Density Bounds for some Degenerate Stable Driven SDEs.

Lorick HUANG
Michigan State University

Abstract

We consider a system of linear differential equations whose first entry is perturbed by an anisotropic non degenerate (eventually tempered) Stable noise. Assuming some continuity on the noise coefficient, and a Hörmander like condition that allows the propagation of the noise through the system, we prove the uniqueness to the martingale problem for the associated generator, under technical restrictions on the number of oscillators and the dimension. Also, we establish density bounds reflecting the multi-scale behavior of the process

The equation

We study degenerate stochastic differential equations driven by a finite range stable process, that is:

$$\begin{aligned}
dX_t^1 &= \left(a_t^{1,1}X_t^1 + \cdots + a_t^{1,n}X_t^n\right)dt + \sigma(t, X_{t^-})dZ_t \qquad (1)\\
dX_t^2 &= \left(a_t^{2,1}X_t^1 + \cdots + a_t^{2,n}X_t^n\right)dt\\
dX_t^3 &= \left(a_t^{3,2}X_t^2 + \cdots + a_t^{3,n}X_t^n\right)dt\\
&\vdots\\
dX_t^n &= \left(a_t^{n,n-1}X_t^{n-1} + a_t^{n,n}X_t^n\right)dt,\ X_0 = x \in \mathbb{R}^{nd},
\end{aligned}$$

where $\sigma : \mathbb{R}_+ \times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$, $a^{i,j} : \mathbb{R}_+ \to \mathbb{R}^d \otimes \mathbb{R}^d$, $i \in [\![1, n]\!]$, $j \in [\![(i-1) \vee 1, n]\!]$. Also, Z is an $\mathbb{R}^d$ valued symmetric α stable (possibly tempered) process ($\alpha \in (0, 2)$), that is a stable process whose Lévy measure is truncated:

$$\nu(dz) = C_{d,\alpha} q(|z|) \frac{d|z|}{|z|^{1+\alpha}} \mu(d\bar{z}),\ z = |z|\bar{z}, \in \mathbb{R}_+ \times S^{d-1}.$$

Assumptions

We will make the following assumptions:

[H-1]: (**Hölder regularity**) $\exists H > 0,\ \eta \in (0, 1],\ \forall x, y \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,

$$||\sigma(t, x) - \sigma(t, y)|| \leq H|x - y|^\eta.$$

[H-2]: (**Non degeneracy of the spectral measure**) $\exists \Lambda_1, \Lambda_2 \in \mathbb{R}_+^*$, $\forall u \in \mathbb{R}^d$,

$$\Lambda_1 |u|^\alpha \leq \int_{S^{d-1}} |\langle u, \varsigma\rangle|^\alpha \mu(d\varsigma) \leq \Lambda_2 |u|^\alpha. \qquad (2)$$

[H-3]: (**Ellipticity**) $\exists\ \bar{c}, \underline{c} > 0$, $\forall \xi \in \mathbb{R}^d$, $\forall z \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,

$$\underline{c}|\xi|^2 \leq \langle \xi, \sigma\sigma^*(t, z)\xi\rangle \leq \bar{c}|\xi|^2. \qquad (3)$$

[H-4]: (**Hörmander-like condition for** $(A_t)_{t\geq 0}$) $\exists \overline{\alpha},\ \underline{\alpha} \in \mathbb{R}_+^*$, $\forall \xi \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,
$\underline{\alpha}|\xi|^2 \leq \langle a_t^{i,i-1}\xi, \xi\rangle \leq \overline{\alpha}|\xi|^2$, $\forall i \in [\![2, n-1]\!]$. Also, for all $(i, j) \in [\![1, n]\!]^2$, $\|a_t^{i,j}\| \leq \overline{\alpha}$.

Links with degenerate FPDE

The function $u(t, x) = \mathbb{E}[f(X_T)|f(X_t) = x]$ solves the fractional PDE:

$$\begin{cases}\partial_t u(t, x) = -L_t u(t, x)\\ u(T, x) = f(x)\end{cases}$$

where L_t is the generator of X. For $n = 2$, $d = 1$ and $\sigma = 1$, the fractional PDE satisfied can be written:

$$\partial_t u(t, x) = x_2 \cdot \nabla_{x_1} u(t, x) + \Delta_{x_1}^{\frac{\alpha}{2}} u(t, x),$$

where the fractional Laplacian $\Delta_{x_1}^{\frac{\alpha}{2}}$ only acts on the first variable of $x = (x_1, x_2)$.
In general, the generator L_t writes for $\varphi : \mathbb{R}^{nd} \to \mathbb{R}$:

$$\begin{aligned}
L_t\varphi(x) &= \langle A_t, \nabla_x \varphi\rangle +\\
&+ \int_{\mathbb{R}^d} \varphi(x + B\sigma(t, x)z) - \varphi(x) + \langle \nabla_x \varphi(x), B\sigma(t, x)z\rangle \mathbf{1}_{\{|z|\leq 1\}} \nu(dz),
\end{aligned}$$

where B is the injection matrix of $\mathbb{R}^d$ into $\mathbb{R}^{nd}$.

The Parametrix Series

We approximate the solution of (1) by the solution of the *Frozen equation*:

$$d\tilde{X}_s^{T,y} = A_s \tilde{X}_s^{T,y} ds + B\sigma(s, R_{s,T}(y))dZ_s. \qquad (4)$$

Denoting $\tilde{p}_\alpha$ the density of (4), and $P_{s,t}$ the transition of (1) ,we have for all $0 \leq t < T$, $(x, y) \in (\mathbb{R}^{nd})^2$ and any bounded measurable $f : \mathbb{R}^{nd} \to \mathbb{R}$:

$$P_{t,T} f(x) = \mathbb{E}[f(X_T)|X_t = x] = \int_{\mathbb{R}^{nd}} \left(\sum_{r=0}^{+\infty} (\tilde{p}_\alpha \otimes H^{(r)})(t, T, x, y)\right) f(y),$$

where H is the parametrix kernel:
$\forall 0 \leq t < T,\ (x, y) \in (\mathbb{R}^{nd})^2,\ H(t, T, x, y) := (L_t - \tilde{L}_t^{T,y})\tilde{p}_\alpha^{T,y}(t, T, x, y).$
The notation $\otimes$ stands for the time space convolution:

$$f \otimes g(t, T, x, y) = \int_t^T du \int_{\mathbb{R}^{nd}} dz f(t, u, x, z) g(u, T, z, y).$$

Main Result

Assume **[H]** holds. When $d(1 - n) + \alpha + 1 > 0$, for every $x \in \mathbb{R}^{nd}$, there is a unique solution to the martingale problem associated with the generator of (1). When $n = 2, d = 1$, the unique weak solution of (1) has a density with:
$\forall m \geq 1,\ \exists C := C([\mathbf{H}], T, K, m, \alpha) \geq 1,\ s.t.\ \forall 0 \leq t < s \leq T,\ \forall (x, y) \in (\mathbb{R}^2)^2,$

$$p(t, s, x, y) \leq C\bar{p}(t, s, x, y) \log(K \vee |(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}(x))|),$$

where

$$\bar{p}(t, s, x, y) = C \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{\{K \vee |(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}(x))|\}^{2+\alpha}} \Theta\Big(C(s - t)^{1/\alpha} |(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}(x))|\Big).$$

Eventually for $0 < T \leq T_0 := T_0([\mathbf{H}], K)$ small enough, the following diagonal lower bound holds:

$$\forall 0 \leq t < s \leq T,\ \forall (x, y) \in (\mathbb{R}^2)^2\ s.t.\ |(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}(x))| \leq K,\ p(t, s, x, y) \geq C^{-1}\det(\mathbb{T}_{s-t}^\alpha)^{-1}.$$

Transport of the initial condition and multi-scale property

Consider the simple case: $dX_t = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} X_t dt + \begin{pmatrix} dZ_t\\ 0\end{pmatrix}$. This writes:

$$\begin{aligned}
X_s^1 &= x_1 + Z_s,\\
X_s^2 &= x_2 + sx_1 + \int_0^s Z_t dt.
\end{aligned}$$

That is $X_s = R_s(x) + \begin{pmatrix} Z_s\\ \int_0^s Z_u du\end{pmatrix}$, where $R_s(x)$ denotes the transport of the initial condition by the Resolvent of the deterministic ODE associated with (1).
We can put all the component at the same scale by normalizing by

$$\mathbb{T}_s^\alpha = \begin{pmatrix} s^{\frac{1}{\alpha}} I_d & & 0\\ & \ddots & \\ 0 & & s^{n-1+\frac{1}{\alpha}} I_d\end{pmatrix}.$$

Typical Sets

In the degenerate framework, the typical behavior is given by

$$\left|(\mathbb{T}_s^\alpha)^{-1}(y - R_s x)\right| \asymp \frac{|y^1 - R_s^1 x|}{s^{\frac{1}{\alpha}}} + \cdots + \frac{|y^n - R_s^n x|}{s^{n-1+\frac{1}{\alpha}}}.$$

Multi-scale Stable Process

From [Watanabe (TAMS) 2007], we know that if the spectral measure is such that $\mu\big(B(x, r)\big) \leq Cr^{\gamma-1}$, then:

$$p_Z(t, x) \leq Ct^{-d/\alpha}\left(1 + \frac{|x|}{t^{/\alpha}}\right)^{-\gamma-\alpha}.$$

And $\tilde{X}_s^{T,y,t,x} \overset{(d)}{=} R_{s,t}x + (s - t)^{-1/\alpha}\mathbb{T}_{s-t}\mathcal{S}_{s-t}$, where $(\mathcal{S}_u)_{u\geq 0} \in \mathbb{R}^{nd}$, is a stable process with degenerate spectral measure, with support of dimension $d + \alpha + 1$ in $\mathbb{R}^{nd}$.