard domain $\xi \in [-1,1]$

$$\psi_p(\zeta) = \begin{cases} \frac{1-\zeta}{2}, & p=0, \\ \left(\frac{1-\zeta}{2}\right)\left(\frac{1+\zeta}{2}\right)P^{1,1}_{p-1}(\zeta), & p=1,2,\cdots,P-1, \\ \frac{1+\zeta}{2}, & p=P, \end{cases}$$



Local v.s. Global Test Functions [2]

Jacobi poly-fractonomials of second kind [3]:

$$ {}^{(2)}\mathcal{P}^{\mu}_{k}(\xi) = (1-\xi)^{\mu} P^{\mu,-\mu}_{k-1}(\xi), \quad \xi \in [-1,1]$$

$$v^{local}_{k}(x) = v^{\varepsilon}_{k}(x) = \begin{cases} {}^{(2)}\mathcal{P}^{\mu}_{k+1}(x^{\varepsilon}), & \forall x \in \Omega_{\varepsilon}, \\ 0, & otherwise, \end{cases}$$

$$v^{global}_{k}(x) = v^{\varepsilon}_{k}(x) = \begin{cases} {}^{(2)}\mathcal{P}^{\mu}_{k+1}(x^{1\sim\varepsilon}), & \forall x \in [0, x_{\varepsilon}], \\ 0, & otherwise \end{cases}$$

References

[1] G. E. Karniadakis, S. J. Sherwin, *Spectral/hp Element Methods for CFD*, Oxford University Press (2nd edition), 2005

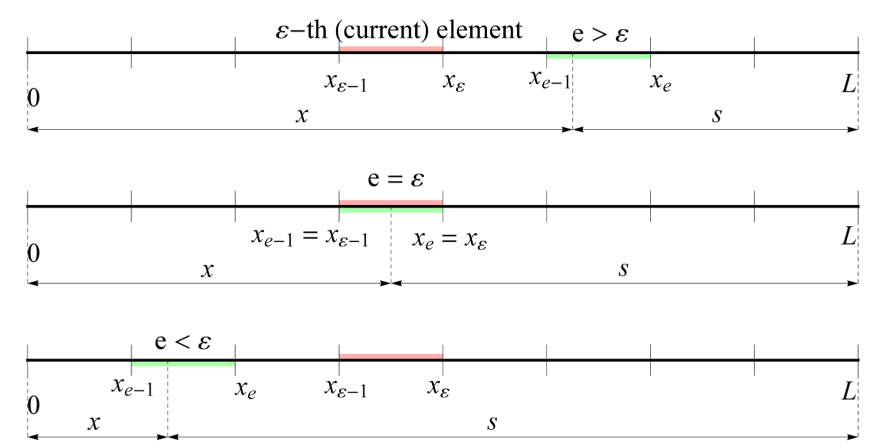
[2] E. Kharazmi, M. Zayernouri, G. E. Karniadakis, *A Petrov-Galerkin Spectral Element Method for Fractional Elliptic Problems*, ArXiv preprint, 2016

[3] M. Zayernouri, G. E. Karniadakis, *Fractional Sturm-Liouville Eigen-Problem: Theory and Numerical Approximations*, J. Comp. Physics, 2013

Elemental Operations

$$\sum_{e=1}^{\varepsilon-1}\sum_{p=0}^{P} \hat{u}^{(e)}_{p}\, \hat{\mathbf{S}}^{(e,\varepsilon)}_{kp} + \sum_{p=0}^{P} \hat{u}^{(\varepsilon)}_{p}\left[\mathbf{S}^{(\varepsilon)}_{kp} - \lambda\, \mathbf{M}^{(\varepsilon)}_{kp}\right] = \mathbf{f}^{(\varepsilon)}_{k}, \qquad \begin{cases} \varepsilon = 1,2,\cdots,N_{el}, \\ k = 0,1,\cdots,P, \end{cases}$$

Fractional derivative of test functions: ${}^{RL}_{x}\mathcal{D}^{\mu}_{L} v^{\varepsilon}_{k}(x)$



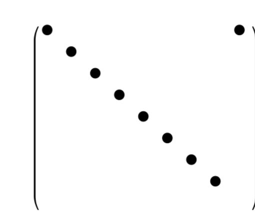
$$\begin{cases} 0, & \forall x \in \Omega_e,\ e > \varepsilon, \\ {}^{RL}_{x}\mathcal{D}^{\mu}_{x_e}\left[{}^{(2)}\mathcal{P}^{\mu}_{k+1}(x)\right], & \forall x \in \Omega_e,\ e = \varepsilon, \\ H^{(\varepsilon)}_{k}(x), & \forall x \in \Omega_e,\ e < \varepsilon. \end{cases}$$

History Matrix (Non-Local Effects): $\hat{\mathbf{S}}^{(e,\varepsilon)}_{kp} = \left(\frac{d\psi_p}{dx}, H^{(\varepsilon)}_{k}(x)\right)_{\Omega_e}$, ⟶ Analytical Expression of Non-Locality (Uniform Grid)

Stiffness Matrix: $\mathbf{S}^{(\varepsilon)}_{kp} = \left(\frac{d\psi_p}{dx}, {}^{RL}_{x}\mathcal{D}^{\mu}_{x_\varepsilon}\left[{}^{(2)}\mathcal{P}^{\mu}_{k+1}(x)\right]\right)_{\Omega_\varepsilon}$, ⟶ Sparse Stiffness Matrix

Mass Matrix: $\mathbf{M}^{(\varepsilon)}_{kp} = \left(\psi_p(x), {}^{(2)}\mathcal{P}^{\mu}_{k+1}(x)\right)_{\Omega_\varepsilon}$,

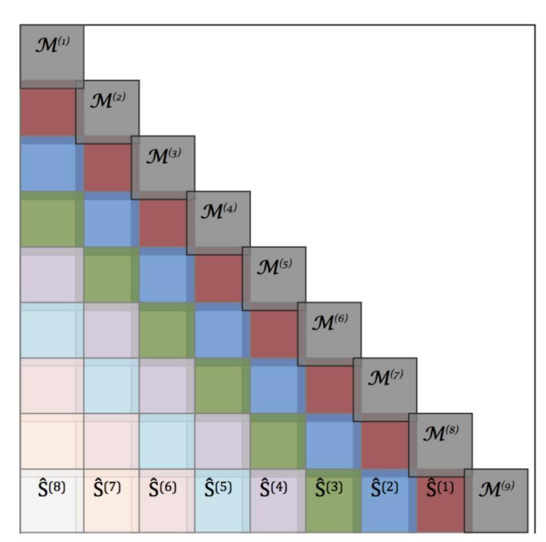
Load Vector: $\mathbf{f}^{(\varepsilon)}_{k} = \left(f, {}^{(2)}\mathcal{P}^{\mu}_{k+1}(x)\right)_{\Omega_\varepsilon}$,



Assembling

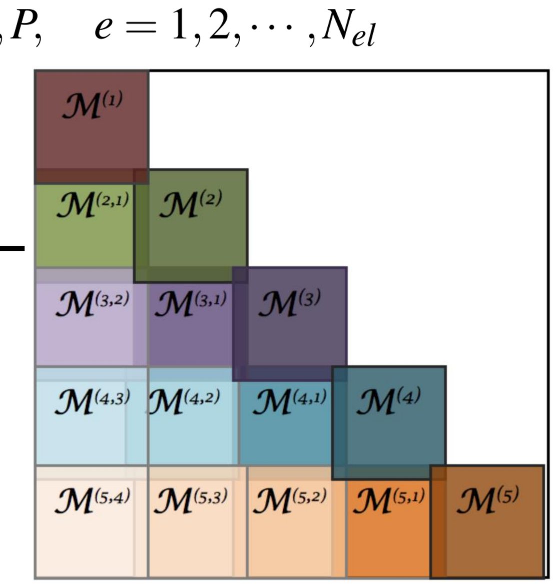
Non-Local Assembling: new procedure of assembling history matrices

Mapping Array: $map[e][p] = P(e-1)+p, \quad p = 1,2,\cdots,P, \quad e = 1,2,\cdots,N_{el}$



Global linear system (local test functions)

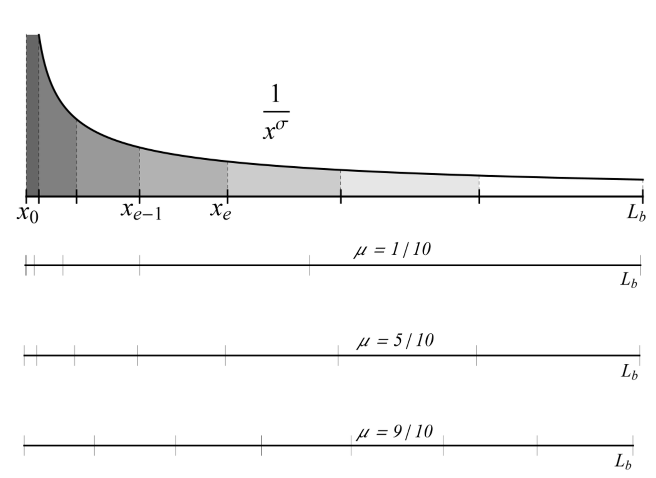
# History Matrices	
$N_{el}-1$	$\frac{N_{el}(N_{el}-1)}{2}$



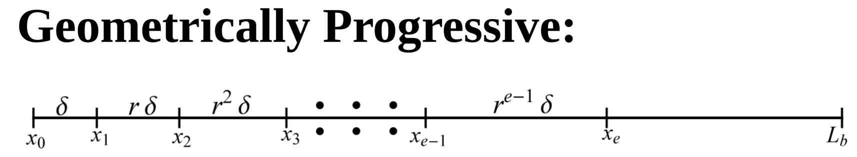
Global linear system (global test functions)

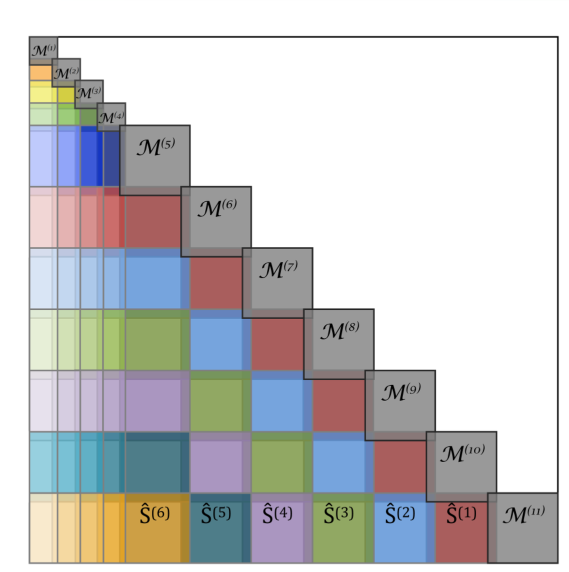
Non-Uniform Grids

Kernel-Based:



Geometrically Progressive:

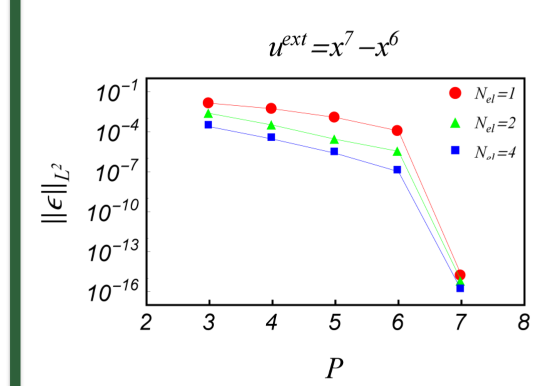


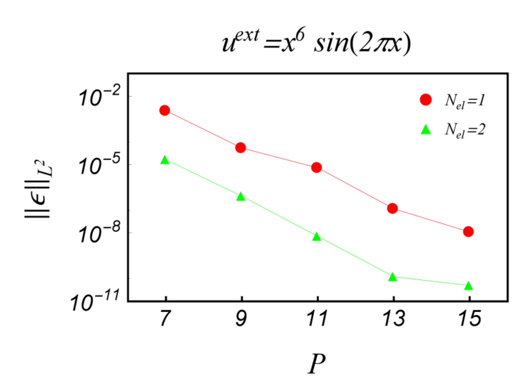


Global linear system (local test functions) Non-Uniform Grid

Numerical Results

Smooth Problems:





Condition Number of Global Linear System ($N_{el} = 10, P = 10$)

Local Basis/Test	Local Basis, Global Test
420.24	5.1×10^{17}

Singular Problems:

Single-Boundary Singularity

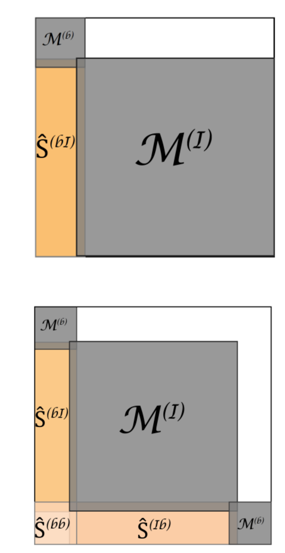
$u^{ext} = (1-x)x^{5+\mu}, \quad \mu = 1/2$

Boundary Element Error

P_b	$L_b = 10^{-1}L$	$L_b = 10^{-2}L$	$L_b = 10^{-4}L$
6	3.94221×10^{-11}	2.96862×10^{-17}	4.8243×10^{-29}
10	7.07024×10^{-13}	2.54089×10^{-18}	2.26939×10^{-29}

Interior Element Error, $P_b = 10$

P_I	$L_b = 10^{-1}L$	$L_b = 10^{-2}L$	$L_b = 10^{-4}L$
6	1.73622×10^{-5}	3.80264×10^{-5}	4.13249×10^{-5}
10	1.3122×10^{-9}	8.76951×10^{-9}	1.10139×10^{-8}
14	4.39611×10^{-12}	1.07775×10^{-10}	1.66044×10^{-10}



Full-Boundary Singularity

$u^{ext} = (1-x)^{3+\mu_1}x^{3+\mu_2} \qquad L_b = 10^{-2}L$

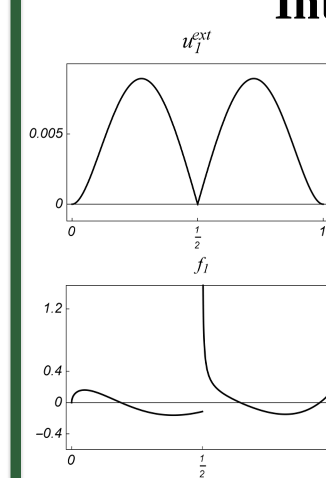
$P_b = 6$

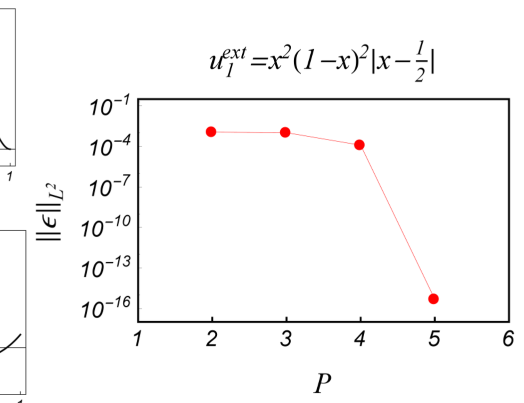
P_I	Left BE Error	IE Error	Right BE Error
6	2.73893×10^{-7}	6.52605×10^{-5}	3.51075×10^{-6}
10	2.46964×10^{-11}	1.52215×10^{-7}	2.2902×10^{-9}
14	3.08719×10^{-12}	9.30483×10^{-9}	2.69541×10^{-10}

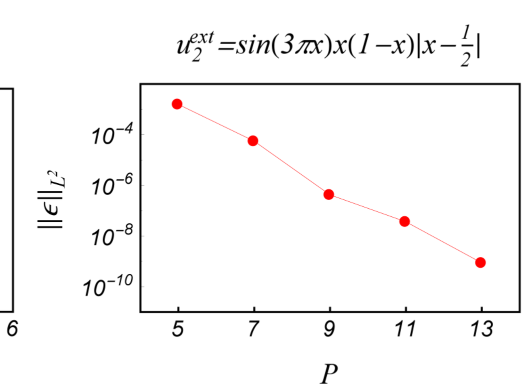
$P_b = 10$

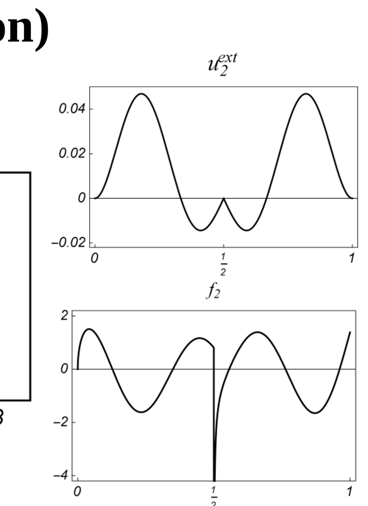
P_I	Left BE Error	IE Error	Right BE Error
6	2.73892×10^{-7}	6.52605×10^{-5}	3.51075×10^{-6}
10	2.48058×10^{-11}	1.52215×10^{-7}	2.29003×10^{-9}
14	3.19684×10^{-12}	9.30511×10^{-9}	2.69506×10^{-10}

Interior Singularity (Discontinuous Force Function)





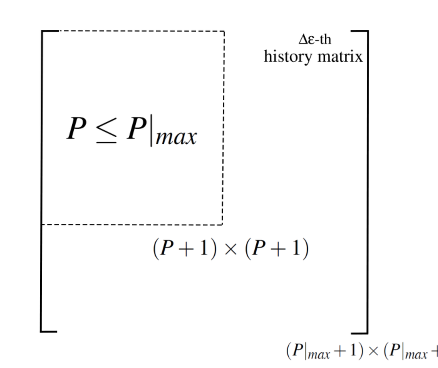




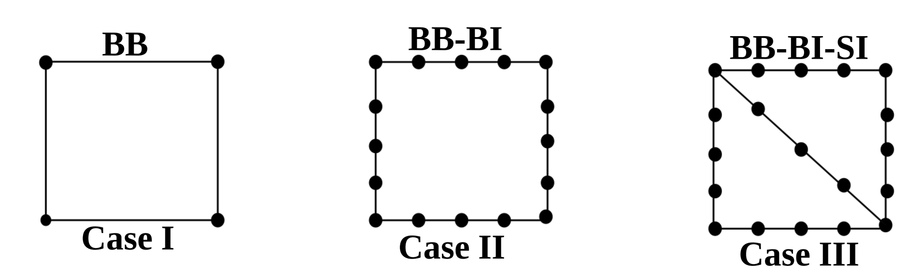
Off-line History Computation and Retrieval:

CPU Time

	$N_{el} = 10$		$N_{el} = 100$		$N_{el} = 500$		$N_{el} = 1000$	
P	Off-line retrieval	On-line computation	Off-line retrieval	On-line computation	Off-line retrieval	On-line computation	Off-line retrieval	On-line computation
2	2.6520	7.2540	24.7105	83.5229	141.3525	429.3147	370.6895	790.3478
3	4.7580	18.9073	46.0826	161.8042	266.0441	1308.8327	746.4959	4423.7671
4	8.8140	32.2922	84.8645	499.9988	485.7715	5599.4062	1392.8705	14709.4902



Memory Fading Analysis:



Full fading

# faded history matrices	$\mu = 1/10$	$\mu = 1/2$	$\mu = 9/10$
0	9.26034×10^{-12}	2.31391×10^{-11}	4.24903×10^{-9}
2	7.8905×10^{-11}	1.26365×10^{-10}	4.25456×10^{-9}
5	1.42423×10^{-8}	6.39474×10^{-8}	1.95976×10^{-8}
8	2.69431×10^{-7}	2.47423×10^{-6}	8.45001×10^{-7}
11	2.09737×10^{-6}	3.19995×10^{-5}	1.37959×10^{-5}
14	9.07427×10^{-6}	2.44911×10^{-4}	1.40684×10^{-4}
17	2.94001×10^{-5}	1.39043×10^{-3}	1.6001×10^{-3}

Partial fading case I

# faded history matrices	$\mu = 1/10$	$\mu = 1/2$	$\mu = 9/10$
0	9.26034×10^{-12}	2.31391×10^{-11}	4.24903×10^{-9}
2	7.8905×10^{-11}	1.26365×10^{-10}	4.25456×10^{-9}
5	1.42423×10^{-8}	6.39474×10^{-8}	1.95976×10^{-8}
8	2.69431×10^{-7}	2.47423×10^{-6}	8.45001×10^{-7}
11	2.09737×10^{-6}	3.19995×10^{-5}	1.37959×10^{-5}
14	9.07427×10^{-6}	2.44911×10^{-4}	1.40684×10^{-4}
17	2.94001×10^{-5}	1.39043×10^{-3}	1.6001×10^{-3}

Partial fading case II

# faded history matrices	$\mu = 1/10$	$\mu = 1/2$	$\mu = 9/10$
0	9.26034×10^{-12}	2.31391×10^{-11}	4.24903×10^{-9}
2	9.27241×10^{-12}	2.34361×10^{-11}	4.2491×10^{-9}
5	3.37716×10^{-11}	6.6476×10^{-10}	4.44832×10^{-9}
8	3.8092×10^{-10}	1.99961×10^{-8}	1.36941×10^{-8}
11	1.47228×10^{-9}	2.2715×10^{-7}	1.47786×10^{-7}
14	1.15821×10^{-8}	1.64103×10^{-6}	1.47098×10^{-6}
17	5.06103×10^{-7}	7.87929×10^{-6}	1.85274×10^{-5}

Partial fading case III

# faded history matrices	$\mu = 1/10$	$\mu = 1/2$	$\mu = 9/10$
0	9.26034×10^{-12}	2.31391×10^{-11}	4.24903×10^{-9}
2	9.26023×10^{-12}	2.3113×10^{-11}	4.24903×10^{-9}
5	1.18462×10^{-11}	7.7854×10^{-11}	4.23683×10^{-9}
8	1.60656×10^{-10}	3.38055×10^{-9}	3.65689×10^{-9}
11	1.32413×10^{-9}	4.35421×10^{-8}	8.84638×10^{-9}
14	7.10271×10^{-9}	3.87096×10^{-7}	1.7226×10^{-7}
17	2.87023×10^{-8}	3.71057×10^{-6}	5.12104×10^{-6}