STT-Research Memoranda #693

On asymptotic distributions of weighted sums of periodograms

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Abstract

We establish asymptotic normality of weighted sums of periodograms of a stationary linear process where weights depend on the sample size. Such sums appear in numerous statistical applications and can be regarded as a discretized versions of the quadratic forms involving integrals of weighted periodograms. Conditions for asymptotic normality of these weighted sums are simple and resemble Lindeberg-Feller condition for weighted sums of independent and identically distributed random variables. Our results are valid for short, long or negative memory processes. The proof is based on sharp bounds derived for Bartlett type approximation of these sums by the corresponding sums of weighted periodograms of independent and identically distributed random variables.

Keywords. Bartlett approximation; Linear process; Quadratic forms; Lindeberg-Feller.

1 Introduction

Let $X_j, j = 0, \pm 1, \cdots$, be a stationary process with a spectral density f_X and let $u_j = 2\pi j/n$, $j = 1, \cdots, [n/2]$, denote discrete Fourier frequencies. In this paper we develop asymptotic distribution theory for the weighted sums

$$\sum_{j=1}^{[n/2]} b_{n,j} I_X(u_j), \quad n \geq 1, \tag{1.1}$$

of periodograms $I_X(u_j) = \left|(2\pi n)^{-1} \sum_{t=1}^{n} e^{\mathbf{i}tu_j} X_t\right|^2$, where $b_{n,j}$ are real weights. In particular, if $b_{n,j} = b_n(u_j)$ where $b_n, n \geq 1$ is a sequence of real valued functions on $\Pi := [-\pi, \pi]$, these sums are estimates of $\sum_{j=1}^{[n/2]} b_n(u_j) f_X(u_j)$ and can be viewed as discretized versions of the integrals

$$\mathcal{I}_n := \int_0^\pi b_n(u) I_X(u) du.$$

Integrals $\mathcal{I}_n$ arise naturally in many situations in statistical inference. For example, the spectral distribution function F can be written as $F(y) = \int_{-\pi}^{y} f_X(u) du$, and the auto-covariance function of $\{X_j\}$ is

$$\mathrm{Cov}(X_k, X_0) \;=\; 2\int_0^\pi \cos(ku) f_X(u) du, \quad k = 0, 1, 2, \cdots.$$

In these two examples b does not depend on n. If one wishes to estimate $f_X(u_0)$ at a point $u_0 \in (0, \pi)$ by kernel smoothing method, then b will typically depend on n.

In the case when b does not depend on n and $\{X_j\}$ is a stationary Gaussian or linear process, asymptotic distribution theory of $\mathcal{I}_n$ is well understood and investigated both for short memory and long memory linear processes: for asymptotic normality results see Hannan (1973), Fox and Taqqu (1987), Giraitis and Surgailis (1990) and Giraitis and Taqqu (1998); for non-Gaussian limits see Terrin and Taqqu (1990) and Giraitis, Taqqu and Terrin (1998). Simple sufficient conditions for central limit theorem (CLT) of quadratic forms that can be written as a sequence of multiple stochastic integrals can be found in Nualart and Peccati (2005).

It is perhaps worth pointing out that even in the case when b does not depend on n, investigation of limit distribution of $\mathcal{I}_n$ is technically involved. As is evident from the works of Hannan (1973) and Bhansali, Giraitis and Kokoszka (2007b), deriving asymptotic distribution of $\mathcal{I}_n$ in case of general weight sequences b_n depending on n will be prohibitively complicated, and conditions for asymptotic normality will lack desirable simplicity.

In comparison, the verification of asymptotic normality of weighted sums of periodograms is relatively simple. In sections 2 and 3 below we provide theoretical tools to establish asymptotic normality of

$$\mathcal{D}_n := \sum_{j=1}^{[n/2]} b_{n,j} I_X(u_j) - \sum_{j=1}^{[n/2]} b_{n,j} f_X(u_j)$$

and to evaluate the mean-squared error $E\mathcal{D}_n^2$, when $\{X_j\}$ is a stationary linear process with i.i.d. innovations, possibly having long memory. Our conditions for asymptotic normality of these weighted sums are formulated in terms of $\{b_{n,j},\ f_X(u_j)\}$. They are simple and resemble Lindeberg-Feller type condition for weighted sums of i.i.d. r.v.'s, regardless of the dependence structure of the process $\{X_j\}$, which can be short, long or negative memory. In contrast, conditions for asymptotic normality of integrated periodogram $\mathcal{I}_n$ given in Bhansali, Giraitis and Kokoszka (2007b) cannot be expressed directly in terms of weights b_n and spectral density f_X. They are more involved and their verification requires significant technical effort.

Secondly, important benefit of using discretization is the translation invariance property of the periodogram, viz, $I_{X+c}(u_j) = I_X(u_j)$, for any $c \in \mathbb{R}$, and $j = 1, \cdots, n-1$, i.e. the data is automatically de-meaned.

A number of papers in the literature deal with more general quadratic forms (sums of weighted periodograms). Generalizations of $\mathcal{D}_n$ usually includes relaxing assumption of linearity of $\{X_j\}$. Hsing and Wu (2004) obtain asymptotic normality of a quadratic form $\sum_{t,s=1}^n b_{t-s} K(X_t, X_s)$ for a non-linear transform K of a linear process $\{X_j\}$ under a set of complex conditions that do not provide a direct answer in term of $\{b_t\}$, K and $\{X_j\}$. Moreover, their weights b_t's are not allowed to depend on n.

Shao and Wu (2007a) derive CLT for discrete Fourier transforms and spectral density estimates under some restrictions on dependence structure of $\{X_j\}$ based on conditional moments. Liu and Wu (2010) consider nonparametric estimation of spectral density of a stationary process using m-dependent approximation of X_j's. Wu and Shao (2007b) establish the CLT for quadratic forms with weights depending on n using martingale approximation method. Generality of these papers requires verification of a number of complex technical conditions which impose *a priori* a rate condition in approximations, that must be verified in each specific case. For example, Shao and Wu (2007a) requires geometric-contraction condition, which implies exponential decay of the autocovariance $\gamma(k)$ function of $\{X_j\}$, whereas in Liu and Wu (2010) the dependence is restricted assuming summability of $|\gamma(k)|$ and the use of a coupling argument. Both papers also restrict the set of $b_{n,j}$'s to specific weights appearing in kernel estimation. Such structural assumptions may be easier to verify than verifying mixing conditions, but they are redundant, not informative and too restrictive in the case when $\{X_j\}$ is a linear process. The present paper establishes asymptotic normality of $\mathcal{D}_n$ in the latter case under minimal conditions, which allow for all types of dependence in $\{X_j\}$ and arbitrary weights $b_{n,j}$ as along as $f_X(u_j)b_{n,j}$'s satisfy condition of uniform negligibility, e.g. (3.6). The main tool of the proof is Bartlett type approximation for discrete Fourier transforms of X_j's which is essentially different from the methods of approximations used in the above works. The obtained conditions are close to being necessary, and simple and easy to verify.

Assumptions. Accordingly, let $\mathbb{Z} := \{0, \pm 1, \cdots\}$,

$$X_j = \sum_{k=0}^{\infty} a_k \zeta_{j-k}, \quad j \in \mathbb{Z}, \quad \sum_{k=0}^{\infty} a_k^2 < \infty, \tag{1.2}$$

be a linear process where $\{\zeta_j,\ j \in \mathbb{Z}\}$ are i.i.d. standardized r.v.'s.

We assume that the spectral density f_X of the process $X_j, j \in \mathbb{Z}$, satisfies

$$f_X(u) = |u|^{-2d} g(u), \quad |u| \le \pi, \tag{1.3}$$

for some $|d| < 1/2$, where $g(u)$ is a continuous function satisfying

$$0 < C_1 \le g(u) \le C_2 < \infty, \quad u \in \Pi, \quad (\exists\, 0 < C_1, C_2 < \infty).$$

As was shown by Hosking (1981), ARFIMA(p, d, q) model satisfies this assumption for all $d \in (-1/2, 1/2)$.

Condition (1.3) allows to derive the mean square error bounds of estimates which is discussed in Theorem 3.3. To derive asymptotic normality and some delicate Bartlett type approximations, we shall additionally need to assume that the transfer function $A_X(u) := \sum_{k=0}^{\infty} e^{-\mathbf{i}ku} a_k$, $u \in \Pi$, is differentiable in $(0,\pi)$ and its derivative $\dot{A}_X$ has the property

$$|\dot{A}_X(u)| \le C|u|^{-1-d}, \quad u \in \Pi. \tag{1.4}$$

Since $f_X(u) = |A_X(u)|^2/2\pi$, $u \in \Pi$, assumptions (1.3) and (1.4) imply

$$|\dot{f}_X(u)| \;\; \le \;\; C\,|u|^{-1-2d}, \qquad u \in \Pi. \tag{1.5}$$

Conditions (1.3) and (1.4) are formulated this way to cover long and negative memory models, with $|d| < 1/2$, $d \ne 0$. They allow spectral density to vanish or to have a singularity point at zero frequency. The standard case where functions f_X and A_X are Lipshitz continuous and bounded away from 0 and ∞ is discussed in section 3.

To proceed further, let

$$w_{X,j} = \frac{1}{\sqrt{2\pi n}} \sum_{k=1}^{n} e^{\mathbf{i}u_j k} X_k, \qquad w_{\zeta,j} = \frac{1}{\sqrt{2\pi n}} \sum_{k=1}^{n} e^{\mathbf{i}u_j k} \zeta_k, \tag{1.6}$$

denote the *discrete Fourier transforms* of $\{X_j\}$ and $\{\zeta_j\}$, respectively, computed at frequencies u_j's, $j = 0, \cdots, [n/2]$. The corresponding periodograms, transfer functions and spectral densities of $\{X_j\}$ and $\{\zeta_j\}$ at frequency u_j are denoted by

$$\begin{aligned} I_{X,j} &= |w_{X,j}|^2, \quad I_{\zeta,j} = |w_{\zeta,j}|^2, \quad A_{X,j} = A_{X,j}(u_j), \quad A_{\zeta,j} = 1, \\ f_{X,j} &:= f_X(u_j), \quad f_{\zeta,j} := f_\zeta(u_j) \equiv \frac{1}{2\pi}, \qquad j = 0, 1, \cdots, n/2. \end{aligned}$$

The main focus of this paper is to establish asymptotic normality of the quadratic forms

$$Q_{n,X} = \sum_{j=1}^{\nu} b_{n,j} I_{X,j}, \quad \nu = \nu_n := [n/2] - 1,$$

where $\{b_{n,j},\, j = 1, \cdots, \nu\}$ is an array of real numbers depending on n. This in turn is facilitated by first developing asymptotic distribution theory for the sums

$$S_{n,X} = \sum_{j=1}^{\nu} b_{n,j} \frac{I_{X,j}}{f_{X,j}}.$$

Moreover, asymptotic analysis of these sums is more illustrative of the methodology used. The asymptotic normality of the sums $S_{n,X}$ is discussed in section 2, and that of $Q_{n,X}$ in section 3. It is based on Bartlett type approximation of $S_{n,X}$ and $Q_{n,X}$ by the corresponding sums of weighted periodograms of the i.i.d. r.v.'s $\{\zeta_j\}$.

Asymptotic normality (CLT) for the quadratic forms $Q_{n,X}$ with weights $b_{n,j} \equiv b_j$ that do not depend on n was investigated by Hannan (1973), see also Proposition 10.8.6. of

Brockwell and Davis (1991). Their proof required restrictive condition $\sum_{k=0}^{\infty} k^{1/2}|a_k| < \infty$ on coefficients a_k of the linear process $\{X_j\}$ of (1.2) and was based on Bartlett approximation of periodogram $I_{X,j}/f_{X,j}$ by periodogram $I_{\zeta,j}/f_{\zeta,j}$ of the noise. Robinson (1995b) established asymptotic normality of the sum $S_{n,X}$ in a particular case of weights $b_{n,j} = \log(j/m) - m^{-1}\sum_{k=1}^{m} \log(k/m)$, $j = 1, \cdots, m$ where $m = m_n \to \infty$, $m = o(n)$.

In the present paper we show that CLT's for $Q_{n,X}$ and $S_{n,X}$ hold under similar conditions as the classical CLT for weighted sums of i.i.d. r.v.'s. It requires Lindeberg-Feller type condition on weights $b_{n,j}$ and minimal restrictions on a linear process $\{X_j\}$ which may have short or long memory. For example, in short memory case it suffices to assume that a_k of the linear process $\{X_j\}$ of (1.2) satisfy $\sum_{k=0}^{\infty} |a_k| < \infty$ and the spectral density f_X is bounded away from ∞ and 0. Results below demonstrate that weighted sums of rescaled periodogram $I_{X,j}/f_{X,j}$ of a linear process behave, to some extend, similarly as the weighted sums of i.i.d. r.v.'s.

We also investigate precision of Bartlett approximation of $Q_{n,X}$ and $S_{n,X}$ by sums of weighted periodograms $I_{\zeta,j}/f_{\zeta,j}$. Approximation Lemma 2.1 and Theorem 3.3 contain sharp bounds and are of independent interest. From these results one sees that the above approximation is extremely precise, and the resulting error is small and can be effectively controlled by the weights $\{b_{n,j}\}$ alone.

In the sequel, $\mathrm{Cum}_k(Z)$ denotes the kth cummulant of the r.v. Z, $IID(0,1)$ denotes the class of i.i.d. standardized r.v.'s, $a \wedge b := \min(a,b)$, and $a \vee b := \max(a,b)$, for any real numbers a, b.

2 Asymptotic normality of $S_{n,X}$

Important role in the asymptotic analysis of $S_{n,X}$ is played by Bartlett type approximation

$$\big(I_{X,j}/f_{X,j}\big) \sim \big(I_{\zeta,j}/f_{\zeta,j}\big) = 2\pi\, I_{\zeta,j}, \quad j = 1, \cdots, \nu, \quad \nu = [n/2] - 1.$$

Our first goal is to approximate $S_{n,X}$ by the weighted sum of $I_{\zeta,j}$,

$$S_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}\big(I_{\zeta,j}/f_{\zeta,j}\big) \equiv \sum_{j=1}^{\nu} b_{n,j} 2\pi\, I_{\zeta,j}. \tag{2.1}$$

Let

$$\begin{aligned} R_n &:= S_{n,X} - S_{n,\zeta}, \qquad b_n := \max_{j=1,\cdots,\nu} |b_{n,j}|, \qquad B_n := \Big(\sum_{j=1}^{\nu} b_{n,j}^2\Big)^{1/2}, \qquad (2.2)\\ q_n^2 &:= B_n^2 + \mathrm{Cum}_4(\zeta_0)\frac{1}{n}\Big(\sum_{j=1}^{\nu} b_{n,j}\Big)^2. \end{aligned}$$

We show later that $\mathrm{Var}(S_{n,\zeta}) = q_n^2$, see (2.19).

By definition $S_{n,X} = S_{n,\zeta} + R_n$. Lemma 2.1 below provides an upper bound of order $b_n \log^2 n$ for ER_n^2. The main term $S_{n,\zeta}$ is a quadratic form in i.i.d. r.v.'s. Its asymptotic normality is established under minimal conditions on the weights $b_{n,j}$ in Lemma 2.2.

The following theorem proves asymptotic normality for $S_{n,X}$ under Lindeberg - Feller type condition (2.3) on weights $b_{n,j}$, which is analogous to the asymptotic normality condition for the sums of i.i.d. r.v.'s $\sum_{j=1}^{\nu} b_{n,j}\epsilon_j$, $\{\epsilon_j\} \sim IID(0,1)$.

Because of the invariance property $I_{X+\mu}(u_j) = I_X(u_j)$, $\mu \in \mathbb{R}$, $j = 1, \cdots, n-1$, all results obtained below remain valid also for a process $\{X_j\}$ of (1.2) that has non-zero mean.

Theorem 2.1 *Suppose the linear process $\{X_j,\, j \in \mathbb{Z}\}$ of (1.2) satisfies assumptions (1.3) and (1.4), and $E\zeta_0^4 < \infty$. About the weights $b_{n,j}$'s assume*

$$\frac{\max_{j=1,\cdots,\nu} |b_{n,j}|}{\big(\sum_{j=1}^{\nu} b_{n,j}^2\big)^{1/2}} = \frac{b_n}{B_n} \to 0. \tag{2.3}$$

Then, the following hold.

$$ES_{n,X} = \sum_{j=1}^{\nu} b_{n,j} + o(q_n), \qquad \mathrm{Var}(S_{n,X}) = q_n^2 + o(q_n^2), \tag{2.4}$$
$$q_n^{-1}\big(S_{n,X} - \sum_{j=1}^{\nu} b_{n,j}\big) \to_D \mathcal{N}(0,1).$$

Moreover,

$$\min\big(1, \mathrm{Var}(\zeta_0^2)/2\big) B_n^2 \le q_n^2 \le (1 + |\mathrm{Cum}_4(\zeta_0)|) B_n^2. \tag{2.5}$$

Proof. The proof uses Lemmas 2.1 and 2.2 given below. To prove (2.5), use definition of q_n and the Cauchy-Schwarz inequality to obtain the upper bound. The lower bound is derived in (2.21) of Lemma 2.2.

By (2.18) of Lemma 2.2 and (2.10) of Lemma 2.1, (2.3) and (2.5),

$$ES_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}, \qquad E|R_n| \le (ER_n^2)^{1/2} = o(B_n) = o(q_n). \tag{2.6}$$

These facts in turn complete the proof of the first claim in (2.4).

To prove the second claim, note that by (2.19), $\mathrm{Var}(S_{n,\zeta}) = q_n^2$, which together with (2.6) yields $\mathrm{Var}(R_n) \le ER_n^2 = o(q_n^2)$, $|\mathrm{Cov}(S_{n,\zeta}, R_n)| = o(q_n^2)$. These facts together with the fact $\mathrm{Var}(S_{n,X}) = \mathrm{Var}(S_{n,\zeta}) + \mathrm{Var}(R_n) + 2\mathrm{Cov}(S_{n,\zeta}, R_n)$ completes the proof of the second claim in (2.4).

Finally, since $ES_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}$ and $ER_n^2 = o(q_n^2)$,

$$S_{n,X} - \sum_{j=1}^{\nu} b_{n,j} \;=\; S_{n,X} - ES_{n,\zeta} = S_{n,\zeta} - ES_{n,\zeta} + o_p(q_n).$$

This together with (2.20) of Lemma 2.2 implies the asymptotic normality result in (2.4). $\Box$

The following lemma provides two types of sharp upper bounds for the mean square error ER_n^2 that are useful in approximating $S_{n,X}$ by $S_{n,\zeta}$. The idea of using Bartlett type approximations to establish the asymptotic normality of an integrated weighted periodogram of a short memory linear process goes back to the work of Hannan and Heyde (1972) and Hannan (1973), whereas for sums of weighted periodograms of an ARMA process it was used in Proposition 10.8.5 of Brockwell and Davis (1991). Their approximations were derived under the assumption that the weight function b did not depend on n, and the bounds they obtain have low-level of sharpness, though they are sufficient to show that the main term dominates the remainder. The sharp bounds for an integrated weighted periodogram established in Bhansali, *et al.* (2007b) technically are more involved and harder to apply than those for sums in the next lemma.

Lemma 2.1 *Assume that $\{X_j\}$ of (1.2) satisfies (1.3) and (1.4), and $E\zeta_0^4 < \infty$. Then, the following hold.*

$$\begin{aligned} E(R_n - ER_n)^2 &\le Cb_n^2 \log^3 n, \quad \textit{and} &(2.7)\\ &\le Cb_nB_n, \\ |ER_n| &\le Cb_n \log^2 n, \quad \textit{and} &(2.8)\\ &= o(B_n), \quad \text{if} \quad b_n = o(B_n). \end{aligned}$$

In particular,

$$E(S_{n,X} - S_{n,\zeta})^2 \le Cb_n^2 \log^4 n; \qquad (2.9)$$

$$E(S_{n,X} - S_{n,\zeta})^2 = o(B_n^2), \quad \textit{if} \quad b_n = o(B_n). \qquad (2.10)$$

To prove Lemma 2.1, we need two auxiliary results. Next proposition provides a general approximation bound.

Proposition 2.1 *Let $\{Y_{n,j}^{(i)},\ j = 1,\cdots,n\}$, $i = 1,2$, $n \ge 1$ be the two sets of moving averages*

$$Y_{n,j}^{(i)} = \sum_{k\in\mathbb{Z}} b_{n,j}^{(i)}(k)\zeta_k, \qquad \sum_{k\in\mathbb{Z}}^{\infty} |b_{n,j}^{(i)}(k)|^2 < \infty, \quad i = 1,2,$$

where $\{b_{n,j}^{(i)}(k)\}$ are possibly complex weights. Assume, $\zeta_k \sim IID(0,1)$, $E\zeta_0^4 < \infty$. Then, for any real weights $c_{n,j}$, $j = 1,\cdots,n$,

$$\begin{aligned} &\mathrm{Var}\Big(\sum_{j=1}^n c_{n,j}\{|Y_{n,j}^{(1)}|^2 - |Y_{n,j}^{(2)}|^2\}\Big) &(2.11)\\ &\le C\sum_{j,k=1}^n |c_{n,j}c_{n,k}|\,\Big|\,|r_{n,jk}^{11}|^2 + |r_{n,jk}^{22}|^2 - 2|r_{n,jk}^{12}|^2\Big|, \end{aligned}$$

where $r_{n,jk}^{il} := E[Y_{n,j}^{(i)}\,\overline{Y_{n,k}^{(l)}}] = \sum_{t\in\mathbb{Z}} b_{n,j}^{(i)}(t)\,\overline{b_{n,k}^{(l)}(t)}$, $\quad i,l = 1,2.$

Proof. Observe that

$$\begin{aligned} Q_n &:= \sum_{j=1}^{n} c_{n,j}\{|Y_{n,j}^{(1)}|^2 - |Y_{n,j}^{(2)}|^2\} \\ &= \sum_{t,s\in\mathbb{Z}} \Big(\sum_{j=1}^{n} c_{n,j}\{b_{n,j}^{(1)}(t)\overline{b_{n,j}^{(1)}(s)} - b_{n,j}^{(2)}(t)\overline{b_{n,j}^{(2)}(s)}\}\Big)\zeta_t\zeta_s \\ &=: \sum_{t,s\in\mathbb{Z}} B_n(t,s)\zeta_t\zeta_s. \end{aligned}$$

Hence,

$$\begin{aligned} & E\big|Q_n - EQ_n\big|^2 \\ &\le 4\Big(E\Big|\sum_{t<s} B_n(t,s)\zeta_t\zeta_s\Big|^2 + E\Big|\sum_{s<t} B_n(t,s)\zeta_t\zeta_s\Big|^2 \\ &\qquad + E\Big|\sum_{t\in\mathbb{Z}} B_n(t,t)(\zeta_t^2 - E\zeta_t^2)\Big|^2\Big) \\ &= 4\sum_{t<s}|B_n(t,s)|^2 + 4\sum_{s<t}|B_n(t,s|^2 + 4\mathrm{Var}(\zeta_0)\sum_{t\in\mathbb{Z}}|B_n(t,t)|^2 \\ &\le (4 + 4\mathrm{Var}(\zeta_0))\sum_{t,s\in\mathbb{Z}}|B_n(t,s)|^2. \end{aligned}$$

But,

$$\begin{aligned} \sum_{t,s\in\mathbb{Z}}|B_n(t,s)|^2 &= \sum_{j,k=1}^{n} c_{n,j}c_{n,k}\sum_{t,s\in\mathbb{Z}}\{b_{n,j}^{(1)}(t)\overline{b_{n,j}^{(1)}(s)} - b_{n,j}^{(2)}(t)\overline{b_{n,j}^{(2)}(s)}\} \\ &\qquad \times\{\overline{b_{n,k}^{(1)}(t)}b_{n,k}^{(1)}(s) - \overline{b_{n,k}^{(2)}(t)}b_{n,k}^{(2)}(s)\} \\ &= \sum_{j,k=1}^{n} c_{n,j}c_{n,k}\Big(|r_{n,jk}^{11}|^2 + |r_{n,jk}^{22}|^2 - |r_{n,jk}^{12}|^2 - |r_{n,kj}^{12}|^2\Big). \end{aligned}$$

This completes the proof of (2.11). □

Now note that

$$\begin{aligned} R_n &= S_{n,X} - S_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}\big(\frac{I_{X,j}}{f_{X,j}} - \frac{I_{\zeta,j}}{f_{\zeta,j}}\big) \\ &= \sum_{j=1}^{\nu}\frac{b_{n,j}}{f_{X,j}}\big\{I_{X,j} - f_{X,j}\frac{I_{\zeta,j}}{f_{\zeta,j}}\big\}. \end{aligned} \tag{2.12}$$

Also, recall that $I_{X,j} = |w_{X,j}|^2$, $I_{\zeta,j} = |w_{\zeta,j}|^2$, and that the discrete Fourier transforms $w_{X,j}$ and $w_{\zeta,j}$ are moving averages with complex valued coefficients. The corollary below, which follows from Proposition 2.1, is useful in analyzing the sums of the type appearing in (2.12). Let $f_{X\zeta}(u) = (2\pi)^{-1}A_X(u)$ denote the cross-spectral density of $\{X_j\}$ and $\{\zeta_j\}$. See (4.2) below for the definition of cross spectral density.

Corollary 2.1 *Suppose that $\{X_j\}$ is a linear process as in (1.2) and $E\zeta_0^4 < \infty$. Then, for any real weights $c_{n,j}$, $j = 1, \cdots, n$,*

$$\operatorname{Var}\Big(\sum_{j=1}^{\nu} c_{n,j}\{I_{X,j} - f_{X,j}\frac{I_{\zeta,j}}{f_{\zeta,j}}\}\Big) \le C(s_{n,1} + s_{n,2}), \tag{2.13}$$

where

$$\begin{aligned} s_{n,1} &:= C\sum_{j,k=1}^{\nu} c_{n,j}^2\Big\{(E|w_{X,j}|^2 - f_{X,j})^2 + f_{X,j}\Big|E|w_{X,j}|^2 - f_{X,j}\Big| \\ &\qquad + f_{X,j}\Big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\Big|^2 + f_{X,j}^{3/2}\Big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\Big|\Big\}, \\ s_{n,2} &:= \sum_{1\le k<j\le\nu} |c_{n,j}c_{n,k}|\Big\{|E[w_{X,j}\overline{w_{X,k}}]|^2 + f_{X,k}|E[w_{X,j}\overline{w_{\zeta,k}}]|^2\Big\}. \end{aligned}$$

Proof. The proof uses some results from section 4 below. Observe that

$$\frac{f_{X,j}I_{\zeta,j}}{f_{\zeta,j}} = |A_{X,j}|^2 I_{\zeta,j} = |A_{X,j}w_{\zeta,j}|^2.$$

Note that the r.v. $Y_{n,j}^{(1)} := w_{X,j}$ and $Y_{n,j}^{(2)} := A_{X,j}w_{\zeta,j}$ can be written as moving averages of ζ_j's with complex weights. Therefore, by Proposition 2.1, the l.h.s. of (2.13) can be bounded above by

$$\begin{aligned} &C\sum_{j,k=1}^{\nu} |c_{n,j}c_{n,k}|\Big|E[w_{X,j}\overline{w_{X,k}}]|^2 + |A_{X,j}|^2|A_{X,k}|^2|E[w_{\zeta,j}\overline{w_{\zeta,k}}]|^2 - 2|A_{X,k}|^2|E[w_{X,j}\overline{w_{\zeta,k}}]|^2\Big| \\ &\quad = C\Big(\sum_{j=k=1}^{n}[\cdots] + \sum_{k\ne j}[\cdots]\Big) := C(s'_{n,1} + s'_{n,2}). \end{aligned}$$

By (4.1) below, $E|w_{\zeta,j}|^2 = 1/2\pi$, $E[w_{\zeta,j}\overline{w_{\zeta,k}}] = 0$, for $1 \le k < j \le \nu$. Recall also that $f_{X,j} = |A_{X,j}|^2/(2\pi)$. Therefore,

$$\begin{aligned} s'_{n,1} &= \sum_{j,k=1}^{\nu} c_{n,j}^2\Big|(E|w_{X,j}|^2)^2 + f_{X,j}^2 - 4\pi f_{X,j}|E[w_{X,j}\overline{w_{\zeta,j}}]|^2\Big|, \\ s'_{n,2} &= \sum_{1\le k<j\le\nu} |c_{n,j}c_{n,k}|\Big(|E[w_{X,j}\overline{w_{X,k}}]|^2 + f_{X,k}|E[w_{X,j}\overline{w_{\zeta,k}}]|^2\Big). \end{aligned}$$

Observe that $s'_{n,2} = s_{n,2}$. To estimate $s'_{n,1}$, let

$$A := (E|w_{X,j}|^2)^2 - f_{X,j}^2, \quad B := |E[w_{X,j}\overline{w_{\zeta,j}}]|^2 - f_{X\zeta,j}.$$

The term within $|\cdots|$ in $s'_{n,1}$ can be written as

$$\begin{aligned} &(E|w_{X,j}|^2)^2 + f_{X,j}^2 - 4\pi f_{X,j}|E[w_{X,j}\overline{w_{\zeta,j}}]|^2 \\ &\quad = \ (A - 4\pi f_{X,j}B) + (2f_{X,j}^2 - 4\pi f_{X,j}|f_{X\zeta,j}|^2) = A - 4\pi f_{X,j}B, \end{aligned}$$

because $4\pi f_{X,j}|f_{X\zeta,j}|^2 = 4\pi f_{X,j}|A_{X,j}|^2/(2\pi)^2 = 2f_{X,j}^2$.

Next, note that $\big||z_1|^2 - |z_2|^2\big| \le |z_1 - z_2|^2 + 2|z_1 - z_2||z_2|$, for any complex numbers z_1, z_2, and that $|f_{X\zeta,j}| = |A_{X,j}|/(2\pi)^2 \le f_{X,j}^{1/2}$. Therefore,

$$\begin{aligned}
|A - 4\pi f_{X,j}B| &\le |A| + 4\pi f_{X,j}|B| \\
&\le (E|w_{X,j}|^2 - f_{X,j})^2 + 2f_{X,j}\Big|E|w_{X,j}|^2 - f_{X,j}\Big| \\
&\quad + 4\pi f_{X,j}\Big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\Big|^2 + 8\pi f_{X,j}^{3/2}\Big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\Big|,
\end{aligned}$$

which shows that $s'_{n,1} \le Cs_{n,1}$ and completes proof of corollary. □

Proof of Lemma 2.1. The proof uses Theorem 4.1 given in section 4. Recall $R_n = S_{n,X} - S_{n,\zeta}$. We shall prove (2.7) and (2.8). Since $ER_n^2 \le 2(E(R_n - ER_n)^2 + (ER_n)^2)$, these two facts together imply (2.9) and (2.10).

Now, we prove (2.7). Note that from (2.12), R_n is like the r.v. in the left hand side of (2.13) with $c_{n,j} = b_{n,j}/f_{X,j}$. Thus, $\mathrm{Var}(R_n) \le s_{n,1} + s_{n,2}$, with

$$\begin{aligned}
s_{n,1} &:= C\sum_{j=1}^{\nu}\frac{b_{n,j}^2}{f_{X,j}^2}\Big[(E|w_{X,j}|^2 - f_{X,j})^2 + f_{X,j}\Big|E|w_{X,j}|^2 - f_{X,j}\Big| \\
&\quad + f_{X,j}\Big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\Big|^2 + f_{X,j}^{3/2}\Big|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}\Big|\Big], \\
s_{n,2} &:= \sum_{1\le k<j\le\nu}\frac{|b_{n,j}b_{n,k}|}{f_{X,j}f_{X,k}}\Big(|E[w_{X,j}\overline{w_{X,k}}]|^2 + f_{X,k}|E[w_{X,j}\overline{w_{\zeta,k}}]|^2\Big).
\end{aligned}$$

It thus suffices to show that these $s_{n,1}$ and $s_{n,2}$ are bounded from the above by the the upper bounds given in the r.h.s. of (2.7).

Part (iii) of Theorem 4.1 below provides bounds for $E[w_{X,j}\overline{w_{X,k}}]$ and $E[w_{X,j}\overline{w_{\zeta,k}}]$. Recall that the spectral density f_X satisfies (1.3), whereas the cross-spectral density $f_{X\zeta}(u) = (2\pi)^{-1}A_X(u)$ has the property $|f_{X\zeta}(u)| \le C|u|^{-d}$, $|\dot{f}_{X\zeta}(u)| \le C|u|^{-1-d}$, $u \in \Pi$. Therefore, they satisfy conditions of part (iii) of Theorem 4.1, and hence

$$\begin{aligned}
&|E|w_{X,j}|^2 - f_{X,j}| \le C|u_j|^{-2d}j^{-1}\log j, \\
&|E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}| \le C|u_j|^{-d}j^{-1}\log j,
\end{aligned} \tag{2.14}$$

where C does not depend on j and n. Since, by (1.3), $1/f_{X,j} \le Cu_j^{2d}$, these bounds yield

$$s_{n,1} \le C\sum_{j=1}^{\nu} b_{n,j}^2(j^{-1}\log j).$$

From this we obtain the bounds

$$\begin{aligned} s_{n,1} &\leq Cb_n^2 \log n \sum_{j=1}^{\nu} j^{-1} \leq Cb_n^2 \log^2 n, \quad \text{and} \\ s_{n,1} &\leq Cb_n \sum_{j=1}^{\nu} |b_{n,j}|(j^{-1} \log j) \\ &\leq Cb_n (\sum_{j=1}^{\nu} b_{n,j}^2)^{1/2} (\sum_{j=1}^{\nu} j^{-2} \log^2 j)^{1/2} \leq Cb_n B_n, \end{aligned}$$

which proves that $s_{n,1}$ satisfies both bounds of (2.7).

Next, by (iii) of Theorem 4.1, for all $1 \leq k < j \leq \nu$,

$$\begin{aligned} \left|E[w_{X,j}\overline{w_{X,k}}]\right| &\leq C(u_j^{-2d} + u_k^{-2d})j^{-1} \log j, \\ \left|E[w_{X,j}\overline{w_{\zeta,k}}]\right| &\leq C(u_j^{-d} + u_k^{-d})j^{-1} \log j. \end{aligned}$$

Since, by (1.3),

$$\begin{aligned} (f_j f_k)^{-1}(u_j^{-2d} + u_k^{-2d})^2 &\leq C(u_j u_k)^{2d}(u_j^{-4d} + u_k^{-4d}) \leq C(j/k)^{2|d|}, \\ f_j^{-1}(u_j^{-d} + u_k^{-d})^2 &\leq C u_j^{2d}(u_j^{-2d} + u_k^{-2d}) \leq C(j/k)^{2|d|}, \end{aligned}$$

we obtain

$$s_{n,2} \leq C \sum_{1 \leq k < j \leq \nu} |b_{n,j} b_{n,k}| \, (\frac{j}{k})^{2|d|} \frac{\log^2 j}{j^2}. \tag{2.15}$$

Bound $|b_{n,j} b_{n,k}|$ by b_n^2 to obtain

$$s_{n,2} \leq Cb_n^2 \log^2 n \sum_{1 \leq k < j \leq \nu} \frac{1}{k^{2|d|} j^{2-2|d|}} \leq Cb_n^2 \log^3 n,$$

which implies the first estimate of (2.7). Next, bound $|b_{n,j}|$ by b_n in (2.15), to obtain

$$\begin{aligned} s_{n,2} &\leq Cb_n \sum_{1 \leq k < j \leq \nu} |b_{n,k}| \frac{\log^2 j}{k^{2|d|} j^{2-2|d|}} \leq Cb_n \sum_{1 \leq k \leq \nu} |b_{n,k}| \, \frac{\log^2 k}{k} \\ &\leq Cb_n \Big(\sum_{1 \leq k \leq \nu} b_{n,k}^2 \Big)^{1/2} \Big(\sum_{1 \leq k \leq \nu} \frac{\log^4 k}{k^2} \Big)^{1/2} \leq Cb_n B_n, \end{aligned}$$

that establishes the second bound of (2.7).

To show (2.8), recall that $f_{X,j} E|w_{\zeta,j}|^2 / f_{\zeta,j} = f_{X,j}$. Therefore,

$$ER_n = \sum_{j=1}^{\nu} \frac{b_{n,j}}{f_{X,j}} \Big(E|w_{X,j}|^2 - \frac{f_{X,j}}{f_{\zeta,j}} E|w_{\zeta,j}|^2 \Big) = \sum_{j=1}^{\nu} \frac{b_{n,j}}{f_{X,j}} \Big(E|w_{X,j}|^2 - f_{X,j} \Big).$$

Then, by (2.14) and (1.3),

$$
\begin{aligned}
|ER_n| &\le C\sum_{j=1}^{\nu}\frac{|b_{n,j}|}{f_{X,j}}u_j^{-2d}j^{-1}\log j \le C\sum_{j=1}^{\nu}|b_{n,j}|j^{-1}\log j \\
&\le Cb_n\sum_{j=1}^{\nu}j^{-1}\log j \le Cb_n\log^2 n,
\end{aligned}
$$

which implies the first bound in (2.8).

To establish the second bound, let $K = (B_n/b_n)^{1/2}$. Because of (2.3), $K \to \infty$, $b_nK = (b_n/B_n)^{1/2}B_n = o(B_n)$. Thus,

$$
\begin{aligned}
|ER_n| &\le C\Big(\sum_{j=1}^{K-1}|b_{n,j}|j^{-1}\log j + \sum_{j=K}^{\nu}|b_{n,j}|j^{-1}\log j\Big) \qquad (2.16)\\
&\le C\Big\{b_nK + \Big(\sum_{j=K}^{\nu}b_{n,j}^2\Big)^{1/2}\Big(\sum_{j=K}^{\infty}j^{-2}\log^2 j\Big)^{1/2}\Big\} \\
&= o(B_n).
\end{aligned}
$$

This completes proof of the second estimate in (2.8). $\square$

Now we return to establishing asymptotic normality of $S_{n,\zeta}$, *a weighted quadratic form in i.i.d. r.v.'s* where weights depend on n. The CLT for quadratic forms in i.i.d. r.v.'s $\zeta_j \sim IID(0,1)$ is well investigated, see Guttorp and Lockhart (1988). The following theorem summarizes useful criterion for asymptotic normality, given in Theorem 2.1 in Bhansali *et al.* (2007a). Let $C_n = \{c_{n,ts},\, t,s = 1,\cdots,n\}$ be a symmetric $n\times n$ matrix of real numbers $c_{n,ts}$, and define the quadratic form

$$
Q_n := \sum_{t,s=1}^{n} c_{n,ts}\zeta_t\zeta_s.
$$

Let $\|C_n\| := (\sum_{t,s=1}^n c_{n,ts}^2)^{1/2}$ and $\|C_n\|_{sp} := \max_{\|x\|=1}\|C_nx\|$ denote Euclidean and spectral norms, respectively, of C_n.

Theorem 2.2 *Suppose $\zeta_j \sim IID(0,1)$ and $E\zeta_0^4 < \infty$. Then*

$$
\frac{\|C_n\|_{sp}}{\|C_n\|} \to 0 \qquad (2.17)
$$

implies $(\mathrm{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \to_D \mathcal{N}(0,1)$. *In addition, if the diagonal of the matrix C_n satisfies $\sum_{t=1}^n c_{n;tt}^2 = o(\|C_n\|^2)$, then $\mathrm{Var}(Q_n) \sim 2\|C_n\|^2$, and condition $E\zeta_0^4 < \infty$ can be replaced by $E|\zeta_0|^{2+\delta} < \infty$, for some $\delta > 0$.*

Next lemma derives asymptotic distribution of the sum $S_{n,\zeta}$ of (2.1). Its proof uses Theorem 2.2 and some ideas of the proof of Theorem 2, Robinson (1995b). Recall the definition of q_n^2 and B_n from (2.2).

Lemma 2.2 *Suppose $\zeta_j \sim IID(0,1)$, $E\zeta_0^4 < \infty$, and $b_{n,j}$ satisfy (2.3). Then*

$$ES_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}, \tag{2.18}$$

$$Var(S_{n,\zeta}) = q_n^2, \tag{2.19}$$

$$q_n^{-1}(S_{n,\zeta} - ES_{n,\zeta}) \to_D \mathcal{N}(0,1), \tag{2.20}$$

Moreover,

$$q_n^2 \geq \min\big(1, \mathrm{Var}(\zeta_0^2)/2\big) B_n^2. \tag{2.21}$$

Proof. Write

$$S_{n,\zeta} = \frac{1}{n}\sum_{t,s=1}^{n}\sum_{j=1}^{\nu} e^{\mathbf{i}(t-s)u_j} b_{n,j}\zeta_s\zeta_t = \sum_{t,s=1}^{n} c_n(t-s)\zeta_s\zeta_t,$$

where $c_n(t) := n^{-1}\sum_{j=1}^{\nu} b_{n,j}\cos(tu_j)$, $t = 1, 2, \cdots$. Matrix $C_n = (c_n(t-s))_{t,s=1,\cdots,n}$ is a symmetric $n \times n$ matrix with real entries. Hence, (2.18) follows because ζ_j's are $IID(0,1)$. For the same reason,

$$\begin{aligned} \mathrm{Var}(S_{n,\zeta}) &= 2\sum_{s,t=1:\, t\neq s}^{n} c_n^2(t-s) + \mathrm{Var}(\zeta_0^2)\sum_{t=1}^{n} c_n^2(t-t) \\ &= 2\|C_n\|^2 + \mathrm{Cum}_4(\zeta_0) n^{-1}(\sum_{j=1}^{\nu} b_{n,j})^2 \\ &\geq \min(2, \mathrm{Var}(\zeta_0^2))\|C_n\|^2, \end{aligned} \tag{2.22}$$

since $\mathrm{Var}(\zeta_0^2) - 2 = E\zeta_0^4 - 3 = \mathrm{Cum}_4(\zeta_0)$, and $c_n(0) = n^{-1}\sum_{j=1}^{\nu} b_{n,j}$.

Next, we show that the weights $c_n(t-s)$ satisfy

$$\|C_n\|^2 = 2^{-1}B_n^2, \tag{2.23}$$

$$\|C_n\|_{sp} = o(\|C_n\|). \tag{2.24}$$

By Theorem 2.2, (2.24) implies

$$(\mathrm{Var}(S_{n,\zeta}))^{-1/2}(S_{n,\zeta} - E[S_{n,\zeta}]) \to_D \mathcal{N}(0,1),$$

$$\mathrm{Var}(S_{n,\zeta}) = B_n^2 + \mathrm{Cum}_4(\zeta_0)n^{-1}(\sum_{j=1}^{\nu} b_{n,j})^2,$$

which proves (2.20), whereas (2.23) with (2.22) prove (2.21).

To prove (2.23), by definition of $c_n(t)$,

$$\begin{aligned} \|C_n\|^2 &= \sum_{t,s=1}^{n} c_n^2(t-s) \\ &= n^{-2}\sum_{j,k=1}^{\nu} b_{n,j}b_{n,k}\sum_{s,t=1}^{n}\cos((t-s)u_j)\cos((t-s)u_k). \end{aligned} \tag{2.25}$$

Since $\nu = [n/2] - 1$, then $j + k < n$ in the above sums. We first recall the following equality: for $1 \le j, k \le m$, $j + k < n$ and $a, b \in \mathbb{R}$,

$$\sum_{t=1}^{n} \cos(tu_j + a)\cos(tu_k + b) = \frac{n}{2}\cos(a-b)I(j=k). \tag{2.26}$$

To prove this equality, use the fact $\cos(x) = (e^{\mathbf{i}x} + e^{-ix})/2$ to write the l.h.s. of (2.26) as

$$\sum_{t=1}^{n} \frac{1}{4}\Big(e^{\mathbf{i}t(u_j+u_k)}e^{\mathbf{i}(a+b)} + e^{-\mathbf{i}t(u_j+u_k)}e^{-\mathbf{i}(a+b)}$$
$$+e^{\mathbf{i}t(u_j-u_k)}e^{\mathbf{i}(a-b)} + e^{-\mathbf{i}t(u_j-u_k)}e^{-\mathbf{i}(a-b)}\Big).$$

Since

$$\sum_{t=1}^{n} e^{\mathbf{i}tu_l} = e^{\mathbf{i}u_l}\frac{e^{\mathbf{i}nu_l} - 1}{e^{\mathbf{i}u_l} - 1} = n\{I(l=0) + I(l=n)\}, \tag{2.27}$$

this expression reduces to $(n/4)(e^{\mathbf{i}(a-b)} + e^{-\mathbf{i}(a-b)})I(j=k) = (n/2)\cos(a-b)I(j=k)$.

Hence, applying (2.26) in (2.25), readily yields (2.23):

$$\|C_n\|^2 = 2^{-1}\sum_{j=1}^{\nu} b_{n,j}^2 = 2^{-1}B_n^2.$$

To prove (2.24), let $x \in \mathbb{R}^n$ be such that $\|x\|^2 = 1$. Then

$$\|C_n x\|^2 = \sum_{t=1}^{n}(\sum_{s=1}^{n} c_n(t-s)x_s)^2 = \sum_{s,v=1}^{n} x_s x_v\Big(\sum_{t=1}^{n} c_n(t-s)c_n(t-v)\Big). \tag{2.28}$$

But, by (2.26),

$$\begin{aligned}\sum_{t=1}^{n} c_n(t-s)c_n(t-v) &= \frac{1}{n^2}\sum_{j,k=1}^{\nu} b_{n,j}b_{n,k}\sum_{t=1}^{n}\cos((t-s)u_j)\cos((t-v)u_k)\\ &= \frac{1}{2n}\sum_{j=1}^{\nu} b_{n,j}^2\cos((s-v)u_j).\end{aligned}$$

Hence

$$\|C_n x\|^2 = \frac{1}{2n}\sum_{j=1}^{\nu} b_{n,j}^2\sum_{s,v=1}^{n}\cos((s-v)u_j)x_s x_v.$$

Thus, by the equality $\sum_{s,v=1}^{n}\cos((s-v)u_j)x_s x_v = |\sum_{s=1}^{n} e^{\mathbf{i}su_j}x_s|^2$,

$$\begin{aligned}\|C_n x\|^2 &= \frac{1}{2n}\sum_{j=1}^{\nu} b_{n,j}^2|\sum_{s=1}^{n} e^{\mathbf{i}su_j}x_s|^2 \le \frac{1}{2n}b_n^2\sum_{j=1}^{n}|\sum_{s=1}^{n} e^{\mathbf{i}su_j}x_s|^2\\ &= \frac{1}{2n}b_n^2\sum_{t,s=1}^{n}\sum_{j=1}^{n} e^{\mathbf{i}(t-s)u_j}x_t x_s.\end{aligned}$$

By (2.27), $\sum_{j=1}^{n} e^{\mathbf{i}(t-s)u_j} = nI(t=s)$. Therefore,

$$\|C_n x\|^2 \leq \frac{1}{2} b_n^2 \sum_{t=s=1}^{n} x_t^2 = \frac{1}{2} b_n^2 \|x\|^2, \qquad \|C_n\|_{sp} \leq (1/\sqrt{2}) b_n.$$

Since $b_n = o(B_n)$, and $B_n = \sqrt{2}\|C_n\|$ by (2.23), this proves (2.24), and also completes the proof of the lemma. □

3 A general case of sums of weighted periodogram

We now focuss on the sums

$$Q_{n,X} := \sum_{j=1}^{\nu} b_{n,j} I_{X,j}.$$

Bartlett approximation $I_{X,j} \sim f_{X,j}\big(I_{\zeta,j}/f_{\zeta,j}\big)$ suggests to approximate $Q_{n,X}$ by the sum

$$Q_{n,\zeta} := \sum_{j=1}^{\nu} (b_{n,j} f_{X,j})(\frac{I_{\zeta,j}}{f_{\zeta,j}}) = \sum_{j=1}^{\nu} b_{n,j} f_{X,j} (2\pi) I_{\zeta,j}.$$

Corollary 2.1 provides tools for establishing approximation to the variance and the mean square error of $Q_{n,X} - Q_{n,\zeta}$.

In Theorem 2.1 above, the spectral density f_X can be unbounded at 0, but is differentiable on $(0,\pi)$. Then the asymptotic normality of the sums $S_{n,X} = \sum_{j=1}^{\nu} b_{n,j}\big(I_{X,j}/f_{X,j}\big)$ holds under Lindeberg-Feller type condition (2.3) on the weights $b_{n,j}$.

Now we turn to case when f_X is bounded and continuous on Π, with no assumptions about its differentiability, i.e. $d = 0$ in (1.3). In addition, we assume that f_X is bounded away from 0 and ∞:

$$0 < C_1 \leq f_X(u) \leq C_2 < \infty, \quad u \in \Pi, \quad (\exists\, 0 < C_1, C_2 < \infty). \tag{3.1}$$

The restriction $f_X(u) \geq C_1 > 0$ can be dropped at an expense of the simplicity of conditions.

Theorem 3.1 below shows that under Lindeberg-Feller type condition (2.3) on weights $b_{n,j}$, continuity of f_X, or more precisely, continuity of the transfer function A_X, suffices for asymptotic normality of the centered sums $Q_{n,X} - EQ_{n,X}$. To obtain an upper bound on the variance $\mathrm{Var}(Q_{n,X})$ it suffices to assume f_X to be continuous, whereas satisfactory asymptotics of $EQ_{n,X}$ requires f_X to be Lipshitz(β), $\beta > 1/2$, see Theorem 3.3.

By Lemma 2.2, $Q_{n,\zeta}$ has the following mean and variance.

$$v_n^2 := \sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 + \mathrm{Cum}_4(\zeta_0) \frac{1}{n} \Big(\sum_{j=1}^{\nu} b_{n,j} f_{X,j}\Big)^2,$$

$$\mathrm{Var}(Q_{n,\zeta}) = v_n^2, \qquad EQ_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j} f_{X,j}.$$

Observe that $Q_{n,X}$, $Q_{n,\zeta}$ and v_n^2, respectively, are like the $S_{n,X}$, $S_{n,\zeta}$ and q_n^2 of (2.2) with $b_{n,j}$ replaced by $b_{n,j}f_{X,j}$. Let

$$b_{f,n} = \max_{j=1,\cdots,\nu} |b_{n,j}| f_{X,j}, \quad B_{f,n}^2 = \sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2.$$

Similarly as in (2.5), one can show that the variance v_n^2 has the same order as $B_{f,n}^2$, i.e. for some $C_1, C_2 > 0$,

$$\begin{aligned} C_1 B_{f,n}^2 &\le v_n^2 \le C_2 B_{f,n}^2, \qquad \text{and} \\ C_1 B_n^2 &\le v_n^2 \le C_2 B_n^2, \quad \text{under (3.1).} \end{aligned} \tag{3.2}$$

Let $\mathcal{C}(\Pi)$ denote the class of bounded (complex valued) continuous functions on Π, and $\Lambda_\beta(\Pi)$ denote Lipschitz continuous functions of order β, $0 < \beta \le 1$. The next theorem establishes asymptotic normality of $Q_{n,X}$.

Theorem 3.1 *Suppose the linear process $\{X_j,\ j \in \mathbb{Z}\}$ of (1.2) is such that $E\zeta_0^4 < \infty$, and the real weights $b_{n,j}$'s satisfy (2.3).*

If f_X satisfies (3.1) and the transfer function A_X of $\{X_j\}$ is continuous, then

$$\mathrm{Var}(Q_{n,X}) = v_n^2 + o(v_n^2), \qquad v_n^{-1}(Q_{n,X} - EQ_{n,X}) \to_D \mathcal{N}(0,1). \tag{3.3}$$

In addition, if $f_X \in \Lambda_\beta(\Pi)$, with $\beta > 1/2$, then

$$EQ_{n,X} = \sum_{j=1}^{\nu} b_{n,j} f_{X,j} + o(v_n), \;\; v_n^{-1}(Q_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j}) \to_D \mathcal{N}(0,1). \tag{3.4}$$

In the next theorem we extend the result of asymptotic normality of $Q_{n,X}$ to the case when the spectral density f_X is not bounded in the neighborhood of 0, i.e. $d > 0$, or is not bounded away from 0, i.e. $d < 0$. Then the second bound of (3.2) does not hold. The Lindeberg-Feller condition (2.3) now has to be formulated using the weights $b_{n,j}f_{X,j}$ and we need to impose some additional smoothness conditions on A_X in a small neighborhood of 0. We assume that A_X can be factored into a product $A_X = hG$ of a differentiable function h, which may have a pole at 0, and a continuous bounded function G. In particular, if A_X satisfies (1.4), we take $G \equiv 1$.

Theorem 3.2 *Suppose $\{X_j,\ j \in \mathbb{Z}\}$ is the linear process (1.2) with $E\zeta_0^4 < \infty$. Assume that f_X satisfies (1.3) with $|d| < 1/2$, the transfer function A_X can be factored as $A_X = hG$, where G is continuous and bounded away from 0 and ∞, and h is differentiable having derivative $\dot{h}$ and satisfying*

$$C_1 |u|^{-d} \le |h(u)| \le C_2 |u|^{-d}, \qquad |\dot{h}(u)| \le C|u|^{-1-d}, \quad 0 < |u| \le \pi, \tag{3.5}$$

for some $0 < C, C_1, C_2 < \infty$. Then, for any real weights $b_{n,j}$'s satisfying

$$\frac{\max_{j=1,\cdots,\nu} |b_{n,j} f_{X,j}|}{(\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2)^{1/2}} \equiv \frac{b_{f,n}}{B_{f,n}} \to 0, \tag{3.6}$$

(3.3) continues to hold.

If, in addition, $G \in \Lambda_\beta(\Pi)$, with $\beta > 1/2$, then also (3.4) holds.

Proofs of Theorems 3.1 and 3.2. Write

$$Q_{n,X} - EQ_{n,X} \;=\; Q_{n,\zeta} - EQ_{n,\zeta} + r_n,$$

where $r_n := Q_{n,X} - Q_{n,\zeta} - E[Q_{n,X} - EQ_{n,\zeta}]$. The proof of both theorems follows from this decomposition and Lemmas 2.2 and 3.1. The latter lemma will be proved shortly. In (i) and (ii) of this lemma it is shown that $Er_n^2 = o(v_n^2)$ under the assumptions of Theorems 3.1 and 3.2. Therefore the result (3.3) of Theorems 3.1 and 3.2 follows, noticing that, by Lemma 2.2, under assumption (3.6), $v_n^{-1}(Q_{n,X} - Q_{n,\zeta}) \to_D \mathcal{N}(0,1)$. The second result (3.4) of these theorems is shown in (3.17) of Theorem 3.3 below. □

Lemma 3.1 below shows that the order of approximation of $Q_{n,X} - EQ_{n,X}$ by $Q_{n,\zeta} - EQ_{n,\zeta}$ is determined by the smoothness of the transfer function A_X. For example, by part (i) of this lemma, if A_X is a bounded continuous function, then

$$Q_{n,X} - EQ_{n,X} \;=\; Q_{n,\zeta} - EQ_{n,\zeta} = o_p(v_n), \tag{3.7}$$

where $v_n^2 = \mathrm{Var}(Q_{n,\zeta})$. If, in addition, A_X has a bounded derivative, then the order improves to $o_p(n^{-1/2}(\log n)\, v_n)$ without requiring any additional assumptions on $b_{n,j}$. Lemma 3.1(ii) shows that if A_X is discontinuous at 0, then approximation (3.7) is valid under additional regularity behavior on A_X in a neighborhood of 0, as long as the weights $b_{n,j}$ satisfy (3.6).

To state the lemma, we need the following notation. For a complex valued function $h(u)$, $u \in \Pi$, define

$$\begin{aligned} \epsilon_{n,h} &:= n^{-1}\log^2 n, && h \in \Lambda_1[\Pi], \\ &= n^{-\beta}, && h \in \Lambda_\beta[\Pi],\ 0<\beta<1, \\ &= \delta_n, && \delta_n \to 0, \quad h \in \mathcal{C}[\Pi]. \end{aligned}$$

Lemma 3.1 *Assume that $\{X_j\}$ is as in (1.2) and $E\zeta_0^4 < \infty$. Then*

$$Q_{n,X} - EQ_{n,X} \;=\; Q_{n,\zeta} - EQ_{n,\zeta} + r_n, \tag{3.8}$$

where r_n satisfies the following.

(i) *If $A_X \in \Lambda_\beta[\Pi]$, $0 < \beta \le 1$, or $A_X \in \mathcal{C}[\Pi]$, then*

$$Er_n^2 \;\le\; C\epsilon_{n,A_X} B_n^2 = o(v_n^2).$$

(ii) *If $A_X = hG$, where h satisfies (3.5) and either $G \in \mathcal{C}(\Pi)$ or $G \in \Lambda_\beta(\Pi)$, $0 < \beta \le 1$, then*

$$\begin{aligned} Er_n^2 &\le C\big(\min(b_{f,n}^2 \log^3 n,\, b_{f,n}B_{f,n}) + \epsilon_{n,G}B_{f,n}^2\big); \\ &\le C\min(b_{f,n}^2 \log^3 n,\, b_{f,n}B_{f,n}), \quad G \in \Lambda_1(\Pi). \end{aligned} \tag{3.9}$$

If, in addition, (3.6) holds, then

$$Er_n^2 = o(v_n^2). \tag{3.10}$$

Proof. Let

$$\tilde{R}_n = Q_{n,X} - Q_{n,\zeta} = \sum_{j=1}^{\nu} b_{n,j}\{I_{X,j} - \frac{f_{X,j} I_{\zeta,j}}{f_{\zeta,j}}\}.$$

By Corollary 2.1,

$$\mathrm{Var}(\tilde{R}_n) \le C(t_{n,1} + t_{n,2}), \tag{3.11}$$

where $t_{n,i}$, $i = 1, 2$ are like $s_{n,i}$, $i = 1, 2$, of Corollary 2.1 with $c_{n,j} \equiv b_{n,j}$.

Proof of (i). We shall show that

$$E(\tilde{R}_n - E\tilde{R}_n)^2 \;\; \le \;\; C\epsilon_{n,A_X} B_n^2, \tag{3.12}$$

which, in view of (3.2), proves (3.8). The proof of (3.12) is similar to that of Lemma 2.1. For the sake of completeness, we provide the details.

We shall now prove part (i) by considering three cases separately. First, observe that f_X and $f_{X\zeta} = A_X/(2\pi)$ are bounded functions.

Case (1). $A_X \in \Lambda_1[\Pi]$. Then, by (i) of Theorem 4.1,

$$\begin{aligned} |E|w_{X,j}|^2 - f_{X,j}| \vee |E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}| &\le Cn^{-1}\log n, \\ |E[w_{X,j}\overline{w_{X,k}}]| \vee |E[w_{X,j}\overline{w_{\zeta,k}}]| \le Cn^{-1}\log n, &\quad 1 \le k < j \le \nu. \end{aligned}$$

Therefore,

$$\begin{aligned} t_{n,1} \;\; &\le \;\; Cn^{-1}\log n \sum_{j=1}^{\nu} b_{n,j}^2 = Cn^{-1}\log n\, B_n^2, \\ t_{n,2} \;\; &:= \;\; Cn^{-2}\log^2 n \sum_{1\le k<j\le \nu} |b_{n,j} b_{n,k}| \\ &\le \;\; Cn^{-1}\log^2 n \sum_{j=1}^{\nu} b_{n,j}^2 = Cn^{-1}\log^2 n B_n^2, \end{aligned}$$

which proves (3.12).

Case (2). $A_X \in \Lambda_\beta[\Pi]$, $0 < \beta < 1$. Then by (i) of Theorem 4.1,

$$\begin{aligned} |E|w_{X,j}|^2 - f_{X,j}| \vee |E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}| &\le Cn^{-\beta} \\ |E[w_{X,j}\overline{w_{X,k}}]| \vee |E[w_{X,j}\overline{w_{\zeta,k}}]| \le Cn^{-\beta}\ell_n(\beta; j-k), &\quad k < j. \end{aligned}$$

Note that for $1 \le k < j \le \nu < n/2$, $j - k \le n - j + k$, and hence bound

$$\begin{aligned} \ell_n(\beta; j-k) \;\; &\le \;\; C\frac{\log(2+j-k)}{(2+j-k)^{1-\beta}}, \\ (n^{-\beta}\ell_n(\beta; j-k))^2 \;\; &\le \;\; C\frac{\log^2(2+j-k)}{n^\beta(2+j-k)^{2-\beta}}. \end{aligned} \tag{3.13}$$

Apply this fact, to obtain, that for $0 < \beta < 1$,

$$\begin{aligned} t_{n,1} &\le Cn^{-\beta} \sum_{j=1}^{\nu} b_{n,j}^2 = Cn^{-\beta} B_n^2, \\ t_{n,2} &\le C \sum_{1 \le k < j \le \nu} |b_{n,j} b_{n,k}| \, (n^{-\beta} \ell_n(\beta; j-k))^2 \\ &\le Cn^{-\beta} \sum_{1 \le k < j \le \nu} |b_{n,j} b_{n,k}| \, \frac{\log^2(2+j-k)}{(2+j-k)^{2-\beta}} \\ &\le Cn^{-\beta} \sum_{j=1}^{\nu} b_{n,j}^2 \sum_{u=0}^{\infty} \frac{\log^2(2+u)}{(2+u)^{2-\beta}} \le Cn^{-\beta} B_n^2, \end{aligned}$$

which proves (3.12).

Case (3). $A_X \in \mathcal{C}[\Pi]$. By (ii) of Theorem 4.1,

$$|E|w_{X,j}|^2 - f_{X,j}| \vee |E[w_{X,j}\overline{w_{\zeta,j}}] - f_{X\zeta,j}| \le C\delta_n$$
$$|E[w_{X,j}\overline{w_{X,k}}]| \vee |E[w_{X,j}\overline{w_{\zeta,k}}]| \le C\delta_n \ell_n(\epsilon; j-k), \quad k < j,$$

for any $0 < \epsilon < 1/2$, with some $\delta_n \to 0$, that does not depend on k, j and n. Next observe, that (3.12) follows by the same argument as in case (2) above. This completes the proof of (i) of the lemma.

Proof of (ii). First, we prove (3.9). As above, for that we need to bound $t_{n,1}$ and $t_{n,2}$ of (3.11).

Recall that $f_X = |A_X|^2/(2\pi)$, $f_{X\zeta} = A_X/(2\pi)$, $A_X = h(u)G(u)$, where h satisfies (3.5), which together with (1.3) implies that G is bounded away from infinity and zero. For $1 \le k \le j \le \nu$, define

$$\begin{aligned} \tilde{r}_{n,jk} &= 0, && G \in \Lambda_1(\Pi), \\ &= n^{-\beta} \frac{\log(2+j-k)}{(2+j-k)^{1-\beta}}, && G \in \Lambda_\beta(\Pi), \;\; 0 < \beta < 1, \\ &= \delta_n \frac{\log(2+j-k)}{(2+j-k)^{1-\epsilon}}, && G \in \mathcal{C}(\Pi), \;\; 0 < \epsilon < 1/2, \end{aligned}$$

where $\delta_n \to 0$.

By (iv) of Theorem 4.1, for $1 \le k \le j$,

$$\begin{aligned} &|E[w_{X,j}\overline{w_{X,k}}] - f_{X,j} I(j=k)| \\ &\qquad \le C\{(u_k^{-2d} + u_j^{-2d}) j^{-1} \log j + (u_k^{-2d} \wedge u_j^{-2d}) \tilde{r}_{n,jk}\} \\ &|E[w_{X,j}\overline{w_{\zeta,k}}] - f_{X\zeta,j} I(j=k)| \\ &\qquad \le C\{(u_k^{-d} + u_j^{-d}) j^{-1} \log j + (u_k^{-d} \wedge u_j^{-d}) \tilde{r}_{n,jk}\}. \end{aligned}$$

Since $f_X = |A_X|^2/(2\pi) = |hG|^2/(2\pi)$, assumptions on h and G used in part (ii) of lemma, imply that for all $u \in \Pi$,

$$\begin{aligned} f_X(u) &\le C|u|^{-2d}, \quad f_X^{-1}(u) \le C|u|^{2d}, \\ |f_{X\zeta}(u)| &\le C|u|^{-d}, \quad |f_{X\zeta}^{-1}(u)| \le C|u|^{d}. \end{aligned}$$

Therefore, for $1 \le k \le j$,

$$\begin{aligned} &(f_{X,j} f_{X,k})^{-1} (u_k^{-2d} + u_j^{-2d})^2 \le C|j/k|^{2|d|}, \\ &(f_{X,j} f_{X,k})^{-1} (u_k^{-2d} \wedge u_j^{-2d})^2 \le C, \\ &(f_{X,j})^{-1} (u_k^{-d} + u_j^{-d})^2 \le C|j/k|^{2|d|}, \quad (f_{X,j})^{-1} (u_k^{-d} \wedge u_j^{-d})^2 \le C. \end{aligned}$$

To prove (3.9) of (ii), we shall use the bound (3.11). It suffices to show that $t_{n,1} + t_{n,2}$ can be bounded above by the r.h.s. of (3.9). The above bounds readily yield that

$$\begin{aligned} t_{n,1} &\le C \sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 (j^{-1} \log j + \tilde{r}_{n,jj}), \\ t_{n,2} &\le C \sum_{1 \le k < j \le \nu} |b_{n,j} f_{X,j}| \, |b_{n,k} f_{X,k}| \big((\frac{j}{k})^{2|d|} \frac{\log^2 j}{j^2} + \tilde{r}^2_{n,jk} \big). \end{aligned}$$

The argument used in evaluating $s_{n,1}$ and $s_{n,2}$ in Lemma 2.1 yields

$$\begin{aligned} &\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 \frac{\log j}{j} + \sum_{1 \le k < j \le \nu} |b_{n,j} f_{X,j}| |b_{n,k} f_{X,k}| (\frac{j}{k})^{2|d|} \frac{\log^2 j}{j^2} \\ &\quad \le C \min \big(b^2_{f,n} \log^3(n), \ b_{f,n} B_{f,n} \big), \end{aligned}$$

whereas estimation in Cases (2) - (3) above yields that

$$\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 \tilde{r}_{n,jk} + \sum_{1 \le k < j \le \nu} |b_{n,j} f_{X,j}| \, |b_{n,k} f_{X,k}| \tilde{r}^2_{n,jk} \le C \epsilon_{n,G} B^2_{f,n}.$$

Therefore,

$$t_{n,1} + t_{n,2} \le C \Big(\min \big(b^2_{f,n} \log^3(n), \ b_{f,n} B_{f,n} \big) + \epsilon_{n,G} B^2_{f,n} \Big), \tag{3.14}$$

which proves (3.9).

Observe that $\epsilon_{n,G} \to 0$. Therefore, (3.9), (3.6) and (3.2) imply (3.10). This completes the proof of the lemma. □

As seen above, proving CLT for $v_n^{-1}(Q_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j})$ required some smoothness of the spectral density f_X and the transfer function A_X. Conditions on A_X can be relaxed if one wishes to establish only an upper bound for the mean square error of the estimator $Q_{n,X}$ of $\sum_{j=1}^{\nu} b_{n,j} f_{X,j}$ as is shown in the next theorem.

Theorem 3.3 *Let $\{X_j\}$ be as in (1.2) with $E\zeta_0^4 < \infty$. Assume that $f_X(u) = |u|^{-2d}g(u)$, $|d| < 1/2$, where g is a continuous function, bounded away from 0 and ∞.*
(i) *Then*

$$E(Q_{n,X} - EQ_{n,X})^2 \le CB_{f,n}^2. \tag{3.15}$$

(ii) *In addition,*

$$E\Big(Q_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j}\Big)^2 \le CB_{f,n}^2, \tag{3.16}$$

in each of the following three cases.

c1) $d = 0, \quad g \in \Lambda_\beta[\Pi], \qquad 1/2 < \beta \le 1;$
c2) $d \ne 0, \quad g \in \Lambda_\beta[\Pi], \qquad 1/2 < \beta \le 1;$
c3) $|\dot f_X(u)| \le C\, u^{-1-2d}, \qquad 0 < u \le \pi.$

Moreover, in case c1),

$$EQ_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_j = o(B_{f,n}). \tag{3.17}$$

If $b_{n,j}$'s satisfy (3.6), then (3.17) holds also in cases c2) *and* c3).

Proof. (i) Recall $I_{X,j} = |w_{X,j}|^2$. By Proposition 2.1,

$$\begin{aligned} E(Q_{n,X} - EQ_{n,X})^2 &= \mathrm{Var}\Big(\sum_{j=1}^{\nu} b_{n,j} I_{X,j}\Big) \\ &\le C \sum_{j,k=1}^{\nu} |b_{n,j} b_{n,k}| |E[w_{X,j}\overline{w_{X,k}}]|^2. \end{aligned}$$

For $j = k$ bounding $(E|w_{X,j}|^2)^2 \le 2(E|w_{X,j}|^2 - f_{X,j})^2 + 2f_{X,j}^2$, and letting

$$\begin{aligned} s'_{n,1} &:= \sum_{j=1}^{\nu} b_{n,j}^2 (E|w_{X,j}|^2 - f_{X,j})^2, \\ s'_{n,2} &:= \sum_{1 \le k < j \le \nu} |b_{n,j} b_{n,k}| |E[w_{X,j}\overline{w_{X,k}}]|^2, \end{aligned}$$

one obtains

$$E(Q_{n,X} - EQ_{n,X})^2 \le C(s'_{n,1} + s'_{n,2} + B_{f,n}^2). \tag{3.18}$$

Under assumptions of this theorem, by (iv) of Theorem 4.1, for $1 \le k < j \le \nu$, $(0 < \epsilon < 1/2)$,

$$\begin{aligned} |E|w_{X,j}|^2 - f_{X,j}| &\le Cu_j^{-2d}\big(j^{-1}\log j + \delta_n\big), \\ |E[w_{X,j}\overline{w_{X,k}}]| &\le C\big((u_k^{-2d} + u_j^{-2d}) j^{-1}\log j + (u_k^{-2d} \wedge u_j^{-2d})\delta_n \ell(\epsilon, j-k)\big), \end{aligned}$$

where $\delta_n \to 0$. Observe that $s'_{n,i} \le t_{n,i}$, $i = 1, 2$, where $t_{n,1}$ and $t_{n,2}$ are as in the proof of Lemma 3.1. Therefore, the same argument as used in proving (3.14) implies that $s'_{n,1} + s'_{n,2}$ satisfies the bound (3.14), which in turn yields

$$s'_{n,1} + s'_{n,2} \le C\big(b_{f,n} B_{f,n} + \epsilon_{n,G} B^2_{f,n}\big) \le C B^2_{f,n},$$

since $b_{f,n} \le B_{f,n}$. This completes proof of (3.15).

(ii) Observe that

$$\begin{aligned} |E|w_{X,j}|^2 - f_{X,j}| &\le Cu_j^{-2d} n^{-\beta}, && \text{in case c1)} \\ &\le Cu_j^{-2d}(j^{-1}\log j + n^{-\beta}), && \text{in case c2)} \\ &\le Cu_j^{-2d}(j^{-1}\log j), && \text{in case c3)} \end{aligned}$$

by parts (i), (iv) and (iii) of Theorem 4.1, respectively. Let

$$q_n := \Big| EQ_{n,X} - \sum_{j=1}^{\nu} b_{n,j} f_{X,j} \Big| = \Big| \sum_{j=1}^{\nu} b_{n,j}(E|w_{X,j}|^2 - f_{X,j}) \Big|.$$

Under assumptions of theorem, $f_{X,j}^{-1} \le Cu_j^{2d}$, $0 < u \le \pi$. Thus, in case c1),

$$\begin{aligned} q_n &\le C \sum_{j=1}^{\nu} |b_{n,j} f_{X,j}| n^{-\beta} \le C n^{1/2-\beta} \big(\sum_{j=1}^{\nu} (b_{n,j} f_{X,j})^2 \big)^{1/2} \\ &= o(B_{f,n}), \end{aligned} \tag{3.19}$$

which proves (3.16) and (3.17).

In case c2),

$$q_n \le C \sum_{j=1}^{\nu} |b_{n,j} f_{X,j}| (j^{-1}\log n + n^{-\beta}).$$

By the same argument as in the proof of (2.16), it follows that

$$\begin{aligned} \sum_{j=1}^{\nu} |b_{n,j} f_{X,j}| j^{-1} \log j &= O(B_{f,n}), \\ &= o(B_{f,n}), \quad \text{if (3.6) holds,} \end{aligned}$$

which together with (3.19) yields (3.16) and (3.17).

In case c3), proof of (3.16) and (3.17) is the same as in case c2). This completes proof of theorem. □

Remark 3.1 Consider now the sum

$$Q_{n,X} = \sum_{j=1}^{\theta n} b_{n,j} I_{X,j}, \qquad (0 < \theta < 1/2), \tag{3.20}$$

where summation is taken over a fraction $\{1, \cdots, \theta n\}$ of the set $\{1, \cdots, \nu\}$. Then periodograms $I_{X,j}$ used in $Q_{n,X}$ are based on frequencies u_j from the zero neighborhood $[0, \Delta]$, $\Delta = 2\pi\theta$ which is a sub-interval of $[0, \pi]$. Observe that smoothness conditions on f_X and A_X are required only to obtain upper bounds on covariances $E[w_{X,j}\overline{w_{X,k}}]$ and $E[w_{X,j}\overline{w_{\zeta,k}}]$ in Theorem 4.1. Therefore, it follows that in order for these bounds to be valid at frequencies $u_j \in [0, \Delta]$ it suffices to impose smoothness conditions on f_X and A_X on a slightly larger interval $[0, a]$, $a > \Delta$, covering $[0, \Delta]$.

Hence, for the sum $Q_{n,X}$ of (3.20), all of the above results derived in this section hold true if conditions on f_X and A_X are satisfied on some interval $[0, a]$, with $a > \Delta$, instead of $[0, \pi]$. For example, Theorem 3.2 holds if on $[0, a]$, $A_X = hG$ and h, f satisfy (3.5) and (1.3). Theorem 3.3 is valid, if on $[0, a]$, $f_X(u) = |u|^{-2d} g(u)$ and g is Lipshitz continuous of order $\beta > 1/2$.

No restrictions are required on f_X on the interval $[a, \pi]$, except the integrability condition $\int_a^\pi f_X(u)du < \infty$.

Eample 3.1 Consider the stationary ARFIMA(p, d, q) model

$$\phi(B)X_j = (1 - B)^{-d}\theta(B)\zeta_j, \quad j \in \mathbb{Z}, \qquad \{\zeta_j\} \sim IID(0, \sigma_\zeta^2).$$

We shall show that this model satisfies the smoothness and differentiability conditions (1.3) and (1.4) pertaining to the spectral density f_X and the transfer function A_X. Indeed, when the complex roots of polynomials $\phi(z)$ and $\theta(z)$ lie outside unite circle $\{|z| \le 1\}$, the operator $\phi(B)^{-1}\theta(B) = \sum_{k=0}^\infty b_k B^k$ can be written as a series of powers B^k and the above equations are rewritten as

$$X_j = (1 - B)^{-d} Y_j, \quad Y_j = \phi(B)^{-1}\theta(B)\zeta_j = \sum_{k=0}^\infty b_k \zeta_{j-k}, \quad j \in \mathbb{Z},$$

where $\{Y_j\}$ is a short memory process with a absolutely summable weights b_k. Its transfer and spectral density functions

$$A_Y(u) = \frac{\theta(e^{-\mathbf{i}u})}{\phi(e^{-\mathbf{i}u})} = \sum_{k=0}^\infty b_k e^{-\mathbf{i}uk}, \quad f_Y(u) = \frac{\sigma_\zeta^2}{2\pi}|A_Y(u)|^2, \quad u \in \Pi,$$

are continuous and bounded away from 0. Moreover, A_Y and f_Y have bounded derivatives. Then, with $h(u) = (1 - e^{-\mathbf{i}u})^{-d}$,

$$\begin{aligned} f_X(u) &= |h(u)|^2 f_Y(u) \equiv |u|^{-2d} g(u), \\ A_X(u) &= h(u)A_Y(u), \end{aligned} \tag{3.21}$$

where $g(u) = (|h(u)|^2/|u|^{-2d}) f_Y(u) = (2|\sin(u/2)|/|u|)^{-2d} f_Y(u)$ is a continuous function on Π, bounded away from 0 and ∞. Whence, f_X satisfies assumption (1.3).

Moreover, h is differentiable and satisfies

$$|h(u)| \le C|u|^{-2d}, \quad |\dot{h}(u)| \le C|u|^{-1-2d}, \quad \forall\, u \in [0,\pi], \tag{3.22}$$
$$|h(u)| \sim |u|^{-2d}, \quad u \to 0.$$

Thus

$$\begin{aligned} |\dot{A}_X(u)| &\le C(|\dot{h}(u)||A_Y(u)| + |h(u)||\dot{A}_Y(u)|) \\ &\le C|1-e^{-\mathbf{i}u}|^{-d-1} \le C|u|^{-d-1}, \qquad 0<|u|<\pi, \end{aligned}$$

and hence A_X satisfies (1.4). Note also that $A_X = hA_Y$ is naturally factored into a differentiable component h and continuous component A_Y as required in Theorem 3.2. Thus Theorems 2.1, 3.1, 3.2 and 3.3 are applicable.

Eample 3.2 Now consider a more general process $\{X_j\}$,

$$X_j = (1-B)^{-d} Y_j, \qquad j \in \mathbb{Z}, \qquad |d| < 1/2,$$

where $Y_j = \sum_{k=0}^{\infty} b_k \zeta_{j-k}$, $\{\zeta_j\} \sim IID(0,1)$, $\sum_{k=0}^{\infty} |b_k| < \infty$, is a short memory process. Observe that $\{X_j\}$ has the spectral density and transfer function as in (3.21). Hence, the same argument as used in (3.21), shows that f_X satisfies (1.3) with parameter $|d| < 1/2$. Although A_X may not satisfy (1.4), because A_Y is only continuous, but A_X is factored as required in Theorems 3.2 and 3.3. Hence, these theorems are applicable.

4 Appendix. Covariances of DFT

Here we shall present some preliminary results needed about the properties of discrete Fourier transforms. Observe, that if $\{\zeta_j\}$ is a white noise process $WN(0,1)$, then its discrete Fourier transforms (DFT's) are uncorrelated:

$$\begin{aligned} E[w_{\zeta,j}\overline{w_{\zeta,k}}] &= \frac{1}{2\pi}, \quad 1 \le k = j \le n, \\ &= 0, \qquad 1 \le k < j \le n, \end{aligned} \tag{4.1}$$

which follows in view of (2.27).

In general, unlike in the white noise case, DFT's of a stationary process $\{X_j\}$ with a spectral density f_X are correlated, i.e. covariances $E[w_{X,j}\overline{w_{X,k}}] \ne 0$, for $k \ne j$.

Consider now the two linear processes

$$X_j = \sum_{k=0}^{\infty} a_k \zeta_{j-k}, \quad Y_j = \sum_{k=0}^{\infty} b_k \zeta_{j-k}, \;\; j \in \mathbb{Z}; \quad \sum_{k=0}^{\infty} a_k^2 < \infty, \; \sum_{k=0}^{\infty} b_k^2 < \infty,$$

with the same white noise innovations $\{\zeta_j\} \sim WN(0,\sigma^2)$. Let

$$A_X(v) := \sum_{k=0}^{\infty} e^{-\mathbf{i}kv} a_k, \quad A_Y(v) := \sum_{k=0}^{\infty} e^{-\mathbf{i}kv} b_k,$$

denote their respective transfer functions and

$$f_X(v) = (\sigma^2/2\pi)|A_X(v)|^2, \qquad f_Y(v) = (\sigma^2/2\pi)|A_Y(v)|^2,$$

their respective spectral densities.

Let $f_{XY}(v)$ denote a (complex valued) cross-spectral density:

$$\begin{aligned} f_{XY}(v) &:= \frac{\sigma^2}{2\pi} A_X(v)\overline{A_Y(v)}, \quad v \in \Pi, \\ E[X_j Y_{j-k}] &= \int_\Pi e^{\mathbf{i}kv} f_{XY}(v)dv = \frac{\sigma^2}{2\pi}\sum_{l=0}^{\infty} a_{l+k} b_l, \quad k \geq 0, \ j \in \mathbb{Z}. \end{aligned} \tag{4.2}$$

Observe, that in the case of white noise $Y_j = \zeta_j$, $j \in \mathbb{Z}$,

$$\begin{aligned} f_{X\zeta}(v) &:= \frac{\sigma^2}{2\pi} A_X(v), \quad v \in \Pi, \\ E[X_j Y_{j-k}] &= \frac{\sigma^2}{2\pi}\int_\Pi e^{\mathbf{i}kv} A_X(v)dv = \sigma^2 a_k, \quad k \geq 0. \end{aligned} \tag{4.3}$$

Theorem 4.1 below summarizes asymptotic properties of cross-covariances $E[w_{X,j}\overline{w_{Y,k}}]$. It generalizes and extends Theorem 2 of Robinson (1995a) for short memory and long memory time series, which enable derivation of the upper bounds based on Bartlett approximation of this paper. Its proof is technical and full details are given in Giraitis and Koul (2010).

In case when Fourier frequencies in covariances $E[w_{X,j}\overline{w_{Y,k}}]$ are from an interval $(-\Delta, \Delta)$, $\Delta < \pi$ (a neighborhood of 0), smoothness conditions on f_X, f_Y, A_X, A_Y are local, i.e. they need to be imposed on an interval $[0, a]$, $a > \Delta$. If the Fourier frequencies are from the entire spectrum over Π, then smoothness conditions have to be imposed on the whole spectrum Π.

To proceed further, let $\mathcal{C}[0, a]$ denote complex valued functions that are continuous on $[0, a]$, and $\Lambda_\beta[0, a]$, $0 < \beta \leq 1$ denote Lipschitz continuous functions with parameter β. We write $h \in \mathcal{C}_{1,\alpha}[0, a]$, $|\alpha| < 1$, if

$$|h(u)| \leq C|u|^{-\alpha}, \quad |\dot{h}(u)| \leq C|u|^{-1-\alpha}, \quad \forall\, u \in [0, a].$$

Class $\mathcal{C}_{1,\alpha}[0, a]$ allows an infinity peak and non-differentiability at 0, whereas $\Lambda_\beta[0, a]$ covers continuous piecewise differentiable functions.

Note that for any $h \in \mathcal{C}[0, a]$,

$$\omega_h(\eta) \ := \sup_{u,v\in[0,a]:|u-v|\leq\eta} |h(u) - h(v)| \to 0, \qquad \eta \to 0,$$

because h is uniformly continuous on $[0, a]$. Define

$$\delta_{n,\epsilon}(h) := \omega_h(n^{-1}\log n) + (\log n)^{-\epsilon}, \quad 0 < \epsilon < 1.$$

For the sake of brevity introduce the functions

$$
\begin{aligned}
\ell_n(\epsilon;k) &:= \frac{\log(2+k)}{(2+k)^{1-\epsilon}} + \frac{\log(2+n-k)}{(2+n-k)^{1-\epsilon}}, \qquad 0\le k\le j\le n,\\
r_{n,jk}(g) &:= 0, \qquad\qquad\qquad\qquad g\in\Lambda_1[0,\,a], \quad \beta=1,\\
&:= n^{-\beta}\,\ell_n(\beta;j-k), \qquad g\in\Lambda_\beta[0,\,a], \quad 0<\beta<1,\\
&:= \delta_{n,\epsilon}(g)\,\ell_n(\epsilon;j-k), \quad g\in\mathcal{C}[0,\,a], \qquad \epsilon\in(0,\,1).
\end{aligned}
$$

Theorem 4.1 *Let either $\Delta < a \le \pi$, or $\Delta = a = \pi$. Then, the following facts (i)-(iv) hold for all $0 < |u_k| \le u_j \le \Delta$.*
(i) *If $f_{XY} \in \Lambda_\beta[0,a]$, $0<\beta\le 1$, then*

$$
\begin{aligned}
\Big|E[w_{X,j}\overline{w_{Y,j}}] - f_{XY}(u_j)\Big| &\le Cn^{-1}\log n, && \beta=1,\\
&\le Cn^{-\beta} && 0<\beta<1.\\
\Big|E[w_{X,j}\overline{w_{Y,k}}]\Big| &\le Cn^{-1}\log n, && \beta=1,\\
&\le Cn^{-\beta}\ell_n(\beta;j-k), && 0<\beta<1,\ k<j.
\end{aligned}
$$

(ii) *If $f_{XY}\in\mathcal{C}[0,a]$, then, $\forall\,\epsilon\in(0,1)$,*

$$
\begin{aligned}
\Big|E[w_{X,j}\overline{w_{Y,j}}] - f_{XY}(u_j)\Big| &\le C\delta_{n,\epsilon}(f_{XY});\\
|E[w_{X,j}\overline{w_{Y,k}}]| &\le C\delta_{n,\epsilon}(f_{XY})\,\ell_n(\epsilon;j-k), \qquad k<j.
\end{aligned}
$$

(iii) *If $f_{XY}\in\mathcal{C}_{1,\alpha}[0,a]$, $|\alpha|<1$, then*

$$
\begin{aligned}
\Big|E[w_{X,j}\overline{w_{Y,j}}] - f_{XY}(u_j)\Big| &\le Cu_j^{-\alpha}\,j^{-1}\log j,\\
|E[w_{X,j}\overline{w_{Y,k}}]| &\le C\big(|u_k|^{-\alpha}+u_j^{-\alpha}\big)j^{-1}\log j, \quad k<j.
\end{aligned}
$$

(iv) *Suppose $f_{XY}=hg$, where $h\in\mathcal{C}_{1,\alpha}[0,a]$, $|\alpha|<1$, and $g\in\Lambda_\beta[0,a]\cup\mathcal{C}[0,a]$, $0<\beta\le 1$. Then, for all $1\le|k|\le j\le\tilde{n}$,*

$$
\begin{aligned}
&\Big|E[w_{X,j}\overline{w_{Y,k}}] - f_{XY}(u_j)I(j=k)\Big|\\
&\quad\le\ C\Big((|u_k|^{-|\alpha|}+u_j^{-|\alpha|})j^{-1}\log j + \big(|u_k|^{-|\alpha|}\wedge u_j^{-|\alpha|}\big)r_{n,jk}(g)\Big).
\end{aligned}
$$

The constant C in the above (i)-(iv) does not depend on k,j and n.

Parts (i)-(ii) of Theorem 4.1 consider the case when f_X is continuous and bounded, whereas part (iv) covers the case when $f_X(u)=|u|^{-2d}g(u)$, $|d|<1/2$ can be factored into component $|u|^{-2d}$ and a bounded continuous part $g(u)$. The case when g has also bounded derivative is covered in part (iii). Obtaining upper bounds in (i)-(iv) of Theorem 4.1 does not require the process $\{X_j\}$ to be linear. For convenience of applications, theorem is formulated for a cross-spectral density of two stationary linear processes X and Y with the same underlying white noise innovation process. This allows to express the cross spectral density f_{XY} via transfer functions A_X and A_Y as indicated in (4.2). In general, the results of Theorem 4.1 are valid for any spectral density or cross-spectral density that satisfies smoothness condition of this theorem.

Acknowledgement

Research supported in part by ESRC grant RES062230790 and the NSF DMS grant 0704130.

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