Answers of Master's Exam, Fall, 2006

1) a) $\dfrac{\binom{4}{0}\binom{4}{3}}{\binom{8}{3}} + \dfrac{\binom{4}{1}\binom{4}{2}}{\binom{8}{3}} = 28/56 = 1/2$

   b) (6/8)(5/7)(4/6)(3/5)(2/4)

   c) 0.0334 (using the ½ correction)

   d) 0.102 (using the ½ correction)

   e) 0.8008

2) a) $F_Y(y) = y^{3/2}$ for $0 \le y \le 1$, $f(y) = (3/2)\, y^{1/2}$ for $0 \le y \le 1$
   b) $F_M(m) = [F_X(m)]^2 = (1/4)(m^3 + 1)^2$ for $-1 \le m \le 1$
   c) $1/\sqrt{2}$
   d) $E(W) = [E(X_1)]^2 = 0^2 = 0$, $Var(W) = E(W^2) = [E(X_1^2)]^2 = 0.36$

   b) $F_M(m) = (1/16)\, m^4$ for $0 \le m \le 2$, 0 for $m < 0$, 1 for $m > 2$

   c) $F_Y(y) = F_X(1 + y^{1/2}) - F_X(1 - y^{1/2}) = (1/4)[(1 + y^{1/2})^2 - (1 - y^{1/2})^2]$
$= y^{1/2}$ for $0 \le y \le 1$. $f_Y(y) = (1/2)\, y^{-1/2}$ for $0 \le y \le 1$..

   d) $Var(X_1) = \sigma^2$, $Cov(X_1, 2 X_1 - 2) = 2 \sigma^2$, $Var(2 X_1 - X_2) = 5 \sigma^2$,
$\rho(X_1, 2 X_1 - X_2) = 2/\sqrt{5}$.

   e) $E(X_1 X_2) = E(X_1)\, E(X_2) = (4/3)^2$.

3) a) Let $X_1, X_2, \ldots$ be independent and identically distributed with mean μ.
Let $\bar{X}_n = \dfrac{X_1 + \ldots + X_n}{n}$ Then for any $\varepsilon > 0$

$$\lim_{n \to \infty} P(| \bar{X}_n - \mu | > \varepsilon) = 0$$

   b) Suppose that $Var(X_1) = \sigma^2 < \infty$. Then $P(| \bar{X}_n - \mu| > \varepsilon) \le Var(\bar{X}_n)/\varepsilon^2 = (\sigma^2 / n)/\varepsilon^2$, whose limit as n approaches infinity is zero.

4) a) $f_{XY}(x, y) = 2 x (1/x) = 2$ for $0 \le y \le x$, $0 < x < 1$.
Therefore, $f_Y(y) = \int_y^1 2\, dx = 2(1 - y)$ for $0 \le y < 1$, 0 otherwise.

   b) $(2 U - 1)^{1/3}$

c) $E(Y \mid X = x) = x/2$. . $E(Y \mid X)$ is obtained by replacing x by X.

d) In my opinion this problem should not have been asked. It's too "messy."

$F_{XY}(x, y) = 0$ if either x or y is less than zero.

$= x^2$ if $0 \le x \le y \le 1$

$= x^2$ if $0 \le x \le 1 \le y$

$= y^2 + 2(x - y)\, y$ if $0 \le y \le x \le 1$

$= y^2 + 2(1 - y)y$ if $0 \le y \le 1 \le x$

$= 1$ for $x \ge 1$ and $y \ge 1$

5) a) $B = (A_1 \cup A_2 \cup A_3) \cap (A_4 \cup A_5)$

b) Let $q_k = 1 - p_k$ for each k.

$P(B) = (1 - q_1 q_2 q_3)(1 - q_4 q_5)$

c) $p_2 / (1 - q_1 q_2 q_3)$

Statistics

6) 6) a) a) $\hat{\lambda} = 2n / \Sigma X_i^2$.

b) $\mu = E(X_1) = 2/\lambda$, so the method of moments estimator is $2/\bar{X}$.

7) 7) a) Let D_i = (Low carb loss) – (High-carb loss) for the ith person, i = 1, 2, … , 7

Suppose that the D_i constitute a random sample from the $N(\mu_D, \sigma_D^2)$ distribution.

Test H_0: $\mu_D \le 0$ vs H_a: $\mu_D > 0$. Reject for $t > 1.943$. $\bar{D}$ 0.7, $S_D^2 = 0.58667$,

$T = 2.418$. Reject H_0.

8) a) Let $\hat{p}_1$ and $\hat{p}_2$ be the sample proportions X_1/n_1 and X_2/n_2. $\hat{\Delta} = \hat{p}_1 - \hat{p}_2$. Then $Var(\hat{\Delta}) = p_1(1 - p_1)/n_1 + p_2(1 - p_2)/n_2$ and we can estimate this variance by replacing the p_i by their estimates. Call this estimator $\hat{\sigma}^2$. A 95% Confidence Interval:

$$[\hat{\Delta} \pm 1.645\ \hat{\sigma}]$$

b) $[-0.3 \pm 0.0493]$

c) In a large number, say 10000, repetitions of this experiments, always with samples sizes 500 and 400, about 95 % of the intervals obtained would contain the true parameter Δ.

d) 4802

8) a) 0.00317

b) Let $W = X_1 + X_2$ be the total number of The Neyman-Pearson Theorem states that the most powerful test of H_0: $p = 0.5$ vs H_a: $p = p_0$, where $p_0 < 0.5$ rejects for $\lambda = p_0^2 (1 - p_0)^{w-2} / 0.5^w (1 - .5)^{w-2} \geq k$ for some k , where $w = x_1 + x_2$, the number of successes. Since $p_0/0.5 < 1$, λ is a decreasing function of w, so, equivalently, we should reject for $w \leq k^*$, where k^* is some constant.

c) Power = 0.1584.

10) a) Let $Q(\beta) = \Sigma (Y_i - \beta / x_i)^2$. Taking the partial derivative wrt to β and setting the result equal to zero, we get $\hat{\beta}$ as given.

b) Replacing each Y_i by $\beta / x_i + \varepsilon_i$, we get $E(\hat{\beta}) = \beta + E(\Sigma \varepsilon_i / x_i) / \Sigma (1/x_i^2)) = \beta$.

c) $Var(\hat{\beta}) = \sigma^2 / \Sigma (1/x_i^2)$.

d) $\hat{\beta} = 2.3$, $S^2 = 1.1$ $[2.3 \pm 0.451]$

11) b) $Var(\mu^*) = \sigma^2 [1/(2^n - 1)] \Sigma 2^{2(n-i)} = \sigma^2 [1/(2^n - 1)] [4^n - 1]/3$

$Var(\bar{X}) = \sigma^2 / n$. The relative efficiency of μ^* to $\bar{X}$ is

$e(\mu^*, \bar{X}) = Var(\bar{X}) / Var(\mu^*) = (1/n) [1/(2^n - 1)] [4^n - 1]/3$. The limit as n approaches ∞ is ∞.