Preliminary Exam: Probability
9:00am - 2:00pm, August 23, 2013
The exam lasts from 9:00am until 2:00pm.
Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step. On you work, label the steps this way: (i), (ii),...
On each page you turn in, write your assigned code number instead of your name. Separate and staple each main part and return each in its designated folder.

Problem 1 **(14 pts)**. Let X be a random variable and denote

$\varphi(\theta) \equiv E\left(e^{\theta X}\right), \theta \in \mathbb{R}$. Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables and denote $S_n = \sum_{k=1}^{n} X_k, n \geq 1$.

a. **(8 points)**

(i). Let a be a real number. Prove that for each $\theta > 0,\ n \geq 1$ we have

$$P(S_n \geq na) \leq \frac{\varphi^n(\theta)}{e^{n\theta a}}.$$

(ii). Show how to modify the LHS of the inequality so that it will hold, with the same RHS, for $\theta < 0$.

b. **(6 points)**

Let $X \sim Poisson(\lambda), \lambda > 0$.

(i). Calculate $\varphi(\theta)$.

(ii). Assume that $a > \lambda$. Use part a to find the smallest upper bound that you can for

$$\operatorname{limsup}_n \frac{\log[P(S_n \geq na)]}{n}.$$

Explain why the assumption $a > \lambda$ is important.

Problem 2 **(14 pts)**. This problem deals with positive random variables that may get the value ∞. Let $X \geq 0$ be a random variable defined on a probability space $(\Omega, \mathcal{G}, P)$. Let $\mathcal{F} \subset \mathcal{G}$ be a σ-algebra. Prove

a. **(6 points)**
   (i) $Y \equiv \lim_n E_{\mathcal{F}}(X \wedge n), a.s.$ exist, and
   (ii). $Y \in \mathcal{F}$ and $E(Y: A) = E(X: A),\ A \in \mathcal{F}$.
   (Namely, we can write $Y = E_{\mathcal{F}}(X)$)

b. **(8 points)**
   If $Z \geq 0$ is a random variable that satisfy (ii) of part a, namely: $Z \in \mathcal{F}$ and $E(Z: A) = E(X: A),\ A \in \mathcal{F}$, then $Y = Z, a.s.$

Problem 3 **(18 pts)**. Let $\{X_k:\ k \geq 1\}$ be a sequence of **integrable** random variables defined on $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_k\}$ be an increasing sequence of σ-algebras, $\mathcal{F}_k \subset \mathcal{F}, k \geq 1$.

a. **(5 points)**

Let τ be a random variable whose values belong to $\{1, 2, \ldots\}$. Prove that $\{X_{\tau\wedge k}:\ k \geq 1\}$ is a sequence of **integrable** random variables, i.e. $X_{\tau\wedge k} \in \mathcal{F}$ and $E|X_{\tau\wedge k}| < \infty, k \geq 1$.

b. **(7 points)**

Let τ be a stopping time with respect to $\{\mathcal{F}_k\}$. Prove that the following holds:

$$E_{\mathcal{F}_k}\left(X_{\tau\wedge(k+1)} - X_{\tau\wedge k}\right) = 1_{\{\tau\geq k+1\}} E_{\mathcal{F}_k}(X_{k+1} - X_k),\ \ k \geq 1.$$

c. **(6 points)**

Assume that $\{X_k, \mathcal{F}_k:\ k \geq 1\}$ is adapted. Let $\{\tau_n\}, n = 0, 1, \ldots,$ be a monotone increasing sequence of stopping-times with respect to $\{\mathcal{F}_k\}$ so that $\tau_n \to \infty$, a.s. **Use part b** to show that if $\{X_{\tau_n\wedge k}, \mathcal{F}_k:\ k \geq 1\}$ is a supermartingale for each *n*, then $\{X_k, \mathcal{F}_k:\ k \geq 1\}$ is a supermartingale as well.

Problem 4 **(19 pts)**. Let $\{W(t):\ 0 \le t \le 1\}$ denote a standard Brownian motion. Denote $W(s,t) \equiv W(t) - W(s),\ 0 < s < t < 1$.

a. **(5 points)** Let $0 \equiv t_0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1} \equiv 1$. Prove that there is a constant $C > 0$ that doesn't depend on n or $\{t_i\}$ so that:

$$E[\{\textstyle\sum_{i=0}^{n} W^2(t_i, t_{i+1})\} - 1]^2 \le C \cdot \max_{0\le i\le n}|t_{i+1} - t_i|.$$

Hint: $1 = \sum_{i=0}^{n}(t_{i+1} - t_i)$.

For the rest of the problem let $(q_i), i \ge 1$, denote the rational numbers in $(0,1)$ and for each $n \ge 1$ let the order statistics of $\{q_i:\ 1 \le i \le n\}$ be denoted by: $0 \equiv q_{(n,0)} < q_{(n,1)} < \cdots < q_{(n,n)} < q_{(n,n+1)} \equiv 1$

[Example: If $q_1 = .5, q_2 = .33, q_3 = .16$ then $q_{(3,0)} = 0, q_{(3,1)} = .16$, $q_{(3,2)} = .33$, $q_{(3,3)} = .5$, $q_{(3,4)} = 1$.]

b. **(5 points)**

(i). Prove: $\sum_{i=0}^{n} W^2\big(q_{(n,i)}, q_{(n,i+1)}\big) \to 1$, in probability as $n \to \infty$.

(ii). How can you improve the result in (i) if there exist $\{\mathcal{F}_n\}$, a decreasing sequence of σ-algebras, so that $\{\sum_{i=0}^{n} W^2\big(q_{(n,i)}, q_{(n,i+1)}\big), \mathcal{F}_n\}$ is a backwards martingale?

c. (5 points) Let $0 < s < u < t < 1$ and denote: $\mathcal{F} \equiv \sigma\{|W(s,u)|,\ |W(u,t)|\}$.
Find

(i) $E_{\mathcal{F}}(W(s,u)W(u,t))$.

Hint. Is the random vector $(\text{sign}W(s,u), \text{sign}W(u,t))$ independent of $\mathcal{F}$?

(ii). $E_{\mathcal{F}}(W^2(s,t))$.

d. **(4 points)**

Define a decreasing sequence of σ-algebras, $\{\mathcal{F}_n\}$, so that $\{\sum_{i=0}^{n} W^2\big(q_{(n,i)}, q_{(n,i+1)}\big), \mathcal{F}_n\}$ is a backwards martingale.

Problem 5 **(18 pts)**. Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables. We assume that $= W \cdot Z$, where W and Z are independent, $P(|W| \geq w) = \frac{1}{w^2}, w \geq 1$ and $P(Z = \pm 1) = 1/2$.

a. **(6 points)**
   Find the density of X.
   Hint: Is X symmetric?

b. **(4 points)**
   Calculate $E(X)$ and $E(X^2)$.

c. **(8 points)**
   Define $X_{n,k} = \frac{X_k}{\sqrt{nlog(n)}}, \ 1 \leq k \leq n, \ n \geq 1$ and find the following limits
   (i). $\lim_n \sum_{k=1}^n P(|X_{n,k}| \geq \ \mathrm{x})$, where $x > 0$.
   (ii). $\lim_{n\to\infty} \sum_{k=1}^n E(X_{n,k} : |X_{n,k}| \leq \ \epsilon)$, $\epsilon > 0$.
   (iii). $\lim_{\epsilon\to 0} \lim_{n\to\infty} \sum_{k=1}^n Var(X_{n,k} : |X_{n,k}| \leq \ \epsilon)$.

Remark. The calculations in part c lead to convergence in distribution of $\frac{\sum_{k=1}^n X_k}{\sqrt{nlog(n)}}$ via the generalized CLT.

Problem 6 **(17 pts)**. Let $X_1, X_2, \ldots$ be random variables and denote

$S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. Let $a > 0$. Prove:

a. **(7 points)**

$P(|S_n - S_m| > a) \geq$ (I), where

(I)$\equiv \sum_{k=m+1}^{n} P\left\{\max_{m<j<k}\{|S_j - S_m|\} \leq 2a, |S_k - S_m| > 2a, |S_n - S_k| \leq a\right\}$

Hint: Are the $n - m$ events in (I) disjoint? Try to express in words what (I) represents.

From now on assume that $X_1, X_2, \ldots$ are **independent**.

b. **(6 points)**

Prove: (I)$\geq P\left(\max_{m<k\leq n}\{|S_k - S_m|\} > 2a\right) \cdot \min_{m<k\leq n} P(|S_n - S_k| \leq a)$

c. **(4 points)**

If S_n converge in probability then $\max_{m<k\leq n}\{|S_k - S_m|\} \to 0$ in probability as $n, m \to \infty$.

Hint: Prove first that $\min_{m<k\leq n} P(|S_n - S_k| \leq a) \to 1$ as $n, m \to \infty$.

Then combine parts a and b.

Remark: It follows from part c that S_n converge a.s.