Answers of Master's Exam, Fall, 2008

To: Master's degree students.
From: Jim Stapleton, Professor Emeritus

I discovered, as I did the problems below, that some of these problems are quite difficult. I suggest that you not try to complete all of them. For example, 1a) and 4c) took me a long time to complete, especially 5c). In general I think the exam was quite long. I doubt very much that future exams will be even approximately as long.

1) a) $\frac{32!30!2^{27}}{60!}$ b) $\frac{\binom{32}{3}\binom{32}{7}}{\binom{64}{10}}$

2) a) 0.1505, b) 0.4756 c) 0.1495 d) 0.00605 (using ½ correction)

3) a) $\lim_{n\to\infty} F_n(x)$ = F(x) for every x on the real line at which F is continuous.

b) $F_n(x) = 0$ for $x < 0$, p_n for $0 \le x < (1+1/n)^n$, 1 for $x \ge (1+1/n)^n$.
$\lim_{n\to\infty} F_n(x)$ = 0 for $x < 0$, ½ for $0 < x \le e^1$, 1 for $x > e^1$. The function

F(x) = 0 for $x < 0$, ½ for $0 \le x < e^1$, 1 for $x \ge 1$ is a cdf and is this limit at all points except $x = e^1$, a point of discontinuity of F(x).

c) For every $\varepsilon > 0$ $\lim_{n\to\infty} P(| X_n - 1| < \varepsilon) = 1$.

4) a) $f_X(x) = (4/ 3 \pi) (4 - x^2)^{1/2}$ for $1 \le x \le 4$,
$= (4/3 \pi)((4 - x^2)^{1/2} - (1 - x^2))^{1/2}$ for $0 \le x < 1$

b) $1/(4 - x^2)^{1/2}$ for $0 \le y \le (4 - x^2)^{1/2}$, $1 < x < 2$.

c) $E(Y \mid X = x) = (1/2) (4- x^2)^{1/2}$ for $1 < x < 2$
$= (1/2)((4 - x^2)^{1/2} - (1 - x^2)^{1/2})$ for $0 \le x < 1$. To get Z simply replace x by X.

5) a) $U^{1/3}$
b) $F_M(m) = 1 - (1 - m^2)(1 - m^3)(1 - m)$ for $0 \le m \le 1$.
The derivative with respect to m is the density. It's messy, too messy to write here.
c) $F_S(s) = (s + 1)^2 - 2/5 - s/2 + s^5/10$ for $-1 < s \le 0$
$= 1 - 2[(1/5)(1 - s)^5 + (s/4)(1 - s)^4]$ for $0 \le s \le 1$
This is a difficult problem, far too long for this exam. Don't spend much time on it.
d) $F_Y(y) = (e^y - 1)^2$ for $0 < y \le \ln(2)$.

$f_Y(y) = 2(e^y – 1)\ e^y$ for $0 < y \le \ln(2)$

e) $Var(X_1) = \sigma_1^2 = 1/18$, $Var(X_2) = \sigma_2^2 = 3/80$, $Var(X_3) = \sigma_3^2 = 1/12$
Let Y_1 and Y_2 be the two linear combinations.
$Cov(Y_1, Y_2) = -18\ \sigma_1^2 - 50\ \sigma_2^2 + 18\ \sigma_3^2 = - 7.8333$
f) Let $W = X_1^{1/2}\ X_2 1/3$ =

6) a) MLE is $\hat{\sigma} = (1/n)\ \Sigma\ (X_i – 1) = \bar{X} - 1$.
b) Same as answer to a).
c) $I(\sigma) = 1/\sigma^2$, so $1/nI(\sigma) = \sigma^2/n$.

7) a) $f_1(x)/f_0(x) = (5/3)\ x^2$ for $0 \le x \le 1$. Reject for large X, say $X \ge k$.
Take $k = (1 - \alpha)^{1/3}$ so that $P(X \ge k) = \alpha$. b) Neyman-Pearson
c) Power $= 1 – (1 - \alpha)^{5/3}$

8) a) Let $Y_1, \dots , Y_6$ be the control observations. Let $X_1, \dots , X_6$ be the steroid observations. Suppose that the 12 random variables are independent. Suppose that $Y_1, \dots , Y_6$ have cdf F_1 and $X_1, \dots, X_6$ have cdf F_2. (The subscripts could be reversed, otr the cdf's could be called F and G.)
H_0: $F_1(x) = F_2(x)$ for all x, or simply $F_1 = F_2$. H_a: H_0 not true.
b) The ranks of the X_i 's are 2, 6, 12, 11, 5, 9, so the Wilcoxon statistic is $W_X = 45$.

Under H_0 $E(W_X) = 6.5(6) = 39$, and $Var(W_X) = 6(6)(13/12) = 39$, $P(W_X \ge 45 \mid H_0)$
$\doteq 1 - \Phi((44.5 – 39)/ 39^{1/2}) = 1 - \Phi(0.8807) = 0.189$, so that the p-value is 0.378. Do not reject at the $\alpha = 0.1$ level.

9) Let D_i = Expenditure – Intake for the ith player, $i = 1, \dots , 7$

Suppose that the D_i's are a random sample from the $N(\mu_D, \sigma_D^2)$ distribution.

H_0: $\mu_D = 0$, H_a: $\mu_D \ne 0$,

$T = (\bar{D} - 0)/[S_D^2/7]^{1/2} = -1.7714/(13.782/7)^{1/2} = -1.2624$.

p-value = 2(0.1268) = 0.2536. Do not reject at the 0.10 level.

10) a) Let $Q(\beta) = \Sigma\ (Y_i - \beta\ x_i^{1/2})^2$. Differentiating wrt to β, we get

$-\Sigma\ x_i^{1/2}(Y_i - \beta\ x_i^{1/2})$. This is zero for $\beta = \hat{\beta} = \Sigma\ x_i^{1/2}\ Y_i / \Sigma\ x_i$.

b) $\hat{\beta} = \Sigma\ x_i^{1/2}(\beta\ x_i^{1/2} + \varepsilon_i) / \Sigma\ x_i^2 = \beta + \Sigma\ x_i^{1/2}\ \varepsilon_i / \Sigma\ x_i$. The second term has expectation zero because $E(\varepsilon_i) = 0$ for each i.

c) $\mathrm{Var}(\hat{\beta}) = (1/\Sigma x_i)^2 \Sigma x_i \sigma^2 = \sigma^2 / \Sigma x_i$.

d) Replacing each $x_i^{1/2}$ by $(c\, x_i^{1/2})$ we get $\hat{\beta}^* = (c^{1/2}/c)\,\hat{\beta} = \hat{\beta}\ c^{-1/2}$.

11) a) Let $\hat{p}_1 = X_1/n_1$ and $\hat{p}_2 = X_2/n_2$, $\hat{\Delta} = 3\,\hat{p}_1 - 2\,\hat{p}_2$, $E(\hat{\Delta}) = \Delta$, $\mathrm{Var}(\hat{\Delta})$

$= (9\, p_1 q_1/n_1 + 4\, p_2 q_2/n_2)$, where $q_i = 1 - p_i$ for $i = 1, 2$.

$Z = (\hat{\Delta} - \Delta)/\mathrm{Var}(\hat{\Delta})^{1/2}$ is approximately distributed as N(0, 1). Replacing the p_i by their estimates in $\hat{\mathrm{V}}\mathrm{ar}\,(\hat{\Delta})$ to get $\hat{Z}$, using Slutsky's Theorem, we conclude that $\hat{Z}$ is approximately N(0, 1).

Therefore the 95% Confidence Interval is $[\hat{\Delta} \pm 1.96\, [\hat{\mathrm{V}}\mathrm{ar}\,(\hat{\Delta})]^{1/2}$.

c) The procedure used to determine the interval has the property that 95% of all possible intervals determined when random samples are taken will produce intervals containing the parameter Δ. We do not know whether this interval contains the parameter.

d) We need n large enough to have $(9\, p_1 q_1/n + 4\, p_2 q_2/n) \le [0.1/1.96]^2$. This is largest for $p_1 = p_2 = ½$, so we need $(1/n)((9/4 + 4/4) = 13/4n < [0.1/1.96]^2$, $n > [1.96/0.1]^2\,(13/4) = 1248$

12) Let X_{ij} be the frequency in cell ij. Suppose that the 6 X_{ij} have the multinomial distribution with parameters pij. We wish to test H_0 : Rows and Columns are independent. We get the estimates

102	62	37
51	31	17.

Pearson's chi-square statistic is 2.472. The 0.90 quantile of the chi=square distribution with (2-1)(3-1) = 2 df is 4.605, so we fail to reject H_0.