STT 872, 867-868 Fall Preliminary Examination
Wednesday, August 26, 2015
12:30 - 5:30 pm

1. Let Y have Gamma distribution $\Gamma(m, \theta)$, i.e. the density is $f(y) = \frac{y^{m-1}e^{-\frac{y}{\theta}}}{\Gamma(m)\theta^m}$ for $y > 0$ and 0 otherwise. Prove that $\mathbb{E}Y^r = \frac{\theta^r\Gamma(m+r)}{\Gamma(m)}, r > -m$.

2. Let $X_1, ..., X_n$ be iid from the density $\lambda x^{\lambda-1}, \lambda > 0, 0 < x < 1$.

(a) Find a complete and sufficient statistics for λ based on $X_1, ..., X_n$.

(b) Let $g(\lambda) = \lambda^{-2}$. Formulate the two methods of finding UMVUE and use one of them to find it for $g(\lambda)$.

(c) Show that $\frac{-n}{\sum_{i=1}^n lnX_i}$ is the MLE of λ. Compute its asymptotical variance, $n > 1$.

(d) Find the MLE of λ^{-2}. Using delta-method find its asymptotical variance, $n > 1$.

(e) Let $g(\lambda) = \lambda^{-2}$ and let the loss function be $\mathbb{E}\{(\lambda^2\delta(X_1, ..., X_n) - 1)^2\}$. Using an exponential prior with parameter $\mu > 0$, $E(\mu)$, find a Bayes estimator of $g(\lambda)$, $n > 1$. Hint: $\Lambda \sim \mu e^{-\mu\lambda}, \lambda > 0$.

(f) Show that the loss of the Bayes estimator in (e) is $\frac{2(2n+5)}{(n+3)(n+4)}$. Which conditions on μ would make the estimator in (e) minimax. Are they satisfied? $n > 1$.

(g) Formulate the conditions needed for an estimator of λ^{-2} to be asymptotically efficient. Are they satisfied for the Bayes estimator in (e) with $\mu = \sqrt{n}$.

(h) Under the loss $\mathbb{E}\{(\lambda^2\delta(X_1, ..., X_n) - 1)^2\}$, show that the MLE of λ^{-2} is inadmissible when n is large enough.

(i) Derive a UMP unbiased test of size $\alpha \in (0, 1)$ for testing $H : \lambda = 2$ vs $K : \lambda \neq 2$ in the fullest possible detail, $n > 1$.

3. Let $X_1, ..., X_n$ be iid from the density $1/\tau$ for $\theta \leq x \leq \theta + \tau$ and 0 otherwise.

(a) Construct the MRE estimator of τ under the loss $\mathbb{E}(\frac{\delta}{\tau} - 1)^2$, $n > 2$.

(b) Construct the MRE estimator of θ under the loss $\mathbb{E}(\delta - \theta)^2$, $n > 2$.

(c) Is the estimator in (b) consistent in probability for θ?

(d) Suppose $\tau > 1$ is known. Derive a UMP test of size $\alpha \in (0, 1)$ for testing $H : \theta = 1$ vs $K : \theta < 1$ in the fullest possible detail.

(e) Derive the power of the test in (d).

4. Consider the following linear regression model

$$Y_{ij} = \mu + \theta_i + (\gamma + \beta_i)x_{ij} + e_{ij} \text{ where } e_{ij} \overset{iid}{\sim} N(0, \sigma^2) \tag{1}$$

for $i = 1, 2, 3$ and $j = 1, \cdots, 4$, where x_{ij} are constants that are unequal to each other.

(a) Find the least square (LS) estimator of $\mu + \theta_1$. Can you find another unbiased estimator which has smaller variance than the LS estimator? Why or why not?

(b) Under the constraint $\beta_1 = \beta_2 = \beta_3$, find the best linear unbiased estimator of $\mu + \theta_1$.

(c) Give the variance of the LS estimator of $\mu + \theta_1$ in part (a) and the variance of the BLUE in $\mu + \theta_1$ in part (b). Which one has the smaller variance? Why?

(d) Construct an F-test for the hypothesis $H_0 : \theta_1 = \theta_2 = \theta_3$ vs H_1 : on the equality is not true in H_0 under model (1). Please provide the F-statistic and its distribution under H_0, and an α level rejection region.

(e) Under the constraint $\beta_1 = \beta_2 = \beta_3$, construct an F-test for testing the hypothesis in part (d). Please provide the F-statistic and its distribution under H_0, and an α level rejection region.

5. We still consider model (1) given in question 4 but assume that θ_i and β_i are random effects. More specifically, we assume the following mixed effects model:

$$Y_{ij} = \mu + \theta_i + (\gamma + \beta_i)x_{ij} + e_{ij} \tag{2}$$

for $i = 1, 2, 3$ and $j = 1, \cdots, 4$. In model (2), assume that μ and γ are fixed effects. In addition, assume that $\theta_i \overset{iid}{\sim} N(0, \sigma_\theta^2)$, $\beta_i \overset{iid}{\sim} N(0, \sigma_\beta^2)$ and $e_{ij} \overset{iid}{\sim} N(0, \sigma^2)$. The random effects θ_i, β_i and random errors e_{ij} are independent. Assume that the variance ratios σ_α^2/σ^2 and σ_β^2/σ^2 are known.

(a) Give the best linear unbiased prediction for $\mu + \theta_1$.

(b) Derive the restricted maximum likelihood estimator of σ^2.

(c) Please provide a $1 - \alpha$ prediction interval for $\mu + \theta_1$.