Preliminary Exam: Probability
9:00am - 2:00pm, August 22, 2014
Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution. If you cannot justify a certain step, you still may use it in a later step. On you work, label the steps this way: (i), (ii),...
On each page you turn in, write your assigned code number instead of your name.
Separate and staple each main part and return each in its designated folder.

Problem 1. Let X be a nonnegative random variable defined on a probability space $(\Omega, \mathcal{G}, \mathrm{P})$. Let $\mathcal{F} \subset \mathcal{G}$ be a σ-algebra.

a. (3 pts.) Prove that there is a unique random variable, denoted by $E_{\mathcal{F}}(X)$, that is $\mathcal{F}$ measurable and satisfies
(i) $0 \le E_{\mathcal{F}}(X) \le \infty$, and (ii) $E(X\colon A) = E(E_{\mathcal{F}}(X)\colon A),\ A \in \mathcal{F}$.

Remark: The case $E(X) < \infty$ is done in the book, so focus on the case $E(X) = \infty$.

For the rest of the problem let $\{X_n, n = 1,2, \dots\}$ denote a sequence of nonnegative random variables defined on $(\Omega, \mathcal{G}, \mathrm{P})$ that is monotone decreasing in *n*, a.s.

b. (3 pts.) Prove that $\lim_{n\to\infty} E_{\mathcal{F}}(X_n)$ exists a.s.

c. (4 pts.) Prove that $\lim_{n\to\infty} E(X_n) \ge E(\lim_{n\to\infty} E_{\mathcal{F}}(X_n))$.

Problem 2. Let $\{B_s\colon s \ge 0\}$ denote a standard Brownian motion. For any process $\{X_s\colon s \ge 0\}$ define the trailing maximum process by $M_t^X = \max_{0\le s\le t} X_s$, $t \ge 0$.

a. (7 pts.) Prove that for each $t > 0$ we have $\{B_s\colon 0 \le s \le t\} = \{Y_s \equiv B_{t-s} - B_t\colon 0 \le s \le t\}$ in distribution.
b. (6 pts.) Prove that for each $t > 0$ we have
(i) $M_t^Y = M_t^B - B_t$, a.s. and
(ii) $M_t^B - B_t = M_t^B = |\, B_t \,|$, where the 2 equalities are in distribution.
c. (6 pts.) For fixed $t > 0$ define $\tau \equiv \inf\{ u \ge t\colon B_u = M_t^B\}$ and $\alpha \equiv \inf\{ u \ge t\colon B_u = 0\}$ (in words: τ is the 1st time after *t* in which the process *B* returns to its maximum before *t*, while α is the time of the first 0 after t). Prove that τ and α are identically distributed.

Problem 3. Let $\{X_n, n = 1,2,\dots\}$ denote a sequence of random variables defined on a probability space. Assume that $E(X_n) = 0, E(X_n^2) = 1, \;\; n = 1,2,\dots$

a. (6 pts.) Assume that $\{X_n\}$ are identically distributed. Prove that

$$\max_{1\le i\le n}\left\{\left|\frac{X_i}{\sqrt{n}}\right|\right\} \underset{n\to\infty}{\Longrightarrow} 0$$

b. (6 pts.) Assume that $\{X_n\}$ are independent and identically distributed. Prove that

$$\prod_{1\le i\le n}\left(1+\frac{X_i}{\sqrt{n}}\right) \underset{n\to\infty}{\Longrightarrow} e^{Z-0.5}$$

where $Z \sim N(0,1)$. Hint: Use part a to justify the existence of $\log\left(1+\frac{X_i}{\sqrt{n}}\right), 1 \le i \le n$ for large n and use an appropriate expansion of $\log(1+x), x > -1$.

Problem 4. Let $\{X, X_n, n = 1,2,\dots\}$ denote a sequence of random variables that are integer valued.

a. (5 pts.) Prove that $X_n \underset{n\to\infty}{\Longrightarrow} X$ if and only if $P(X_n = k) \xrightarrow[]{n\to\infty} P(X = k),\; k \in \mathbb{Z}$.

For the rest of the problem assume $X_n \underset{n\to\infty}{\Longrightarrow} X$. Also, denote

$$p_k \equiv P(X = k), \quad p_{n,k} \equiv P(X_n = k).$$

b. (5 pts.) Prove
   (i) $\sum_k (p_k - p_{n,k})^+ \xrightarrow[]{n\to\infty} 0$
   (ii) $\sum_k (p_k - p_{n,k})^+ = \sum_k (p_k - p_{n,k})^- \;, n = 1,2,\dots$

c. (5 pts.) Prove: $\sum_k |\,p_k - p_{n,k}| \xrightarrow[]{n\to\infty} 0$s

Problem 5. Let $\{X_n, n = 1,2,\dots\}$ denote a sequence of nonnegative, strictly stationary and ergodic random variables.

a. (6 pts.) Let $M > 0$ be a constant. Prove that the sequence $\{X_n \wedge M, n = 1,2,\dots\}$ is strictly stationary and ergodic.

b. (6 pts.) Assume that $\frac{\sum_{k=1}^{n} X_k}{n} \xrightarrow[]{n\to\infty} c < \infty$, a.s. Prove that $E(X_1) < \infty$.

Problem 6. Let $\{M_n, n = 0,1,2,\dots\}$ be a squared-integrable martingale sequence adapted to the filtration $\mathcal{F} = \{\mathcal{F}_n, n = 0,1,2,\dots\}$.

a. (4 pts.) Prove that $\{M_n^2\}$ is submartingale with respect to (w.r.t) $\mathcal{F}$.
b. (8 pts.) Doob's decomposition says that there is a representation $M_n^2 = X_n + Y_n, n = 0,1,\dots$, where $\{X_n\}$ is a martingale w.r.t $\mathcal{F}$ and $\{Y_n\}$ is a sequence that is increasing in n, predictable w.r.t $\mathcal{F}$ (i.e. $Y_n \in \mathcal{F}_{n-1}$) with $Y_0 \equiv 0$. Show how to calculate $\{X_n\}$ and $\{Y_n\}$ from $\{M_n\}$.