Asymptotic distribution of the bias corrected LSEs in measurement error linear regression models under long memory[1]

Hira L. Koul and Donatas Surgailis
Michigan State University and Vilnius University

April 30, 2018

Abstract

This paper discusses the consistency and asymptotic distribution of the bias corrected least squares estimators of regression parameters in linear regression models when covariates have measurement error and errors and covariates form mutually independent long memory moving average processes. In the structural measurement error linear regression model, the nature of the asymptotic distribution of suitably standardized bias corrected least squares estimators depends on the range of the values of $\delta_{\max} = \max\{d_X + d_\varepsilon, d_X + d_u, d_u + d_\varepsilon, 2d_u\}$, where d_X, d_u, d_ε, are the long memory parameters of the covariate, measurement error and regression error processes, respectively. This limiting distribution is Gaussian when $\delta_{\max} < 1/2$ and non-Gaussian in the case $\delta_{\max} > 1/2$. The paper also discusses the asymptotic distribution of these estimators in some functional measurement error linear regression models, where the unobservable covariate is non-random. In these models, the limiting distribution of LSEs is always a Gaussian distribution determined by the range of the values of $d_\varepsilon - d_u$.

1 Introduction

The classical regression analysis often assumes that both the response variable and the predicting variables are fully observable and that the errors are independent. But, as is evidenced in the monographs of Fuller (1987), Cheng and Van Ness (1999), Carroll, Ruppert, Stefanski and Craineceanu (2006), and the references therein, there are numerous examples of practical importance where the predicting variables are not observable. Instead one observes surrogates that provide estimates of the true predictors. Such models are known as the regression models with measurement error. On the other hand there are examples from the various scientific disciplines where observed data do not obey the assumption of

[1] *MSC:* primary 62G08; secondary 62G10
Key words and phrases: Gaussian, Non-Gaussian, Wiener-Itô integrals.

independence. Instead one observes data that is generated by some long memory processes. In economics the first authors to point out the usefulness of long memory processes were Granger and Joyeux (1980) and Hosking (1981). The monographs of Giraitis, Koul and Surgailis (2012) and Beran, Feng, Ghosh and Kulik (2013), and the references therein, contain numerous other examples of long memory processes and relevant theoretical results.

The focus of this paper is to study the consistency and asymptotic distribution theory of the bias corrected least squares estimators (LSEs) of the parameters in linear regression models when predicting variables are measured with error and when the covariate and the regression and measurement error processes have long memory. We discuss both structural and functional models. In the former, the predicting variables are random while in the latter they are non-random.

For the sake of relative transparency, we first discuss the simple structural measurement error (ME) linear regression model in the next section, where the long memory models along with the needed assumptions are also described. It also contains the proof of the consistency of the bias corrected LSEs in this model. The derivation of the asymptotic distribution of suitably standardized versions of these estimators is facilitated by the derivation of the limiting distributions of some general quadratic forms of long memory moving average processes given in Section 3. These results in turn are used in Sections 4 and 5 to derive the limiting distributions of the bias corrected LSEs in the simple and multiple structural ME linear regression models, respectively. Section 6 derives similar results for the functional ME simple linear regression model where the true unobservable predicting variable is nonrandom. Section 7 contains the proofs of some of the results of Sections 3 and 4.

2 Simple structural ME linear regression model and long memory

In this section we shall focus on the simple structural ME linear regression model and establish the consistency of the bias corrected LSEs. In this model the unobserved predicting r.v. X_i, the observable random surrogate Z_i and the response Y_i are related to each other by the following relations.

$$(2.1) \quad Y_i = \alpha + \beta X_i + \varepsilon_i, \quad Z_i = X_i + u_i, \quad E\varepsilon_i = 0, \quad Eu_i = 0, \quad i \in \mathbb{Z} := \{0, \pm 1, \cdots\}.$$

Moreover, we assume that the process $\{(\varepsilon_i, X_i, u_i); i \in \mathbb{Z}\}$ is strictly stationary and ergodic and each of these processes form a long memory moving average (LMMA) as in the following assumptions.

Assumption (E) Errors $\{\varepsilon_i\}$ form a moving average process

$$\varepsilon_i = \sum_{k=0}^{\infty} b_k \zeta_{i-k}, \quad i \in \mathbb{Z}, \tag{2.2}$$

where $\{\zeta_s; s \in \mathbb{Z}\}$ are i.i.d., with zero mean and unit variance, with coefficients

$$b_j \sim \kappa_\varepsilon \, j^{-(1-d_\varepsilon)}, \quad \text{as } j \to \infty, \text{ for some } 0 < \kappa_\varepsilon < \infty \text{ and } 0 < d_\varepsilon < 1/2. \tag{2.3}$$

Assumption (X) Covariates $\{X_i\}$ form a LMMA process

$$X_i = \mu_X + \sum_{k=0}^{\infty} a_k \xi_{i-k}, \quad i \in \mathbb{Z}, \quad \text{with MA coefficients} \quad a_j \sim \kappa_X j^{-(1-d_X)}, \ j \to \infty, \tag{2.4}$$

for some $\mu_X \in \mathbb{R}, \ \kappa_X > 0, 0 < d_X < 1/2$, and standardized i.i.d. innovations $\{\xi_s\}$.

Assumption (U) Measurement errors $\{u_i\}$ form a LMMA process

$$u_i = \sum_{k=0}^{\infty} c_k \eta_{i-k}, \quad i \in \mathbb{Z}, \quad \text{with MA coefficients} \quad c_j \sim \kappa_u j^{-(1-d_u)}, \ j \to \infty \tag{2.5}$$

for some $\kappa_u > 0, 0 < d_u < 1/2$, and standardized i.i.d. innovations $\{\eta_s\}$. Moreover, $\mathrm{Var}(u_0)$ is known.

Assumption (I) The innovation sequences $\{\zeta_s; s \in \mathbb{Z}\}$, $\{\xi_s; s \in \mathbb{Z}\}$ and $\{\eta_s; s \in Z\}$ are mutually independent.

From now on let ε, X, u denote copies of ε_0, X_0, u_0, respectively. For any r.v. η with finite variance, let $\sigma_\eta^2 := \mathrm{Var}(\eta)$.

The above assumptions imply that the r.v.'s ε_i, X_i, u_i are mutually independent for each $i \in \mathbb{Z}$ and

$$0 < \sigma_\varepsilon^2 = E\varepsilon^2 = \sum_{k=0}^{\infty} b_k^2 < \infty, \quad 0 < \sigma_X^2 = EX^2 = \sum_{k=0}^{\infty} a_k^2 < \infty.$$
$$0 < \sigma_u^2 = Eu^2 = \sum_{k=0}^{\infty} c_k^2 < \infty.$$

Let $B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$, $a > 0, b > 0$ denote Beta function. From (7.2.10) of Giraitis, Koul and Surgailis (2012) (GKS), we obtain that

$$\begin{aligned}\mathrm{Cov}(\varepsilon_0, \varepsilon_k) &\sim \kappa_\varepsilon^2 B(d_\varepsilon, 1-2d_\varepsilon) k^{-(1-2d_\varepsilon)}, \ \mathrm{Cov}(X_0, X_k) \sim \kappa_X^2 B(d_X, 1-2d_\varepsilon) k^{-(1-2d_X)}, \\ \mathrm{Cov}(u_0, u_k) &\sim \kappa_u^2 B(d_u, 1-2d_u) k^{-(1-2d_u)}, \quad k \to \infty.\end{aligned} \tag{2.6}$$

The sum of the absolute values of these covariance diverge. Hence each of these processes has long memory.

We shall now describe the bias corrected LSEs. For any two sets of variables $U_i, V_i, 1 \leq i \leq n$, let

$$\bar{U} := \frac{1}{n}\sum_{i=1}^{n} U_i, \quad S_{UV} := \frac{1}{n}\sum_{i=1}^{n}(U_i - \bar{U})(V_i - \bar{V}).$$

Then the naive LSEs of α, β, where one simply replaces X_i's in the classical LSE by Z_i's, are

$$\tilde{\beta} := S_{ZY}/S_{ZZ}, \quad \tilde{\alpha} := \bar{Y} - \tilde{\beta}\bar{Z}.$$

As argued say in Fuller (1987), under the classical i.i.d. and finite variance set up,

$$\tilde{\beta} - \beta \to -\beta\,\frac{\sigma_u^2}{\sigma_X^2 + \sigma_u^2}, \quad \text{a.s.}$$

Hence these estimators are inconsistent. The bias correct estimators suitable here are

$$\hat{\beta} := \frac{S_{ZY}}{S_{ZZ} - \sigma_u^2}, \quad \hat{\alpha} := \bar{Y} - \hat{\beta}\bar{Z}. \tag{2.7}$$

We shall first establish the consistency of these estimators under the assumed stationarity, ergodicity and long memory set up. Rewrite

$$Y_i = \alpha + \beta Z_i + \varepsilon_i - \beta u_i, \quad Z_i = X_i + u_i. \tag{2.8}$$

Let

$$T_n := \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})(\varepsilon_i - \beta u_i).$$

Use the relation $Z_i = X_i + u_i$, to obtain the decomposition

$$\begin{aligned} T_n &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(\varepsilon_i - \beta u_i) + \frac{1}{n}\sum_{i=1}^{n}(u_i - \bar{u})\varepsilon_i - \beta\frac{1}{n}\sum_{i=1}^{n}(u_i - \bar{u})^2 \\ &= S_{X\varepsilon} - \beta S_{Xu} + S_{u\varepsilon} - \beta S_{uu}. \end{aligned} \tag{2.9}$$

By the mutual independence of ε_i, X_i, u_i and the assumption that $E\varepsilon_i \equiv 0$, $Eu_i \equiv 0$,

$$E(T_n) = -\beta E(S_{uu}) = -\beta\left[\sigma_u^2 - \mathrm{Var}(\bar{u})\right].$$

By (2.6), $\mathrm{Var}(\bar{u}) = O(n^{2d_u - 1}) \to 0$ and by the Ergodic Theorem and the assumed stationarity,

$$T_n \to -\beta\sigma_u^2, \qquad S_{ZZ} - \sigma_u^2 \to \sigma_X^2 > 0, \quad \text{a.s.}$$

These facts now clearly imply that

$$\hat{\beta} - \beta = \frac{S_{ZY}}{S_{ZZ} - \sigma_u^2} - \beta = \frac{T_n + \beta\sigma_u^2}{S_{ZZ} - \sigma_u^2} \to 0, \quad \text{a.s.,} \tag{2.10}$$

thereby proving the strong consistency of $\hat{\beta}$ for β. This fact and the Ergodic Theorem in turn imply that $\hat{\alpha} \to \alpha$, a.s.

The derivation of the asymptotic distribution of suitably standardized versions of these estimators and their analogs in multiple linear regression models is facilitated by the more general asymptotic distributional results about certain quadratic forms established in the next section.

3 Limit theorem for quadratic forms

Let $\gamma_{t,i} = \sum_{k=0}^{\infty} b_{k,i}\xi_{t-k,i}, t \in \mathbb{Z}, i = 1, \cdots, m$ be m mutually independent LMMA processes with MA coefficients $b_{k,i} \sim \kappa_i k^{d_i - 1}$, $d_i \in (0, 1/2), \kappa_i > 0$ with i.i.d. mutually independent innovations $\{\xi_{s,i}\} \sim IID(0,1), i = 1, \cdots, m$. Let $\Pi_m \subset \{(i,j); 1 \le i \le j \le m\}$ be a non-empty subset of the set of all ordered pairs $(i,j), 1 \le i \le j \le m$ and $\gamma_i := \{\gamma_{t,i};\, t \in \mathbb{Z}\}$. Define the sample cross-covariance between γ_i and γ_j to be

$$S_{\gamma_i,\gamma_j} = n^{-1}\sum_{t=1}^{n}(\gamma_{t,i} - \bar{\gamma}_i)(\gamma_{t,j} - \bar{\gamma}_j), \quad (i,j) \in \Pi_m.$$

We also need to define the normalizing sequence as follows.

$$\delta_{\max} := \max\{d_i + d_j; (i,j) \in \Pi_m\}, \tag{3.1}$$

$$A(n) := \begin{cases} n^{1-\delta_{\max}}, & \delta_{\max} > 1/2, \\ n^{1/2}, & \delta_{\max} < 1/2, \\ (n/\log n)^{1/2}, & \delta_{\max} = 1/2. \end{cases}$$

We are interested in deriving the asymptotic joint distribution of normalized quadratic forms

$$\mathcal{S}_n := \big\{A(n)\big(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}\big);\, (i,j) \in \Pi_m\big\}. \tag{3.2}$$

As shown below, the limit distribution of $\mathcal{S}_n$ is Gaussian or non-Gaussian depending on whether $\delta_{\max} \le 1/2$ or $\delta_{\max} > 1/2$. Before describing this distribution, we need to recall some preliminaries. From GKS, pp.410-411, we recall the definition of the stochastic integrals

$$I_i(f) = \int_{\mathbb{R}} f(s)W_i(ds), \qquad I_{ij}(g) = \int_{\mathbb{R}^2} g(s_1, s_2)W_i(ds_1)W_j(ds_2) \tag{3.3}$$

w.r.t. independent Brownian motions $W_i, i = 1, \cdots, m$ (for $i = j$ the second integral in (3.3) coincides with the usual double Wiener-Itô integral w.r.t. W_i). The integrals $I_i(f), I_{ij}(g)$, $(i,j) \in \Pi_m$ are jointly defined for any non-random integrands $f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}^2)$. Moreover, $EI(f) = EI_{ij}(g) = 0$ and

$$(3.4) \qquad EI_i(f)I_{i'}(f') = \begin{cases} 0, & i \neq i', \\ \langle f, f'\rangle, & i = i', \end{cases} \quad f, f' \in L^2(\mathbb{R}),$$

$$EI_i(f)I_{i'j'}(g) = 0, \qquad \forall i, i', j', \quad f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}^2),$$

$$EI_{ij}(g)I_{i'j'}(g') = \begin{cases} 0, & (i,j) \neq (i',j'), \\ \langle g, g'\rangle, & (i,j) = (i',j'),\ i \neq j, \\ 2\langle g, \mathrm{sym} g'\rangle, & i = i' = j = j', \end{cases} \qquad g, g' \in L^2(\mathbb{R}^2),$$

where $\langle f, f'\rangle = \int_{\mathbb{R}} f(s)f'(s)ds$ ($\|f\| := \sqrt{\langle f, f\rangle}$), $\langle g, g'\rangle = \int_{\mathbb{R}^2} g(s_1, s_2)g'(s_1, s_2)ds_1 ds_2$ ($\|g\| := \sqrt{\langle g, g\rangle}$) denote scalar products (norms) in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, respectively, and sym denotes the symmetrization, see GKS, sections 11.5 and 14.3.

Let $\Pi_m^+ := \{(i,j) \in \Pi_m; d_i + d_j > 1/2\}$. Introduce

$$(3.5) \qquad f_{d_i}(s) := \kappa_i \int_0^1 (t-s)_+^{d_i - 1} dt, \qquad 1 \le i \le m,$$

$$\widetilde{g}_{d_i,d_j}(s_1, s_2) := \kappa_i \kappa_j \int_0^1 (t-s_1)_+^{d_i-1}(t-s_2)_+^{d_j-1} dt,$$

$$g_{d_i,d_j}(s_1, s_2) := \widetilde{g}_{d_i,d_j}(s_1, s_2) - f_{d_i}(s_1) f_{d_j}(s_2), \qquad (i,j) \in \Pi_m^+.$$

Then $f_{d_i} \in L^2(\mathbb{R})$, $\widetilde{g}_{d_i,d_j} \in L^2(\mathbb{R}^2)$, $g_{d_i,d_j} \in L^2(\mathbb{R}^2)$, see GKS, Prop.11.5.6. Observe that

$$\begin{aligned}
&\langle \widetilde{g}_{d_i,d_j}, f_{d_i} \otimes f_{d_j}\rangle / \kappa_i^2 \kappa_j^2 \\
&= \int_{\mathbb{R}^2} ds_1 ds_2 \int_0^1 (t-s_1)_+^{d_i-1}(t-s_2)_+^{d_j-1} dt \int_0^1 (t_1 - s_1)_+^{d_i-1} dt_1 \int_0^1 (t_2 - s_2)_+^{d_j - 1} dt_2 \\
&= \int_{(0,1]^3} dt dt_1 dt_2 \int_{\mathbb{R}} (t-s_1)_+^{d_i-1}(t_1 - s_1)_+^{d_i-1} ds_1 \int_{\mathbb{R}} (t-s_2)_+^{d_j-1}(t_2-s_2)_+^{d_j-1} ds_2 \\
&= B(d_i, 1-2d_i)B(d_j, 1-2d_j) \int_{(0,1]^3} \frac{dt dt_1 dt_2}{|t-t_1|^{1-2d_i}|t-t_2|^{1-2d_j}} \\
&= \frac{B(d_i, 1-2d_i)B(d_j, 1-2d_j)}{4d_i d_j} \int_0^1 (t^{2d_i} + (1-t)^{2d_i})(t^{2d_j} + (1-t)^{2d_j}) dt \\
&= \frac{B(d_i, 1-2d_i)B(d_j, 1-2d_j)}{2d_i d_j} \Big(\frac{1}{1 + 2(d_i + d_j)} + B(2d_i + 1, 2d_j + 1) \Big).
\end{aligned}$$

From this fact we obtain

$$(3.6) \qquad \|f_{d_i}\|^2 = \frac{\kappa_i^2 B(d_i, 1-2d_i)}{d_i(1+2d_i)},$$

$$\|\widetilde{g}_{d_i,d_j}\|^2 = \frac{\kappa_i^2 \kappa_j^2 B(d_i, 1-2d_i)B(d_j, 1-2d_j)}{(d_i + d_j)(2(d_i + d_j) + 1)},$$

$$\|g_{d_i,d_j}\|^2 = \|\widetilde{g}_{d_i,d_j}\|^2 - 2\langle \widetilde{g}_{d_i,d_j}, f_{d_i} \otimes f_{d_j}\rangle + \|f_{d_i}\|^2 \|f_{d_j}\|^2,$$

$$= \kappa_i^2\kappa_j^2 B(d_i, 1-2d_i)B(d_j, 1-2d_j)\Big\{\frac{1}{(d_i+d_j)(2(d_i+d_j)-1)} + \frac{1}{d_id_j(1+2d_i)(1+2d_j)} - \frac{1}{d_id_j(1+2(d_i+d_j))} - \frac{B(2d_i+1, 2d_j+1)}{d_id_j}\Big\}.$$

Consequently, the r.v.'s $I_{ij}(g_{d_i,d_j}),\ d_i + d_j > 1/2$ in (3.10) below are jointly well-defined and their second order characteristics can be obtained from (3.4).

We are now ready to state the main result of this section. Its proof appears in Section 7.

Theorem 3.1 *Let $\gamma_i = \{\gamma_{t,i}; t \in \mathbb{Z}\}$, $i = 1, \cdots, m$, be m stationary LMMA processes as above and $\mathcal{S}_n$ be as in (3.2). Assume in addition that*

$$E|\xi_{0,i}|^{2+\epsilon} < \infty, \quad (\exists\, \epsilon > 0) \quad \textit{for all } 1 \le i \le m, \tag{3.7}$$

and

$$E\xi_{0,i}^4 < \infty, \qquad \textit{for any } 1 \le i \le m \textit{ such that } (i,i) \in \Pi_m. \tag{3.8}$$

Then

$$\mathcal{S}_n \to_D \mathcal{R}_m = \big\{R_{ij}; (i,j) \in \Pi_m\big\}, \tag{3.9}$$

where, for any $(i,j) \in \Pi_m$,

$$R_{ij} \;:=\; \begin{cases} I_{ij}(g_{d_i,d_j})\mathbf{1}(d_i + d_j = \delta_{\max}), & \delta_{\max} > 1/2, \\ \sigma_{ij}Z_{ij}\mathbf{1}(d_i + d_j = 1/2), & \delta_{\max} = 1/2, \\ \sigma_{ij}Z_{ij}, & \delta_{\max} < 1/2, \end{cases} \tag{3.10}$$

with $g_{d_i,d_j} \in L^2(\mathbb{R}^2)$, $f_i \in L^2(\mathbb{R})$ as in (3.5), $\sigma_{ij} \ge 0$ as in (7.5) below, and Z_{ij} as independent $N(0,1)$ r.v.'s, $EZ_{ij}Z_{i'j'} = 0$, for $(i,j) \neq (i',j'), (i,j), (i',j') \in \Pi_m$.

Let $\Pi_{0m} \subset \{1, \cdots, m\}$ be a non-empty set, $d_{\max} := \max\{d_k; k \in \Pi_{0m}\}$ and $\mathcal{S}_{0n} := \{n^{(1/2)-d_{\max}}\bar{\gamma}_k; k \in \Pi_{0m}\}$ be a collection of normalized sample means. Then from Remark 4.3.1 in GKS, we obtain

$$\begin{aligned} \mathcal{S}_{0n} \to_D \mathcal{R}_{0m} = \{R_{0k}, k \in \Pi_{0m}\} \;&:=\; \{I_k(f_{d_k})\mathbf{1}(d_k = d_{\max}); k \in \Pi_{0m}\} \\ &=_D\; \{\sigma_k Z_k \mathbf{1}(d_k = d_{\max}); k \in \Pi_{0m}\}, \end{aligned} \tag{3.11}$$

where Z_k are independent standard normal r.v.'s and $\sigma_k^2 = \|f_{d_k}\|^2$ as in (3.6). The following corollary extends Theorem 3.1 to joint convergence of normalized sample means $\mathcal{S}_{0n}$ and sample cross-covariances $\mathcal{S}_n$.

Corollary 3.1 *Under the assumptions of Theorem 3.1,*

$$(\mathcal{S}_{0n}, \mathcal{S}_n) \to_D (\mathcal{R}_{0m}, \mathcal{R}_m). \tag{3.12}$$

The joint distribution of $(\mathcal{R}_{0m}, \mathcal{R}_m)$ *is Gaussian if* $\delta_{\max} \le 1/2$. *Moreover, for any* $k \in \Pi_{0m}$, $(i,j) \in \Pi_m$,

(3.13)

$$E(R_{0k}R_{ij}) = \begin{cases} (\kappa_k/d_k(1+d_k))E(\xi_{0,k}\xi_{0,i}\xi_{0,j})\sum_{s=0}^{\infty} b_{s,i}b_{s,j}\mathbf{1}(d_k = d_{\max}), & \delta_{\max} < 1/2, \\ 0, & \delta_{\max} \ge 1/2. \end{cases}$$

Remark 3.1 Note that under the assumption of independence of $\gamma_i, i = 1, \cdots, m$ the covariance in (3.13) when $\delta_{\max} < 1/2$ vanishes unless $k = i = j$ and $E\xi_{0,k}^3 \ne 0$ and the Z_k, Z_{ij} in (3.10), (3.11) are independent $N(0,1)$ r.v.'s.

Remark 3.2 Theorem 3.1 and Remark 3.1 can be extended to *mutually dependent* LMMA processes $\gamma_{t,i} = \sum_{k=0}^{\infty} b_{k,i}\xi_{t-k,i}, i = 1, \cdots, m$ with MA coefficients $b_{k,i} \sim \kappa_i k^{d_i - 1}, d_i \in (0, 1/2), \kappa_i > 0$ with innovations forming a $\mathbb{R}^m$-valued i.i.d. sequence $\{(\xi_{s,1}, \cdots, \xi_{s,m}); s \in \mathbb{Z}\}$ with zero mean, whose components are mutually dependent, viz., $E\xi_{0,i}\xi_{0,j} =: \sigma_{\xi,ij}, i,j = 1, \cdots, p$ where $\Sigma_\xi = E\xi_0\xi_0'$ is a general positive definite matrix. In such a case if (3.8) is strengthened to $E\xi_{0,i}^2\xi_{0,j}^2 < \infty$, $(i,j) \in \Pi_m$ the convergences in (3.9) and (3.12) hold under the same normalizations except that the limit r.v.'s there are generally correlated and have a representation w.r.t. *mutually correlated* Brownian motions $W_i, W_j, EW_i(t)W_j(t) = t\,\sigma_{\xi,ij}$. The double stochastic integral

$$I_{ij}(g) = \int_{\mathbb{R}^2} g(s_1, s_2)W_i(ds_1)W_j(ds_2) \tag{3.14}$$

w.r.t. such Brownian motion is well-defined for any $g \in L^2(\mathbb{R}^2)$ and has zero mean and a finite variance $EI_{ij}^2(g) = \sigma_{\xi,ii}\sigma_{\xi,jj}\|g\|^2 + \sigma_{\xi,ij}^2\langle g, g^*\rangle$ where $g^*(s_1,s_2) := g(s_2,s_1)$. In particular, the variance of the double Wiener-Itô integral $I_{ij}(g_{d_i,d_j}) = \int_{\mathbb{R}^2} g_{d_i,d_j}(s_1,s_2)W_i(ds_1)W_j(ds_2)$ in (3.6) equals

$$EI_{ij}^2(g_{d_i,d_j}) = \sigma_{\xi,ii}\sigma_{\xi,jj}\|g_{d_i,d_j}\|^2 + \sigma_{\xi,ij}^2\langle g_{d_i,d_j}, g_{d_j,d_i}\rangle, \tag{3.15}$$

where (with $B_{ij} := B(d_i, 1-d_i-d_j), B_{ji} := B(d_j, 1-d_i-d_j)$)

$$\begin{aligned} \frac{\langle g_{d_i,d_j}, g_{d_j,d_i}\rangle}{\kappa_i^2\kappa_j^2} &= \frac{B_{ij}B_{ji}}{(d_1+d_j)(2(d_i+d_j)-1)} + \Big(\frac{B_{ij}+B_{ji}}{(d_i+d_j)(d_i+d_j+1)}\Big)^2 \\ &\quad - \frac{2}{(d_i+d_j)^2}\Big(\frac{2B_{ij}B_{ji}}{2(d_i+d_j)+1} + (B_{ij}^2 + B_{ji}^2)B(d_i+d_j+1, d_i+d_j+1)\Big). \end{aligned}$$

Note that for $i = j$ the last expression agrees with $\|g_{d_i,d_j}\|^2/\kappa_i^2\kappa_j^2$ in (3.6).

Remark 3.3 The 4th moment condition in (3.8) is required only for those LMMA processes γ_i which enter sample variances S_{γ_i,γ_i} in the collection $\mathcal{S}_n$ (3.2). For instance for Π_3 in (4.2) the 4th moment condition applies to the innovations of the measurement errors $\{u_t\}$ alone whereas $\{X_t\}$ and $\{\varepsilon_t\}$ may have infinite 4th moment. Condition (3.8) is crucial for the validity of (3.9). Indeed if $E\xi_{0,i}^4 = \infty$ for some $i = 1, \cdots, m$ then $ES^2_{\gamma_i,\gamma_i} = \infty$ and the limit distribution of S_{γ_i,γ_i} may be α-stable with $\alpha < 2$, see Surgailis (2004), and Horvath and Kokoszka (2008).

4 Limit distribution of $\hat{\alpha}$, $\hat{\beta}$

In this we shall use the results of the previous section to derive the limiting distribution of a suitably standardized $\hat{\alpha}$, $\hat{\beta}$.

To begin with note that from (2.9) we obtain

$$\begin{aligned} T_n + \beta\sigma_u^2 &= S_{X\varepsilon} - \beta S_{Xu} + S_{u\varepsilon} - \beta(S_{uu} - \sigma_u^2) \\ &= S_{X\varepsilon} - \beta S_{Xu} + S_{u\varepsilon} - \beta(S_{uu} - ES_{uu}) + \beta E\bar{u}^2. \end{aligned} \tag{4.1}$$

According to (2.10), (4.1), the limit distribution of $\hat{\beta} - \beta$ coincides with that of the quadratic form $\widetilde{T}_n := (T_n + \beta\sigma_u^2)/\sigma_X^2$. Under Assumptions (E), (X), and (U), $\widetilde{T}_n$ is a particular case of the quadratic forms studied in Theorem 3.1. More specifically, $\widetilde{T}_n$ corresponds to the case $m = 3$, $\gamma_{t,1} \equiv \varepsilon_t$, $\gamma_{t,2} \equiv X_t$, $\gamma_{t,3} \equiv u_t$ and the set

$$\Pi_3 = \{(X, \varepsilon),\ (X, u),\ (u, \varepsilon),\ (u, u)\}. \tag{4.2}$$

Accordingly, the limit distribution of $\widetilde{T}_n$ and $\hat{\beta} - \beta$ is essentially determined by the maximum

$$\delta_{\max} = \max\{d_X + d_\varepsilon, d_X + d_u, d_u + d_\varepsilon, 2d_u\}, \tag{4.3}$$

with the convergence rate $\hat{\beta} - \beta = O_p\big(n^{-(1-\min\{1/2, 1-\delta_{\max}\})}(1 + \mathbf{1}(\delta_{\max} = 1/2)\log n)\big)$. From (2.7) we obtain

$$\hat{\alpha} - \alpha = \bar{\varepsilon} - \beta\bar{u} - (\hat{\beta} - \beta)\bar{Z}. \tag{4.4}$$

Note that in the decomposition (4.4), the linear term $\bar{\varepsilon} - \beta\bar{u} = O_p(n^{\max\{d_\varepsilon, d_u\} - 1/2})$, where

$$(1/2) - \max\{d_\varepsilon, d_u\} < \min\{1/2, 1 - \delta_{\max}\}.$$

Since $\bar{Z} = \bar{X} + \bar{u} = O_p(1)$ $(\mu_X := EX \neq 0)$, $= o_p(1)$ $(\mu_X = 0)$, the above facts imply that *the term* $(\hat{\beta} - \beta)\bar{Z}$ *in* (4.4) *is asymptotically negligible independent of the value of* μ_X, and the limit distribution of $\hat{\alpha} - \alpha$ is determined by that of $\bar{\varepsilon} - \beta\bar{u}$.

Under suitable assumptions on the innovations, see (4.6) below, Theorem 3.1 and Remark 3.1 completely describes the limit distribution of $(\bar{\varepsilon} - \beta\bar{u}, \widetilde{T}_n)$, or that of $(\hat{\alpha} - \alpha, \hat{\beta} - \beta)$. The description of this limiting distribution is relatively simpler and more transparent if we assume that the LM parameters d_X, d_ε *and* d_u *are all different*, i.e.,

$$d_u \neq d_\varepsilon \neq d_X. \tag{4.5}$$

This assumption guarantees that the maximum in (4.3) is achieved by a single pair in Π_3 of (4.2), i.e., either by (X, ε), or by (X, u), or by (u, ε), or by (u, u).

In order to apply Theorem 3.1, in addition to Assumptions (E), (X), (U), we need the following conditions on the innovations:

$$E|\zeta_0|^{2+\epsilon} + E|\xi_0|^{2+\epsilon} < \infty \quad (\exists\, \epsilon > 0), \qquad E|\eta_0|^4 < \infty. \tag{4.6}$$

Corollary 4.1 *Let Assumptions (E), (X), (U) and (I) be satisfied. In addition, assume* (4.5) *and* (4.6) *hold. Let* $d_{\max} := \max\{d_\varepsilon, d_X, d_u\},\ d_{\min} := \min\{d_\varepsilon, d_X, d_u\}$.

(i) Case $\delta_{\max} = 2d_u > 1/2$ *(this implies* $d_{\max} = d_u$*). Then*

$$\big(n^{1/2-d_u}(\hat{\alpha} - \alpha), n^{1-2d_u}(\hat{\beta} - \beta)\big) \to_D \Big(-\beta I_u(f_u), \frac{\beta}{\sigma_X^2}\big(\frac{\kappa_u^2 B(d_u, 1-d_u)}{d_u(1+2d_u)} - I_{uu}(g_{d_u,d_u})\big)\Big),$$

where I_{uu} *(*I_u*) are the double (single) Wiener-Itô integrals in* (3.3) *w.r.t. the same standard Brownian motion* $W_i = W_j \equiv W_u$ *and the integrand* $g_{d_u,d_u} = g_{d_i,d_j}$ *(*$f_{d_u} = f_{d_i}$*) in* (3.5)*, where* $d_i = d_j = d_u, \kappa_i = \kappa_j = \kappa_u$.

(ii) Case $\delta_{\max} = d_X + d_u > 1/2$ *(this implies* $d_{\max} = d_X > d_u > d_\varepsilon$*). Then*

$$\big(n^{1/2-d_u}(\hat{\alpha} - \alpha), n^{1-d_X-d_u}(\hat{\beta} - \beta)\big) \to_D \Big(-\beta I_u(f_u), -\frac{\beta}{\sigma_X^2} I_{Xu}(g_{d_X,d_u})\Big), \tag{4.7}$$

where I_{Xu} *(*I_u*) is the double (single) Wiener-Itô integral in* (3.3) *w.r.t. independent standard Brownian motions* $W_i \equiv W_X,\ W_j \equiv W_u$ *and the integrand* $g_{d_X,d_u} = g_{d_i,d_j}$ *(*$f_{d_u} = f_{d_j}$*) in* (3.5)*, where* $d_i = d_X,\ \kappa_i = \kappa_X,\ d_j = d_u,\ \kappa_j = \kappa_u$.

(iii) Case $\delta_{\max} = d_u + d_\varepsilon > 1/2$ *(this implies* $d_{\max} = d_\varepsilon > d_u > d_X$*). Then*

$$\big(n^{1/2-d_\varepsilon}(\hat{\alpha} - \alpha), n^{1-d_u-d_\varepsilon}(\hat{\beta} - \beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}), \frac{1}{\sigma_X^2} I_{u\varepsilon}(g_{d_u,d_\varepsilon})\Big), \tag{4.8}$$

where $I_{u\varepsilon}$ *(*I_ε*) is the double (single) Wiener-Itô integral in* (3.3) *w.r.t. independent standard Brownian motions* $W_i \equiv W_u,\ W_j \equiv W_\varepsilon$ *and the integrand* $g_{d_u,d_\varepsilon} = g_{d_i,d_j}$ *(*$f_{d_\varepsilon} = f_{d_j}$*) in* (3.5) *where* $d_i = d_u,\ \kappa_i = \kappa_u,\ d_j = d_\varepsilon,\ \kappa_j = \kappa_\varepsilon$.

(iv) Case $\delta_{\max} = d_X + d_\varepsilon > 1/2$ *(this implies* $d_{\min} = d_u < d_\varepsilon$*). Then*

$$(4.9) \qquad \big(n^{1/2-d_\varepsilon}(\hat\alpha - \alpha),\ n^{1-d_X-d_\varepsilon}(\hat\beta - \beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}), \frac{1}{\sigma_X^2} I_{X\varepsilon}(g_{d_X,d_\varepsilon})\Big),$$

where $I_{X\varepsilon}$ *(*I_ε*) is the double (single) Wiener-Itô integral in* (3.3) *w.r.t. independent standard Brownian motions* $W_i \equiv W_X,\ W_j \equiv W_\varepsilon$ *and the integrand* $g_{d_X,d_\varepsilon} = g_{d_i,d_j}$ *(*$f_{d_\varepsilon} = f_{d_j}$*) in* (3.5)*, where* $d_i = d_X,\ \kappa_i = \kappa_X,\ d_j = d_\varepsilon,\ \kappa_j = \kappa_\varepsilon$*.*

(v) Case $\delta_{\max} < 1/2$*. In addition, assume that the innovations of* u_t *have 3rd moment zero:* $E\eta^3 = 0$ *when* $d_u > d_\varepsilon$*. Then*

$$(4.10) \qquad \big(n^{1/2-(d_u \vee d_\varepsilon)}(\hat\alpha - \alpha),\ n^{1/2}(\hat\beta - \beta)\big) \to_D \big(\sigma_\alpha Z_\alpha,\ \sigma_\beta Z_\beta\big),$$

where Z_α, Z_β *are independent* $N(0,1)$ *r.v.'s,*

$$\sigma_\alpha^2 := \begin{cases} \beta^2 \|f_{d_u}\|^2, & d_u > d_\varepsilon, \\ \|f_{d_\varepsilon}\|^2, & d_\varepsilon > d_u, \end{cases}, \qquad \sigma_\beta^2 := \sigma_R^2/\sigma_X^4,$$

where $\sigma_R^2 := \sum_{t\in\mathbb{Z}} Cov(R_0, R_t)$ *and*

$$R_t := (\varepsilon_t - \beta u_t)(X_t + u_t) = (\varepsilon_t - \beta u_t) Z_t, \qquad t \in \mathbb{Z}$$

is a stationary process with $ER_t = -\beta\sigma_u^2$ *and* $\sum_{t\in\mathbb{Z}} |Cov(R_0, R_t)| < \infty$*.*

Remark 4.1 It is of some interest to compare the above asymptotic distributional results with those available in the case of i.i.d. set up. For that reason we shall first recall the results available in the i.i.d. case. Accordingly, suppose $\{\varepsilon, \varepsilon_i\}, \{X, X_i\}, \{u, u_i\}$ are mutually independent sequences of i.i.d.r.v.'s with positive and finite variances $\sigma_\varepsilon^2,\ \sigma_X^2,\ \sigma_u^2$, respectively. Suppose further that $E\varepsilon = Eu = 0$ and $\mu_4 = Eu^4 < \infty$. Let $\mu_X = EX$, $\mu_3 = Eu^3$. Let

$$\varphi := \frac{1}{\sigma_X^4}\Big[\sigma_X^2(\sigma_\varepsilon^2 + \beta^2\sigma_u^2) + \sigma_u^2\sigma_\varepsilon^2 + \beta^2(\mu_4 - \sigma_u^4)\Big].$$

$$\Gamma := \begin{pmatrix} (\sigma_\varepsilon^2 + \beta^2\sigma_u^2) + 2\frac{1}{\sigma_X^2}\beta^2\mu_3\mu_X + \varphi\mu_X^2 & -\frac{1}{\sigma_X^2}\beta^2\mu_3 - \varphi\,\mu_X \\ -\frac{1}{\sigma_X^2}\beta^2\mu_3 - \varphi\,\mu_X & \varphi \end{pmatrix}.$$

Using the classical CLT, we obtain

$$(4.11) \qquad n^{1/2}(\hat\alpha - \alpha, \hat\beta - \beta) \to_D N(0, \Gamma).$$

For the sake of completeness a sketch of the proof of (4.11) is included in the last section.

In the case of no measurement errors, $\sigma_u^2 = 0$, $\mu_4 = \mu_3 = 0$, $\varphi = \sigma_\varepsilon^2/\sigma_X^2$ and

$$\Gamma = \begin{pmatrix} \sigma_\varepsilon^2 + \mu_X^2(\sigma_\varepsilon^2/\sigma_X^2) & -\mu_X(\sigma_\varepsilon^2/\sigma_X^2) \\ -\mu_X(\sigma_\varepsilon^2/\sigma_X^2) & (\sigma_\varepsilon^2/\sigma_X^2) \end{pmatrix} = \frac{\sigma_\varepsilon^2}{\sigma_X^2}\begin{pmatrix} \sigma_X^2 + \mu_X^2 & -\mu_X \\ -\mu_X & 1 \end{pmatrix}.$$

Now suppose $\mu_3 = 0$. Then in the i.i.d. set up the above LSEs are asymptotically correlated and normally distributed, regardless of whether there is measurement error in the covariate or not. But, surprisingly, under the above assumed long memory set up with $\delta_{\max} < 1/2$, by (4.10), these estimators are asymptotically independent and normally distributed even when there is no measurement error. In the case $\mu_3 \neq 0$, then the limiting r.v.'s in (4.10) are correlated. The correlation can be obtained from (3.13).

For $\delta_{\max} > 1/2$, Corollary 3.1 and (3.13) yield that these r.v.'s are still asymptotically uncorrelated but have non-Gaussian distribution.

5 Structural ME multiple linear regression model

Here we shall now discuss the asymptotic distributions of the bias adjusted LSEs in the structural multiple linear regression model. Accordingly, now β, X_t, Z_t, u_t are p-dimensional random vectors and the model of interest is

$$(5.1) \qquad Y_t = \alpha + X_t'\beta + \varepsilon_t, \quad Z_t = X_t + u_t, \quad t \in \mathbb{Z},$$

where X and u are vector-valued LMMA processes satisfying the following assumptions and x' denotes the transpose of a vector $x \in \mathbb{R}^p$.

Assumption (X)$_p$ Covariates $X_t = (X_{t,1}, \cdots, X_{t,p})'$ form a LMMA process

$$(5.2) \qquad X_{t,i} \;=\; \mu_{X,i} + \sum_{k=0}^{\infty} a_{k,i}\xi_{t-k,i}, \quad t \in \mathbb{Z}, \quad \text{with} \quad a_{k,i} \sim \kappa_{X,i} k^{-(1-d_{X,i})}, \; k \to \infty,$$

where $\mu_{X,i} \in \mathbb{R}$, $\kappa_{X,i} > 0, 0 < d_{X,i} < 1/2$, and i.i.d. innovations $\{\xi_s = (\xi_{s,1}, \cdots, \xi_{s,p})'; s \in \mathbb{Z}\}$ with $E\xi_{0,i} = 0, E\xi_{0,i}\xi_{0,j} = \sigma_{\xi,ij}$, $i, j = 1, \cdots, p$.

Assumption (U)$_p$ Measurement errors $u_t = (u_{t,1}, \cdots, u_{t,p})'$ form a LMMA process

$$(5.3) \qquad u_{t,i} = \sum_{k=0}^{\infty} c_{k,i}\eta_{t-k,i}, \quad t \in \mathbb{Z}, \quad \text{with} \quad c_{k,i} \sim \kappa_{u,i} k^{-(1-d_{u,i})}, \; k \to \infty,$$

where $\kappa_{u,i} > 0, 0 < d_{u,i} < 1/2$, and i.i.d. innovations $\{\eta_s = (\eta_{s,1}, \cdots, \eta_{s,p})'; s \in \mathbb{Z}\}$ with $E\eta_{0,i} = 0, E\eta_{0,i}\eta_{0,j} = \sigma_{\eta,ij}$, $i, j = 1, \cdots, p$. Moreover, $\Sigma_u := E(u_0 u_0')$ is known and positive definite.

Assumption (I)$_p$ The innovation sequences $\{\zeta_s; s \in \mathbb{Z}\}$, $\{\xi_s; s \in \mathbb{Z}\}$, and $\{\eta_s; s \in \mathbb{Z}\}$ in Assumptions (E), (X)$_p$ and (U)$_p$ are mutually independent.

We also assume that

$$(5.4) \qquad E|\zeta_{0,i}|^{2+\epsilon} + E|\xi_{0,i}|^{2+\epsilon} < \infty \quad (\exists\, \epsilon > 0), \qquad E\eta_{0,i}^4 < \infty, \quad \forall\, 1 \leq i \leq p.$$

Similarly as in the simple linear regression case, the bias corrected estimators of α, β in the multiple linear regression model (5.1) are defined as

$$
\begin{aligned}
&S_{ZZ} := \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})(Z_i - \bar{Z})', \quad S_{ZY} := \frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})(Y_i - \bar{Y}),\\
&\widehat{\beta} := \big(S_{ZZ} - \Sigma_u\big)^{-1} S_{ZY}, \quad \widehat{\alpha} := \bar{Y} - \bar{Z}'\widehat{\beta}.
\end{aligned}
$$

Whence as in the case of simple linear regression model we obtain

$$
\begin{aligned}
\widehat{\beta} - \beta &= (S_{ZZ} - \Sigma_u)^{-1}\big(S_{X\varepsilon} + S_{u\varepsilon} - S_{Xu}\beta - (S_{uu} - \Sigma_u)\beta\big),\\
\widehat{\alpha} - \alpha &= \bar{\varepsilon} - \bar{u}'\beta - \bar{Z}'(\widehat{\beta} - \beta)
\end{aligned}
\tag{5.5}
$$

Since $S_{ZZ} - \Sigma_u \to_p \Sigma_X := EX_0X_0'$, we see from (5.5) that the limit distribution of $\widehat{\beta} - \beta$ coincides with that of $\widetilde{T}_n := \Sigma_X^{-1}(T_n + E(\bar{u}\bar{u}')\beta)$, where

$$
T_n := S_{X\varepsilon} + S_{u\varepsilon} - S_{Xu}\beta - (S_{uu} - ES_{uu})\beta
$$

is a zero-mean quadratic form in LMMA satisfying Assumptions (E), $(X)_p$ and $(U)_p$. As it follows from Theorem 3.1 and Remark 3.2, under these assumptions the limit distribution of T_n and $\widetilde{T}_n$ is essentially determined by the maximum

$$
\delta_{\max} := \max\{d_{X,i} + d_\varepsilon, d_{u,i} + d_\varepsilon, d_{X,i} + d_{u,j}, d_{u,i} + d_{u,j};\ 1 \le i, j \le p\}. \tag{5.6}
$$

Accordingly, the limit distribution of $\widehat{\beta}-\beta$ is non-gaussian or Gaussian depending on whether $\delta_{\max} > 1/2$ or $\delta_{\max} \le 1/2$. In general, $\widehat{\alpha}$ and $\widehat{\beta}_i, 1 \le i \le p$ may have different convergence rates and a complicated joint limit distribution. We first discuss the case $\delta_{\max} < 1/2$ where the limit result admits a relatively simple formulation as seen in the following corollary.

Corollary 5.1 *Let Assumptions (E), $(X)_p$, $(U)_p$ and $(I)_p$ be satisfied and $\delta_{\max} < 1/2$. In addition, assume that $d_{u,i}, 1 \le i \le p$ are all different, $d_{u,\max} := \max\{d_{u,i}, 1 \le i \le p\}$, the 3rd moment of the innovations of $u_{t,i}$ with $d_{u,i} = d_{u,\max}$ is zero when $d_{u,\max} > d_\varepsilon$, and (5.4) hold. Then*

$$
\big(n^{1/2-(d_\varepsilon \vee d_{u,\max})}(\widehat{\alpha} - \alpha),\, n^{1/2}(\widehat{\beta} - \beta)\big) \to_D \big(\sigma_\alpha Z_\alpha,\, \Sigma_X^{-1} Z_\beta\big), \tag{5.7}
$$

where $Z_\alpha \sim N(0,1)$,

$$
\sigma_\alpha^2 := \begin{cases} \beta_i^2 \|f_{d_{u,i}}\|^2, & d_{u,\max} = d_{u,i} > d_\varepsilon, i = 1, \cdots, p, \\ \|f_{d_\varepsilon}\|^2, & d_\varepsilon > d_{u,\max}, \end{cases},
$$

and Z_β is a normal vector independent of Z_α, with $EZ_\beta = 0$ and covariance matrix $EZ_\beta Z_\beta' := \sum_{t\in\mathbb{Z}} Cov(R_0, R_t)$ and

$$
R_t := (\varepsilon_i - \beta' u_t)(X_t + u_t) = (\varepsilon_t - \beta' u_t) Z_t, \qquad t \in \mathbb{Z}
$$

is a stationary $\mathbb{R}^p$-valued process with $ER_t = -\Sigma_u\beta$ and $\sum_{t\in\mathbb{Z}} \|Cov(R_0, R_t)\| < \infty$.

Next, we discuss the limit distribution of the LSE $(\widehat{\alpha}, \widehat{\beta})$ in (5.5) when $\delta_{\max} > 1/2$. The description of this limit distribution is complicated for the case $p \geq 2$ and when long memory parameters of components of $\{X_t\}$ and $\{u_t\}$ are all different. For this reason we shall describe these distributions only in the case when these long memory parameters are equal, viz., $d_{X,i} \equiv d_X, d_{u,i} \equiv d_u, i = 1, \cdots, p$, and in the case when $p = 2$ but $d_{X,,1} \neq d_{X,2}, d_{u,1} \neq d_{u,2}$. We note that in the latter case, the convergence rates of $\widehat{\beta}_1, \widehat{\beta}_2$ are generally different.

Consider first the former case, $p \geq 1$ arbitrary. Let $\Sigma_\eta = E\eta_0\eta_0', \Sigma_\xi = E\xi_0\xi_0'$ denote the respective covariance matrices of innovations in Assumption $(\mathrm{U})_p$ and $(\mathrm{X})_p$. Introduce a scalar-valued standard Brownian motion $W_\varepsilon = W_\varepsilon(t), t \in \mathbb{R}$, and vector-valued Brownian motions $W_X(t) = (W_{X,1}(t), \cdots, W_{X,p}(t))'$, $W_u(t) = (W_{u,1}(t), \cdots, W_{u,p}(t))', t \in \mathbb{R}$ with respective covariance matrices $EW_X(t)W_X(t)' = |t|\Sigma_\xi, EW_u(t)W_u(t)' = |t|\Sigma_\eta$, W_ε, W_X, W_u mutually independent. Recall from (3.3), (3.14) the definition of the stochastic integrals with respect to these Brownian motions: $I_u(f) = \big(\int_{\mathbb{R}} f_i(s)W_{u,i}(ds)\big)_{1\leq i\leq p}$, $f = (f_1, \cdots, f_p)'$, $I_{uu}(g) = \big(\int_{\mathbb{R}^2} g_{ij}(s_1, s_2)W_{u,i}(ds_1)W_{u,j}(ds_2)\big)_{1\leq i,j\leq p}$, $g = (g_{ij})_{1\leq i,j\leq p}$ defined for vector- and matrix-valued valued integrands from $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, respectively, the stochastic integrals $I_X(f), I_{X\varepsilon}(g)$, $I_{u\varepsilon}(g)$ defined in a similar fashion. Note I_{uu}, I_{Xu} are matrix-valued and $I_X, I_u, I_{X\varepsilon}, I_{X\varepsilon}$ are vector-valued r.v.'s.

Corollary 5.2 *Let Assumptions (E), $(X)_p$, $(U)_p$ and $(I)_p$ be satisfied. In addition, assume that $d_{u,i} = d_u,\ d_{X,i} = d_X,\ 1 \leq i \leq p$ and (5.4) hold.*

(i) Case $\delta_{\max} = 2d_u > 1/2$. Then

$$(5.8)\qquad \big(n^{1/2-d_u}(\widehat{\alpha} - \alpha),\, n^{1-2d_u}(\widehat{\beta} - \beta)\big) \to_D \Big(- \beta' I_u(f_u),\, \Sigma_X^{-1}\big(\langle f_u, f_u'\rangle - I_{uu}(g_{d_u,d_u})\big)\beta\Big),$$

where $f_u = (f_{d_1}, \cdots, f_{d_p})'$ and $g_{d_u,d_u} = (g_{d_i,d_j})_{1\leq i,j\leq p}$ are defined as in (3.5) where $d_i = d_j := d_u,\ \kappa_i := \kappa_{u,i},\ \kappa_j := \kappa_{u,j}$.

(ii) Case $\delta_{\max} = d_X + d_u > 1/2$. Then

$$(5.9)\qquad \big(n^{1/2-d_u}(\widehat{\alpha} - \alpha),\ n^{1-d_X-d_u}(\widehat{\beta} - \beta)\big) \to_D \Big(- \beta' I_u(f_u), -\Sigma_X^{-1} I_{Xu}(g_{d_X,d_u})\beta\Big),$$

where f_u is the same as in (5.8) and $g_{d_X,d_u} = (g_{d_i,d_j})_{1\leq i,j\leq p}$ as in (3.5) where $d_i := d_X,\ d_j := d_u, \kappa_i := \kappa_{X,i},\ \kappa_j := \kappa_{u,j}$.

(iii) Case $\delta_{\max} = d_u + d_\varepsilon > 1/2$. Then

$$(5.10)\qquad \big(n^{1/2-d_\varepsilon}(\widehat{\alpha} - \alpha),\ n^{1-d_u-d_\varepsilon}(\widehat{\beta} - \beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}),\ \Sigma_X^{-1} I_{u\varepsilon}(g_{d_u,d_\varepsilon})\Big),$$

where $g_{d_u,d_\varepsilon} = (g_{d_i,d_j})_{1\leq i\leq p}$ and $f_{d_\varepsilon} = f_{d_j}$ as in (3.5) where $d_i := d_{u,i} = d_u,\ \kappa_i := \kappa_{u,i},\ d_j := d_\varepsilon,\ \kappa_j := \kappa_\varepsilon$.

(iv) Case $\delta_{\max} = d_X + d_\varepsilon > 1/2$. Then

$$\big(n^{1/2-d_\varepsilon}(\widehat{\alpha}-\alpha),\, n^{1-d_X-d_\varepsilon}(\widehat{\beta}-\beta)\big) \to_D \Big(I_\varepsilon(f_{d_\varepsilon}), \Sigma_X^{-1} I_{X\varepsilon}(g_{d_X,d_\varepsilon})\Big), \tag{5.11}$$

where f_{d_ε} is the same as in (5.10) *and $g_{d_X,d_\varepsilon} = (g_{d_i,d_j})_{1\le i\le p}$ as in* (3.5) *where $d_i := d_{X,i} = d_X,\ \kappa_i := \kappa_{X,i},\ d_j := d_\varepsilon,\ \kappa_j := \kappa_\varepsilon$.*

Next, consider the case $p = 2, \widehat{\beta} = (\widehat{\beta}_1, \widehat{\beta}_2)',\ d_{X,1} \ne d_{X,2},\ d_{u,1} \ne d_{u,2}$ and $\delta_{\max} > 1/2$, where $\delta_{\max}$ is defined in (5.6). Let $\Sigma_X^{-1} = (\rho_{X,ij})_{1\le i,j\le 2}$. As noted above, the limit distribution of $\widehat{\beta}-\beta$ coincides with that of $\widetilde{T}_n := \Sigma_X^{-1}(T_n + E(\bar{u}\bar{u}')\beta) = (\widetilde{T}_{n1}, \widetilde{T}_{n2})'$ where

$$\begin{aligned}\widetilde{T}_{n1} \;=\; & \rho_{X,11}\big(S_{X_1,\varepsilon} + S_{u_1,\varepsilon} - S_{X_1u_1}\beta_1 - S_{X_1u_2}\beta_2 - (S_{u_1u_1} - ES_{u_1u_1})\beta_1 \\ & \qquad - (S_{u_1u_2} - ES_{u_1u_2})\beta_2 + (\bar{u}_1)^2\beta_1 + \bar{u}_1\bar{u}_2\beta_2\big) \\ & + \rho_{X,12}\big(S_{X_2,\varepsilon} + S_{u_2,\varepsilon} - S_{X_2u_1}\beta_1 - S_{X_2u_2}\beta_2 - (S_{u_2u_1} - ES_{u_2u_1})\beta_1 \\ & \qquad - (S_{u_2u_2} - ES_{u_2u_2})\beta_2 + \bar{u}_1\bar{u}_2\beta_1 + (\bar{u}_2)^2\beta_2\big).\end{aligned} \tag{5.12}$$

We omit a similar expression for $\widetilde{T}_{n2}$, where the only difference is that $\rho_{X,11}, \rho_{X,12}$ in (5.12) are replaced by $\rho_{X,21}, \rho_{X,22}$, respectively. We have the two cases: (a) $\rho_{X,12} = \rho_{X,21} \ne 0$ (or Σ_X is not a diagonal matrix), and (b) $\rho_{X,12} = \rho_{X,21} = 0$ (Σ_X is diagonal). From these formulas it is easy to see that in case (a) that the convergence rate of $\widetilde{T}_{ni}, i = 1,2$ hence also of $\widehat{\beta}_i, i = 1,2$ is the same and is equal to $n^{1-\delta_{\max}}$. In case (b), $\widehat{\beta}_i, i = 1,2$ may have different convergence rates and their limit distribution is more complex. As an illustration, the following corollary details this limit distribution when $\delta_{\max} = 2d_{u,1}$. In other cases (when $\delta_{\max}$ is achieved at other pairs of LM indices in (5.6)) this limit distribution can be derived in a similar fashion.

Corollary 5.3 *Let $p = 2$ and Assumptions (E), $(X)_2$, $(U)_2$, $(I)_2$ and* (5.4) *be satisfied. In addition, assume that $d_{u,1} > \max\{d_{u,2},\ d_{X,1},\ d_{X,2},\ d_\varepsilon\}$ and $\delta_{\max} = 2d_{u,1} > 1/2$.*
(a) Let $\sigma_{X,12} = Cov(X_{0,1}, X_{0,2}) \ne 0$. Then

$$\begin{aligned}&\big(n^{1/2-d_{u,1}}(\widehat{\alpha}-\alpha),\, n^{1-2d_{u,1}}(\widehat{\beta}_i-\beta_i), i = 1,2\big) \\ &\quad \to_D \quad \beta_1\Big(-I_{u_1}(f_{d_{u,1}}),\, \rho_{X,1i}\big(\|f_{d_{u,1}}\| - I_{u_1u_1}(g_{d_{u_1},d_{u_1}})\big), i = 1,2\Big).\end{aligned}$$

(b) Let $\sigma_{X,12} = Cov(X_{0,1}, X_{0,2}) = 0$ and $d_{X,2} \ne d_{u,2}$. Then

$$\begin{aligned}&\big(n^{1/2-d_{u,1}}(\widehat{\alpha}-\alpha),\, n^{1-2d_{u,1}}(\widehat{\beta}_1-\beta_1), n^{1-d_{u,1}-d_{X,2}\vee d_{u,2}}(\widehat{\beta}_2-\beta_2)\big) \\ &\quad \to_D \quad \beta_1\Big(-I_{u_1}(f_{d_{u,1}}),\, \rho_{X,11}\big(\|f_{d_{u,1}}\| - I_{u_1u_1}(g_{d_{u_1},d_{u_1}})\big), \rho_{X,22}\mathcal{W}\Big),\end{aligned}$$

where

$$\mathcal{W} := \begin{cases} -I_{u_1,X_2}(g_{d_{u,1},d_{X,2}}), & d_{X,2} > d_{u,2}, \\ \langle f_{d_{u,1}}, f_{d_{u,2}}\rangle - I_{u_1u_2}(g_{d_{u_1},d_{u_2}}), & d_{X,2} < d_{u,2}. \end{cases}$$

6 Functional ME model: nonrandom design

In this section we describe the analogs of the previous results in the functional linear regression model with LMMA regression and measurement errors, and nonrandom design satisfying the following assumption. For clarity of exposition, the subsequent discussion is confined to the case $p = 1$, or the simple linear regression model in (2.1).

Assumption (X)$_{det}$ There exists a (nonrandom) piece-wise continuous function $V : [0, 1] \to \mathbb{R}$ such that $X_t = V(t/n),\ t = 1, \cdots, n$.

The above form of regressors also assumes that V is not a constant so that $\sigma_V^2 := \int_0^1 (V(t) - \bar{V})^2 dt > 0$, where $\bar{V} := \int_0^1 V(t)dt$. As shown below, the limit behavior of LSE $(\widehat{\alpha}, \widehat{\beta})$ in the nonrandom design case is Gaussian and generally simpler than in the random design case. The dominating role in the limit distribution now is being played by terms $S_{X\varepsilon}, S_{Xu}, \bar{\varepsilon}, \bar{u}$ in (4.1) and (4.4).

Note first that Assumption (X)$_{det}$ implies $\bar{X} \to \bar{V}$ and $S_{XX} \to \sigma_V^2$ as $n \to \infty$. Moreover, $S_{Xu} = O_p(n^{d_u - 1/2}) = o_p(1),\ S_{X\varepsilon} = O_p(n^{d_\varepsilon - 1/2}) = o_p(1)$, see (6.2) below, while $S_{uu} \to \sigma_u^2$. Therefore the normalization matrix $S_{ZZ} - \sigma_u^2$ in (2.10) tends to σ_V^2, viz.,

$$S_{ZZ} - \sigma_u^2 \to_p \sigma_V^2. \tag{6.1}$$

Let $V_c(t) := V(t) - \bar{V},\ t \in [0, 1]$. Assumptions (X)$_{det}$, (E), and (U) imply

$$n^{1/2 - d_\varepsilon} S_{X\varepsilon} \to_D I_\varepsilon(f_{V_c,\varepsilon}), \qquad n^{1/2 - d_u} S_{Xu} \to_D I_u(f_{V_c,u}), \tag{6.2}$$

where I_ε, I_u are the same (Gaussian) stochastic integrals as in Corollary 4.1 with respective integrands

$$f_{V_c,\varepsilon}(s) \ := \ \kappa_\varepsilon \int_0^1 V_c(t)(t - s)_+^{d_\varepsilon - 1} dt, \qquad f_{V_c,u}(s) := \kappa_u \int_0^1 V_c(t)(t - s)_+^{d_u - 1} dt. \tag{6.3}$$

Note $I_\varepsilon(f_{V_c,\varepsilon}), I_u(f_{V_c,u})$ in (6.2) are independent and have a Gaussian distribution with zero mean and respective variances

$$\begin{aligned} EI_\varepsilon^2(f_{V_c,\varepsilon}) &= \|f_{V_c,\varepsilon}\|^2 = \kappa_\varepsilon^2 B(d_\varepsilon, d_\varepsilon) \langle f_{V_c,\varepsilon}, f_{V_c,\varepsilon} \rangle_{d_\varepsilon}, \\ EI_u^2(f_{V_c,u}) &= \|f_{V_c,u}\|^2 = \kappa_u^2 B(d_u, d_u) \langle f_{V_c,u}, f_{V_c,u} \rangle_{d_u}, \end{aligned} \tag{6.4}$$

where

$$\langle f, g \rangle_d := \int_{[0,1]^2} f(t) g(s) |t - s|^{2d - 1} dt ds \tag{6.5}$$

is a strictly positive definite quadratic form. The convergences in (6.2) can be proved by using the criterion in GKS, Cor.4.7.1, for linear forms in i.i.d.r.v.'s. Moreover, $\bar{Z} \to_p \bar{V}$ and $S_{uu} - \sigma_u^2 = O_p(n^{-(1 - 2d_u - 1) \vee (1/2)}(1 + \mathbf{1}(d_u = 1/2) \log^{1/2} n) = o_p(n^{d_u - 1/2})$ and $S_{u\varepsilon} = o_p(n^{d_\varepsilon - 1/2})$ follow from Theorem 3.1. These facts together with (4.1), (4.4), (6.1), (6.2) result in the following corollary.

Corollary 6.1 *Let Assumptions (E), $(X)_{det}$, (U) and (I) be satisfied. In addition, assume that* (5.4) *hold.*

(i) Suppose $d_\varepsilon > d_u$. Then

$$n^{1/2-d_\varepsilon}(\widehat{\alpha}-\alpha,\, \widehat{\beta}-\beta) \to_D (W_{1,\varepsilon}, W_{2,\varepsilon}), \tag{6.6}$$

where $(W_{1,\varepsilon}, W_{2,\varepsilon})$ have a bivariate Gaussian distribution with zero mean and (co)variances

$$\begin{aligned} EW_{1,\varepsilon}^2 &= \|f_{d_\varepsilon}\|^2 + \bar{V}^2\sigma_V^{-4}\|f_{V_c,\varepsilon}\|^2 - 2\bar{V}\sigma_V^{-2}\langle f_{d_\varepsilon}, f_{V_c,\varepsilon}\rangle, \\ EW_{2,\varepsilon}^2 &= \sigma_V^{-4}\|f_{V_c,\varepsilon}\|^2, \qquad EW_{1,\varepsilon}W_{2,\varepsilon} = \sigma_V^{-2}\big(\langle f_{d_\varepsilon}, f_{V_c,\varepsilon}\rangle - \bar{V}\|f_{V_c,\varepsilon}\|^2\big), \end{aligned} \tag{6.7}$$

(ii) Suppose $d_\varepsilon < d_u$. Then

$$n^{1/2-d_u}(\widehat{\alpha}-\alpha,\, \widehat{\beta}-\beta) \to_D -(W_{1,u}, W_{2,u})\beta, \tag{6.8}$$

where $(W_{1,u}, W_{2,u})$ have a similar bivariate Gaussian distribution as in (6.6)-(6.7) *with the only difference that $f_{d_\varepsilon}, f_{V_c,\varepsilon}$ in* (6.7) *are replaced by $f_{d_u}, f_{V_c,u}$, respectively.*

7 Proofs of Theorem 3.1 and Corollary 3.1

Proof of Theorem 3.1. Let $\widetilde{S}_{\gamma_i,\gamma_j} := \frac{1}{n}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$ so that $S_{\gamma_i,\gamma_j} = \widetilde{S}_{\gamma_i,\gamma_j} - \bar{\gamma}_i\bar{\gamma}_j$. Note for any $t, t' \in \mathbb{Z}$

$$\begin{aligned} &\mathrm{Cov}(\gamma_{t,i}\gamma_{t,j}, \gamma_{t',i'}\gamma_{t',j'}) \\ &= \begin{cases} \mathrm{Cov}(\gamma_{t,i}, \gamma_{t',i})\mathrm{Cov}(\gamma_{t,j}, \gamma_{t',j}), & (i,j) = (i',j'), i \ne j, \\ \mathrm{Cov}(\gamma_{t,i}^2, \gamma_{t',i}^2), & i = j = i' = j', \\ 0, & (i,j) \ne (i',j') \end{cases} \end{aligned} \tag{7.1}$$

From (7.1) we obtain

$$\mathrm{Cov}(S_{\gamma_i,\gamma_j}, S_{\gamma_{i'},\gamma_{j'}}) = \mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_j}, \widetilde{S}_{\gamma_{i'},\gamma_{j'}}) = 0, \qquad (i,j) \ne (i',j'). \tag{7.2}$$

From GKS, Prop.3.2.1(ii), it follows that

$$\mathrm{Cov}(\gamma_{t,i}, \gamma_{0,i}) = \sum_{s\le 0} b_{t-s,i}b_{-s,i} \sim \chi_i t^{2d_i-1}, \qquad t \to \infty, \tag{7.3}$$

where $\chi_i := \kappa_i^2 \int_0^\infty (1+s)^{d_i-1}s^{d_i-1}ds = \kappa_i^2 B(d_i, 1-2d_i)$. We shall prove that

$$\begin{aligned} \mathrm{Var}(S_{\gamma_i,\gamma_j}) \sim \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) &\sim \sigma_{ij}^2/n, \quad d_i + d_j < 1/2, \\ \mathrm{Var}(S_{\gamma_i,\gamma_j}) \sim \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) &\sim \sigma_{ij}^2(\log n)/n, \quad d_i + d_j = 1/2, \\ \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \sim \widetilde{\sigma}_{ij}^2 n^{2(d_i+d_j-1)},\ \mathrm{Var}(S_{\gamma_i,\gamma_j}) &\sim \sigma_{ij}^2 n^{2(d_i+d_j-1)}, \quad d_i + d_j > 1/2, \end{aligned} \tag{7.4}$$

where $\widetilde{\sigma}_{ij}^2 := (1+\delta_{ij})\|\widetilde{g}_{d_i,d_j}\|^2$,

$$
\sigma_{ij}^2 \;:=\; \begin{cases} \sum_{t\in\mathbb{Z}} \mathrm{Cov}(\gamma_{t,i}\gamma_{t,j}, \gamma_{0,i}\gamma_{0,j}), & d_i+d_j<1/2, \\ 2(1+\delta_{ij})\chi_i\chi_j, & d_i+d_j=1/2, \\ \widetilde{\sigma}_{ij}^2 & d_i+d_j>1/2; \end{cases} \tag{7.5}
$$

and $\|g_{d_i,d_j}\|^2$ is defined in (3.6), $\delta_{ij} := \mathbf{1}(i=j)$.

Consider (7.4) for $d_i+d_j>1/2$. Here, the asymptotics of $\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j})$ is immediate from (7.1), (7.3) and GKS, Prop.3.3.1(i). To check the asymptotics of $\mathrm{Var}(S_{\gamma_i,\gamma_j})$, consider first the case of $i\neq j$. Write $\mathrm{Var}(S_{\gamma_i,\gamma_j}) = \mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) - 2\mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_j}, \bar{\gamma}_i\bar{\gamma}_j) + \mathrm{Var}(\bar{\gamma}_i\bar{\gamma}_j)$, where the variance $\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j})$ satisfies (7.4) and $\mathrm{Var}(\bar{\gamma}_i\bar{\gamma}_j) = \mathrm{Var}(\bar{\gamma}_i)\mathrm{Var}(\bar{\gamma}_j) \sim \|f_{d_i}\|^2\|f_{d_j}\|^2 n^{2(d_i+d_j-1)}$, see (3.6). The asymptotics of the covariance

$$
\begin{aligned}
\mathrm{Cov}(\widetilde{S}_{\gamma_i,\gamma_j}, \bar{\gamma}_i\bar{\gamma}_j) &= n^{-3}\sum_{t,t_1,t_2=1}^n E\gamma_{t,i}\gamma_{t_1,i}E\gamma_{t,j}\gamma_{t_2,j} \\
&\sim \chi_i\chi_j n^{-3}\sum_{t,t_1,t_2=1}^n |t-t_1|^{2d_i-1}|t-t_2|^{2d_j-1} \\
&\sim n^{2(d_i+d_j-1)}\langle \widetilde{g}_{d_i,d_j}, f_{d_i}\otimes f_{d_j}\rangle
\end{aligned}
$$

follows by integral approximation and a calculation as in (3.6). This proves (7.4) for $d_i+d_j>1/2$ and $i\neq j$. The case $i=j$ follows similarly using the fact that $\mathrm{Cov}(\gamma_{t,i}^2, \gamma_{0,i}^2) = 2(\mathrm{Cov}(\gamma_{t,i},\gamma_{0,i}))^2 + \chi_{4,i}\sum_{s\le t} b_{t-s,i}^2 b_{-s,i}^2$, where $\chi_{4,i} = E(\xi_{0,i}^2-1)^2 - 2 = E\xi_{0,i}^4 - 3$ is the 4th cumulant of $\xi_{0,i}$, see GKS, (6.2.25).

Consider (7.4) for $d_i+d_j=1/2$. Let $i\neq j$. Then by (7.3)

$$
\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j}) \;\sim\; \chi_i\chi_j n^{-2}\sum_{t,s=1}^n |t-s|^{-1} \;\sim\; \sigma_{ij}^2 n^{-1}\log n,
$$

with $\sigma_{ij}^2 = 2\chi_i\chi_j$. The case $i=j$ follows similarly. Finally, (7.4) for the case $d_i+d_j<1/2$ follows from (7.3), (7.1) and the fact that the r.h.s. of (7.1) is summable.

Next, we prove the convergence in (3.9). Because of the differences in the normalization and the limit distribution, the cases $\delta_{\max}>1/2$, $\delta_{\max}=1/2$, and $\delta_{\max}<1/2$, where $\delta_{\max}$ is as in (3.1), will be discussed separately. Let $\Pi_{\max} := \{(i,j)\in\Pi_m; d_i+d_j=\delta_{\max}\}$.

Proof of (3.9)**: Case** $\delta_{\max}>1/2$**.** Since (7.4) imply $A(n)(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}) \to_D 0$ for $(i,j)\notin\Pi_{\max}$, relation (3.9) follows from

$$
\{n^{1-\delta_{\max}}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}); (i,j)\in\Pi_{\max}\} \to_D \{I_{ij}(g_{d_i,d_j}); (i,j)\in\Pi_{\max}\}, \tag{7.6}
$$

where I_{ij} are the double Wiener-Itô integrals in (3.3). Assume first that that $\Pi_{\max}$ consists of a single element $(i,j), i\neq j$. Then, because $\delta_{\max} = d_i+d_j$ and $ES_{\gamma_i,\gamma_j} = 0$ for

$i \neq j$, $n^{1-\delta_{\max}}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}) = n^{1-d_i-d_j} S_{\gamma_i,\gamma_j}$ can be written as a quadratic form in i.i.d. innovations $\{\xi_{s,i}, \xi_{s,j}, s \in \mathbb{Z}\}$, viz.,

$$
\begin{aligned}
Q(g_n) &:= \sum_{s_1,s_2 \in \mathbb{Z}} g_n(s_1,s_2)\xi_{s_1,i}\xi_{s_2,j}, \quad \text{with coefficients} \\
g_n(s_1,s_2) &:= n^{-d_1-d_j} \sum_{t=1}^{n} b_{t-s_1,i} b_{t-s_2,j} - n^{-1-d_i-d_j} \sum_{t_1,t_2=1}^{n} b_{t_1-s_1,i} b_{t_2-s_2,j}.
\end{aligned}
$$

We use GKS, Prop.11.5.5, according to which $n^{1-d_i-d_j} S_{\gamma_i,\gamma_j} \to_D I_{ij}(g_{d_i,d_j})$ follows from the convergence in $L^2(\mathbb{R}^2)$:

$$
\|\widetilde{g}_n - g_{d_i,d_j}\| \to 0 \tag{7.7}
$$

where

$$
\begin{aligned}
\widetilde{g}_n(x_1,x_2) &:= n g_n([nx_1],[nx_2]) \\
&= \frac{n}{n^{d_i+d_j}} \sum_{t=1}^{n} b_{t-[nx_1],i} b_{t-[nx_2],j} - \frac{1}{n^{d_i+d_j}} \sum_{t_1,t_2=1}^{n} b_{t_1-[nx_1],i} b_{t_2-[nx_2],j}.
\end{aligned}
$$

Since $b_{k,i} \sim \kappa_i k^{d_i-1}, k \to \infty$ the point-wise convergence

$$
\begin{aligned}
\widetilde{g}_n(x_1,x_2) \to g_{d_i,d_j}(x_1,x_2) &= \kappa_i\kappa_j \Big\{ \int_0^1 (t-s_1)_+^{d_i-1}(t-s_2)_+^{d_j-1} dt \\
&\qquad - \int_0^1 (t_1-x_1)_+^{d_i-1} dt_1 \int_0^1 (t_2-x_2)_+^{d_j-1} dt_2 \Big\},
\end{aligned}
$$

see (3.5), for any fixed $(x_1,x_2) \in \mathbb{R}^2, x_i \neq 0,1, i = 1,2$ follows by integral approximation. Then, (7.7) follows by the DCT similarly as GKS, Prop.11.5.6. The general case in (7.6) follows similarly and we omit the details.

Proof of (3.9)**: case** $\delta_{\max} = 1/2$**.** Let $\widetilde{\Pi}_{1/2} := \{(i,j) \in \Pi_m : d_i + d_j = 1/2\}$. Then by (7.4) relation (3.9) reduces to

$$
\big\{(n/\log n)^{1/2}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j}); (i,j) \in \widetilde{\Pi}_{1/2}\big\} \to_D \big\{\sigma_{ij} Z_{ij}; (i,j) \in \widetilde{\Pi}_{1/2}\big\}, \tag{7.8}
$$

where $Z_{ij}, (i,j) \in \widetilde{\Pi}_{1/2}$ are independent $N(0,1)$ r.v.'s and $\sigma_{ij}^2 = 2(1+\delta_{ij})\chi_i\chi_j$, see (7.5). Moreover, since $\bar{\zeta}_i = O_p(n^{d_i-1/2})$, $i = 1, \cdots, m$ so $\bar{\gamma}_i\bar{\gamma}_j = O_p(n^{d_i+d_j-1}) = O_p(n^{-1/2})$, $(i,j) \in \Pi_{1/2}$ and hence $(n/\log n)^{1/2}\bar{\gamma}_i\bar{\gamma}_j = O_p((\log n)^{-1/2}) = o_p(1)$, $(i,j) \in \Pi_{1/2}$. Thus, (7.8) follows from

$$
\big\{(n/\log n)^{1/2}(\widetilde{S}_{\gamma_i,\gamma_j} - E\widetilde{S}_{\gamma_i,\gamma_j});\ (i,j) \in \Pi_{1/2}\big\} \to_D \big\{\sigma_{ij} Z_{ij};\ (i,j) \in \Pi_{1/2}\big\}, \tag{7.9}
$$

where $\widetilde{S}_{\gamma_i,\gamma_j} = n^{-1}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$ as above. We shall prove (7.9) for a single pair $(i,j) \in \widetilde{\Pi}_{1/2}$. Let $i \neq j$. Then $E\widetilde{S}_{\gamma_i,\gamma_j} = 0$. Moreover, $\widetilde{S}_{\gamma_i,\gamma_j} = \widetilde{S}'_{\gamma_i,\gamma_j} + \widetilde{S}''_{\gamma_i,\gamma_j}$ where

$$\begin{aligned}\widetilde{S}'_{\gamma_i,\gamma_j} &:= n^{-1}\sum_{t=1}^n \sum_{s_i \le t, i=1,2, s_1 \neq s_2} b_{t-s_1,i} b_{t-s_2,j} \xi_{s_1,i}\xi_{s_2,j}, \\ \widetilde{S}''_{\gamma_i,\gamma_j} &:= n^{-1}\sum_{t=1}^n \sum_{s \le t} b_{t-s,i} b_{t-s,j}\xi_{s,i}\xi_{s,j}\end{aligned}$$

are off-diagonal and diagonal terms, respectively. Moreover, $\sum_{t=1}^\infty |b_{t,i}b_{t,j}| \le C\sum_{t=1}^\infty t^{-3/2} < \infty$ implies $\widetilde{S}''_{\gamma_i,\gamma_j} = O_p(n^{-1/2}) = o_p(1)$. Hence it suffices to prove

$$(n/\log n)^{1/2}\widetilde{S}'_{\gamma_i,\gamma_j} \to_D N(0,\sigma_{ij}^2). \tag{7.10}$$

To prove (7.10), as in in Bhansali et al. (2007), we use martingale CLT in Hall and Heyde (1980). Towards this aim rewrite the l.h.s. of (7.10) as the martingale transform

$$(n\log n)^{-1/2}\sum_{s<n} v_n(s), \quad \text{where} \quad v_n(s) := u_{n,i}(s)\xi_{s,j} + u_{n,j}(s)\xi_{s,i}, \tag{7.11}$$

$$u_{n,i}(s) := \sum_{s'<s} c_n(s',s)\xi_{s',i}, \quad u_{n,j}(s) := \sum_{s'<s} c_n(s,s')\xi_{s',j}, \quad c_n(s',s) := \sum_{t=1}^n b_{t-s',i}\, b_{t-s,j}.$$

Let $\mathcal{F}_t := \sigma\{\xi_{s,i},\xi_{s,j}, s \le t\}$ be the σ-field generated by innovations. Then $E[v_n(s)|\mathcal{F}_{s-1}] = 0$, $E[v_n^2(s)|\mathcal{F}_{s-1}] = u_{n,i}^2(s) + u_{n,j}^2(s)$. By the classical martingale CLT, (7.10) follows from

$$B_{ij}(n) := \mathrm{Var}(\sum_{s<n} v_n(s)) = n^2\mathrm{Var}(\widetilde{S}'_{\gamma_i,\gamma_j}) \sim \sigma_{ij}^2 n\log n, \tag{7.12}$$

$$B_{ij}^{-1}(n)\sum_{s<n} E[v_n^2(s)|\mathcal{F}_{s-1}] \to_D 1, \tag{7.13}$$

$$B_{ij}^{-1}(n)\sum_{s<n} E[v_n^2(s) I(|v_n(s)| > \delta B_{ij}^{1/2}(n))] \to_D 0, \quad \forall\delta > 0. \tag{7.14}$$

The proof of (7.12) follows easily from (7.4). Consider (7.13). Using $B_{ij}(n) = \sum_{s<n} Ev_n^2(s)$, the relation (7.13) follows from (7.12) and

$$\textstyle\sum_{s<n}(E[v_n^2(s)|\mathcal{F}_{s-1}] - Ev_n^2(s)) = o_p(n\log n),$$

or

$$\textstyle\sum_{s<n}(u_{n,k}^2(s) - Eu_{n,k}^2(s)) = o_p(n\log n), \qquad k = i,j. \tag{7.15}$$

Consider (7.15) for $k = i$; the proof for $k = j$ is analogous. By writing the l.h.s. of (7.15) as a centered quadratic form $Q_n = \sum_{s',s''<n}\xi_{s',i}\xi_{s'',i}\sum_{s'\vee s''<s<n} c_n(s',s)c_n(s'',s)$ in i.i.d. r.v.'s $\xi_{s',i}$'s, (7.15) and (7.13) follow from $\mathrm{Var}(Q_n) \le 8E\xi_{0,i}^4 R_n$, and

$$R_n := \sum_{s''\le s'<n}\Big(\sum_{s'<s<n} c_n(s',s)c_n(s'',s)\Big)^2 = O(n^2) = o(n^2\log^2 n), \tag{7.16}$$

see also GKS, (4.5.4). Using the definition of $c_n(s',s)$ in (7.11) it follows that

$$R_n \ \leq \ C\int_{-\infty<s''<s'<n} ds'ds''\Big(\int_{s'<s<n}\widetilde{c}_n(s',s)\widetilde{c}_n(s'',s)ds\Big)^2 =: C\widetilde{R}_n,$$

where $\widetilde{c}_n(s',s) := \int_0^n (t-s')_+^{d_i-1}(t-s)_+^{d_j-1}dt$. By change of variables, $\widetilde{R}_n = n^2\widetilde{R}_1$ and hence (7.16) follows from

$$\widetilde{R}_1 < \infty. \tag{7.17}$$

To check (7.17) use the following bound: for any $-\infty < s' < s < 1$

$$\begin{aligned}
\widetilde{c}_1(s',s) \ &\leq \ \mathbf{1}(s'\in(-1,1))\int_{\mathbb{R}}(t-s')_+^{d_i-1}(t-s)_+^{d_j-1}dt + \mathbf{1}(s'<-1)|s'|^{d_i-1}\int_0^1(t-s)_+^{d_j-1}dt \\
&\leq \ C\mathbf{1}(s'\in(-1,1))|s-s'|^{d_i+d_j-1} + C\mathbf{1}(s'<-1)|s'|^{d_i-1}(1+|s|)^{d_j-1} \\
&= \ C\mathbf{1}(s'\in(-1,1))|s-s'|^{-1/2} + C\mathbf{1}(s'<-1)|s'|^{d_i-1}(1+|s|)^{d_j-1} \qquad (7.18)
\end{aligned}$$

since $d_i + d_j = 1/2$. Then

$$\begin{aligned}
\widetilde{R}_1 \ \leq \ C\int_{(-\infty,1)^2} ds'ds''\Big\{\int_{s'\vee s''}^1 \Big(\frac{\mathbf{1}(|s'|<1)}{|s-s'|^{1/2}} + \frac{\mathbf{1}(s'<-1)}{|s'|^{1-d_i}(1+|s|)^{1-d_j}}\Big) \\
\times\Big(\frac{\mathbf{1}(|s''|<1)}{|s-s''|^{1/2}} + \frac{\mathbf{1}(s''<-1)}{|s''|^{1-d_i}(1+|s|)^{1-d_j}}\Big)ds\Big\}^2 \leq C\sum_{k=1}^4 J_k,
\end{aligned}$$

where

$$\begin{aligned}
J_1 \ &:= \ \int_{(-1,1)^2} ds'ds''\Big\{\int_{-1}^1 \frac{ds}{|s-s'|^{1/2}|s-s''|^{1/2}}\Big\}^2, \\
J_2 \ &:= \ \int_{(-\infty,-1)\times(-1,1)} |s'|^{-2(1-d_i)}ds'ds''\Big\{\int_{-1}^1 \frac{ds}{(1+|s|)^{1-d_j}|s-s''|^{1/2}}\Big\}^2, \\
J_3 \ &:= \ \int_{(-1,1)\times(-\infty,-1)} |s''|^{-2(1-d_i)}ds'ds''\Big\{\int_{-1}^1 \frac{ds}{(1+|s|)^{1-d_j}|s-s'|^{1/2}}\Big\}^2, \\
J_4 \ &:= \ \int_{(-\infty,-1)^2} |s's''|^{-2(1-d_i)}ds'ds''\Big\{\int_{s'\vee s''}^1 \frac{ds}{(1+|s|)^{2(1-d_j)}}\Big\}^2.
\end{aligned}$$

The fact that $J_k < \infty, k = 1,2,3,4$ is elementary by $0 < d_i, d_j < 1/2$. This proves (7.17) and (7.16), (7.13).

To prove (7.14) we use condition (3.7). By the Markov inequality, $E[v_n^2(s)I(|v_n(s)| > \delta B_{ij}^{1/2}(n))] \leq E|v_n(s)|^{2+\epsilon}(\delta B_{ij}^{1/2}(n))^{-\epsilon}$ and (7.14) follows from

$$\sum_{s<n} E|v_n(s)|^{2+\epsilon} = o(B_{i,j}^{(2+\epsilon)/2}(n)) = O((n\log n)^{(2+\epsilon)/2}). \tag{7.19}$$

We have $E|v_n(s)|^{2+\epsilon} \le C(E|u_{n,i}(s)|^{2+\epsilon} + E|u_{n,j}(s)|^{2+\epsilon}) \le C(L_i(s) + L_j(s))$, where $L_i(s) := E\big|\sum_{s'<s} c_n(s', s)\xi_{s,i}\big|^{2+\epsilon}$, $L_j(s) := E\big|\sum_{s'<s} c_n(s, s')\xi_{s,j}\big|^{2+\epsilon}$. By Rosenthal's inequality, see GKS, Lemma 2.5.2,

$$L_i(s) \le C\Big(\sum_{s'<s} c_n^2(s', s)\Big)^{(2+\epsilon)/2}. \tag{7.20}$$

We use the following bound similar to (7.18).

$$|c_n(s', s)| \le C \begin{cases} \frac{n|s'|^{d_i-1}}{n^{1-d_j}+|s|^{1-d_j}}, & s' < -n, \\ |s'-s|_+^{-1/2}, & |s'| \le n. \end{cases} \tag{7.21}$$

From (7.20), (7.21) we obtain

$$\sum_{s<n} L_i(s) \;\le\; C\Big\{\sum_{s\le -n} + \sum_{|s|<n}\Big\}\Big(\sum_{s'<s} c_n^2(s', s)\Big)^{(2+\epsilon)/2} =: C\{J_1 + J_2\},$$

where

$$\begin{aligned} J_1 &\le C\int_{-\infty}^{-n} ds\Big(\int_{-\infty}^{-n} (n|s'|^{d_i-1}|s|^{d_j-1})^2 ds'\Big)^{(2+\epsilon)/2} \\ &= Cn\int_{-\infty}^{-1} |s|^{2(d_j-1)(2+\epsilon)/2} ds\Big(\int_{-\infty}^{-1} |s'|^{2(d_i-1)} ds'\Big)^{(2+\epsilon)/2} = Cn \end{aligned}$$

since the last integral converges. On the other hand, since $d_i + d_j = 1/2$,

$$\begin{aligned} J_2 &\le C\sum_{|s|\le n}\Big(\sum_{s'\le -n} n^{2d_j}|s'|^{2(d_i-1)}\Big)^{(2+\epsilon)/2} + C\sum_{|s|\le n}\Big(\sum_{|s'|\le n} |s-s_+|_+^{-1}\Big)^{(2+\epsilon)/2} \\ &\le Cn + Cn(\log n)^{(2+\epsilon)/2}, \end{aligned}$$

implying $\sum_{s<n} L_i(s) = O(n(\log n)^{(2+\epsilon)/2})$. Since $\sum_{s<n} L_j(s) = O(n(\log n)^{(2+\epsilon)/2})$ follows exactly similarly, we obtain $\sum_{s<n} E|v_n(s)|^{2+\epsilon} = O(n(\log n)^{(2+\epsilon)/2}) = o((n\log n)^{(2+\epsilon)/2})$ for $\epsilon > 0$, proving (7.19), (7.14) and completing the proof of (7.10).

Proof of (3.9)**: Case** $\delta_{\max} < 1/2$**.** Then by (7.4) relation (3.9) is equivalent to

$$\big\{n^{1/2}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j});\ (i,j) \in \Pi_m\big\} \to_D \big\{\sigma_{ij}Z_{ij}; (i,j) \in \Pi_m\big\}, \tag{7.22}$$

where Z_{ij}, $(i,j) \in \Pi_m$ are independent $N(0,1)$ r.v.'s and σ_{ij}^2 are defined in (7.5). Moreover since $\bar X_i \bar X_j = O_p(n^{d_i+d_j-1}) = o_p(n^{-1/2})$ for $d_i + d_j < 1/2$, so S_{γ_i,γ_j} in (7.22) can be replaced by $\widetilde S_{ij} = n^{-1}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$ and (7.22) follows from

$$\big\{n^{1/2}(\widetilde S_{\gamma_i,\gamma_j} - E\widetilde S_{\gamma_i,\gamma_j}); (i,j) \in \Pi_m\big\} \to_D \big\{\sigma_{ij}Z_{ij}; (i,j) \in \Pi_m\big\}. \tag{7.23}$$

We shall prove (7.23) for a single pair $(i,j) \in \Pi_m$. Let $i \neq j$. Then $E\widetilde{S}_{\gamma_i,\gamma_j} = 0$. Hence it suffices to prove

$$n^{1/2}\widetilde{S}_{\gamma_i,\gamma_j} \to_D N(0,\sigma_{ij}^2) \tag{7.24}$$

We use the argument as in GKS, Thm.4.8.1. For $\ell \geq 1$ introduce 'truncated' processes:

$$\gamma_{t,i}^{(\ell)} := \sum_{s \leq t} b_{t-s,i}\mathbf{1}(t-s \leq \ell)\xi_{s,i}, \qquad i = 1,\cdots,m,$$

and the corresponding $\widetilde{S}_{\gamma_i^{(\ell)},\gamma_j^{(\ell)}} := n^{-1}\sum_{t=1}^n \gamma_{t,i}^{(\ell)}\gamma_{t,j}^{(\ell)}$. Thus, for each $1 \leq \ell < \infty$ fixed, $Y_{ij}^{(\ell)}(t) := \gamma_{t,i}^{(\ell)}\gamma_{t,j}^{(\ell)}$, $t \in \mathbb{Z}$ is a ℓ-dependent stationary process with autocovariance $\rho_{ij}^{(\ell)}(t) := \mathrm{Cov}(Y_{ij}^{(\ell)}(t), Y_{ij}^{(\ell)}(0))$ such that

$$\begin{aligned}\rho_{ij}^{(\ell)}(t) &= \big(\sum_{s=0}^{\infty} b_{s,i}b_{t+s,i}\mathbf{1}(t+s \leq \ell)\big)\big(\sum_{s=0}^{\infty} b_{s,j}b_{t+s,j}\mathbf{1}(t+s \leq \ell)\big)\\ &\leq C\big(\sum_{s=0}^{\infty} |b_{s,i}b_{t+s,i}|\big)\big(\sum_{s=0}^{\infty} |b_{s,j}b_{t+s,j}|\big) \leq Ct^{2(d_i+d_j-1)}, \quad t \geq 1,\end{aligned}$$

and $\rho_{ij}^{(\ell)}(t) \to \rho_{ij}(t) := \mathrm{Cov}(Y_{ij}(t), Y_{ij}(0)$, as $\ell \to \infty$, where $Y_{ij}(t) := \gamma_{t,i}\gamma_{t,j}$. These facts and the CLT for ℓ-dependent stationary processes, see e.g. GKS, Prop.4.3.2, imply that

$$\begin{aligned}n^{1/2}\widetilde{S}_{\gamma_i^{(\ell)},\gamma_j^{(\ell)}} &\to_D N(0,(\sigma_{ij}^{(\ell)})^2), \qquad n \to \infty,\\ (\sigma_{ij}^{(\ell)})^2 &:= \sum_{t\in\mathbb{Z}} \rho_{ij}^{(\ell)}(t) \to \sigma_{ij}^2, \quad \ell \to \infty.\end{aligned}$$

Hence, (7.24) follows provided we show that uniformly in $n \geq 1$

$$n\mathrm{Var}(\widetilde{S}_{\gamma_i,\gamma_j} - \widetilde{S}_{\gamma_i^{(\ell)},\gamma_j^{(\ell)}}) = \sum_{|t|<n}\big(1-\frac{|t|}{n}\big)\mathrm{Cov}\big(Y_{ij}(t) - Y_{ij}^{(\ell)}(t), Y_{ij}(0) - Y_{ij}^{(\ell)}(0)\big) \to, \tag{7.25}$$

as $\ell \to \infty$. The proof of (7.25) mimics that of (GKS, (4.8.7)). We omit the details. This proves (7.24) and the extension to (7.23) seems straightforward. Theorem 3.1 is proved. □

Proof of Corollary 3.1. Assume for concreteness that the sets $\Pi_{0m} = \{k\}, \Pi_m = \{(i,j)\}$ each consist of a single element, $d_{\max} = d_k, \delta_{\max} = d_i + d_j$. Let $\delta_{\max} > 1/2$. Following the proof of Theorem 3.1 in this case, write $n^{1/2-d_k}\bar{\gamma}_k = \sum_{s\in\mathbb{Z}} f_n(s)\xi_{s,k}$ as a linear form in innovations with coefficients $f_n(s) = n^{-1/2-d_k}\sum_{t=1}^n b_{t-s,k}, s \in \mathbb{Z}$. Let $\tilde{f}_n(x) := n^{1/2}f_n([sx]), x \in \mathbb{R}$ and $\|\cdot\|_1$ denote the norm in $L^2(\mathbb{R})$. According to (GKS, Propositions 11.5.5, 14.3.3), the joint convergence in (3.12), or $(n^{1/2-d_k}\bar{\gamma}_k, n^{1-d_i-d_j}(S_{\gamma_i,\gamma_j} - ES_{\gamma_i,\gamma_j})) \to_D (I_k(f_{d_k}), I_{ij}(g_{d_i,d_j}))$ follows from (7.7) and $\|\tilde{f}_n - f_{d_k}\|_1 \to 0$, where the last relation can be verified similarly to (7.7).

This proves (3.12) for $\delta_{\max} > 1/2$. For $\delta_{\max} = d_i + d_j \le 1/2$ the joint convergence in (3.12) can be proved similarly as in the proof of Theorem 3.1 and we omit the details.

Consider (3.13). For $\delta_{\max} > 1/2$ (3.13) follows the orthogonality of single and double Wiener-Itô integrals, see (3.4). Suppose $\delta_{\max} \le 1/2$. As in the proof of Theorem 3.1, let $\widetilde{S}_{\gamma_i,\gamma_j} = n^{-1}\sum_{t=1}^n \gamma_{t,i}\gamma_{t,j}$. It suffices to prove that

$$\lim_{n\to\infty} n^{1-d_k} E\big(\bar{\gamma}_k(\widetilde{S}_{\gamma_i,\gamma_j} - E\widetilde{S}_{\gamma_i,\gamma_j})\big) = \frac{\kappa_k}{d_k(1+d_k)} E(\xi_{0,k}\xi_{0,i}\xi_{0,j}) \sum_{s=0}^{\infty} b_{s,i}b_{s,j}. \tag{7.26}$$

To show (7.26), split $\widetilde{S}_{\gamma_i,\gamma_j} - E\widetilde{S}_{\gamma_i,\gamma_j} = S'_n + S''_n$, where $nS'_n := \sum_{s\le n}\sum_{t=1\vee s}^n b_{t-s,i}b_{t-s,j}(\xi_{s,i}\xi_{s,j} - E\xi_{s,i}\xi_{s,j})$, $S''_n := n^{-1}\sum_{s_1,s_2\le n, s_1\ne s_2}\sum_{t=1\vee s_1\vee s_2}^n b_{t-s_1,i}b_{t-s_2,j}\xi_{s_1,i}\xi_{s_2,j}$. Since $E\bar{\gamma}_k S''_n = 0$, it suffices to prove (7.26) with $\widetilde{S}_{\gamma_i,\gamma_j} - E\widetilde{S}_{\gamma_i,\gamma_j}$ replaced by S'_n. We have

$$n^{1-d_k}E(\bar{\gamma}_k S'_n) \;=\; E(\xi_{0,k}\xi_{0,i}\xi_{0,j})n^{-1-d_k}\sum_{s\le n}\sum_{t=1\vee s}^n b_{t-s,k}L_{s,ij}(n), \tag{7.27}$$

where $L_{s,ij}(n) := \sum_{t=1\vee s}^n b_{t-s,i}b_{t-s,j} \to L_{ij} := \sum_{t=0}^\infty b_{t,i}b_{t,j} < \infty$ for any $1 \le s \le n$ and $|L_{s,ij}(n)| \le C\sum_{t=|s|}^\infty t^{d_i+d_j-2} \le C(1+|s|)^{d_i+d_j-1}$, $s \le 0$. Thus, by (10.2.53) of GKS,

$$\begin{aligned}
n^{-1-d_k}\sum_{s=1}^n\sum_{t=s}^n b_{t-s,k}L_{s,ij}(n) &\sim L_{ij}\kappa_k n^{-1-d_k}\sum_{s=1}^n\sum_{t=s}^n (t-s)_+^{d_k-1} \to (\kappa_k/d_k(1+d_k))L_{ij},\\
\Big|\sum_{s\le 0}\sum_{t=s}^n b_{t-s,k}L_{s,ij}(n)\Big| &\le C\sum_{t=1}^n\sum_{s=0}^\infty (t+s)^{d_k-1}(1+s)^{d_i+d_j-1}\\
&\le C\sum_{t=1}^n t^{d_k+d_i+d_j-1} \le Cn^{d_k+d_i+d_j} = o(n^{1+d_k}).
\end{aligned}$$

This completes the proof of (7.26). The last relation also implies the statement (3.13) of the corollary when $\delta_{\max} < 1/2$ and also when $\delta_{\max} = 1/2$ due to the presence of the logarithmic factor in the normalization $A(n)$ (3.1). □

Proof of (4.11). Let $\widetilde{S}_{UV} := n^{-1}\sum_{i=1}^n (U_iV_i - EU_iV_i)$, $U_i^c := U_i - EU_i$. Then (4.1) can be rewritten as $T_n + \beta\sigma_u^2 = T'_n - T''_n$, where

$$T'_n \;:=\; \widetilde{S}_{X^c\varepsilon} - \beta\widetilde{S}_{X^c u} + \widetilde{S}_{u\varepsilon} - \beta\widetilde{S}_{uu}, \qquad T''_n := \overline{X^c}\bar{\varepsilon} - \beta\overline{X^c}\bar{u} + \bar{u}\bar{\varepsilon} - \beta(\bar{u})^2.$$

Note all summands in T''_n are uncorrelated, implying

$$\mathrm{Var}(T''_n) = \mathrm{Var}(\overline{X^c})\mathrm{Var}(\bar{\varepsilon}) + \beta^2\mathrm{Var}(\overline{X^c})\mathrm{Var}(\bar{u}) + \mathrm{Var}(\bar{u})\mathrm{Var}(\bar{\varepsilon}) + \beta^2\mathrm{Var}((\bar{u})^2) = O(n^{-2}).$$

Hence and from (2.10) and (4.1),

$$n^{1/2}(\hat{\beta} - \beta) \;=\; n^{1/2}T'_n/\sigma_X^2 + o_p(1). \tag{7.28}$$

Similarly from (4.4) and (7.28) we obtain

$$
\begin{aligned}
n^{1/2}(\hat{\alpha}-\alpha) &= n^{1/2}(\bar{\varepsilon}-\beta\bar{u}) - n^{1/2}(\hat{\beta}-\beta)(\mu_X + o_p(1)) \\
&= n^{1/2}(\bar{\varepsilon}-\beta\bar{u}) + (\mu_X/\sigma_X^2)n^{1/2}T_n' + o_p(1).
\end{aligned}
\tag{7.29}
$$

Note $n^{1/2}T_n'$ and $n^{1/2}(\bar{\varepsilon}-\beta\bar{u}) + (\mu_X/\sigma_X^2)T_n'$ are sums of i.i.d.r.v.s with zero mean and finite variance. Moreover, since all terms in T_n' are mutually uncorrelated,

$$
\begin{aligned}
\mathrm{Var}(T_n') &= \mathrm{Var}(\widetilde{S}_{X^c\varepsilon}) + \beta^2\mathrm{Var}(\widetilde{S}_{X^c u}) + \mathrm{Var}(\widetilde{S}_{u\varepsilon}) + \beta^2\mathrm{Var}(\widetilde{S}_{uu}) \\
&= n^{-1}\big(\sigma_X^2\sigma_\varepsilon^2 + \beta^2\sigma_X^2\sigma_u^2 + \sigma_u^2\sigma_\varepsilon^2 + \beta^2(\mu_4 - \sigma_u^4)\big).
\end{aligned}
$$

Hence, $\mathrm{Var}(n^{1/2}T_n'/\sigma_X^2) = \varphi$, see (4.11). We also find that the covariance matrix of $(n^{1/2}(\bar{\varepsilon}-\beta\bar{u}) + (\mu_X/\sigma_X^2)T_n', n^{1/2}T_n'/\sigma_X^2)$ (the main terms in (7.28), (7.29)) coincides with Γ in (4.11). Then (4.11) follows from (7.28), (7.29) and the classical CLT for sums of i.i.d.r.v.'s. $\square$

8 References

Beran, J., Feng, Y., Ghosh, S. and Kulik, R. (2013). *Long-memory Processes. Probabilistic Properties and Statistical Methods.* Springer, Heidelberg.

Bhansali, R.J., Giraitis, L. and Kokoszka, P.S. (2007). Convergence of quadratic forms with nonvanishing diagonal. *Statist. Probab. Lett.*, **77**, 726–734.

Carroll, R.J., Ruppert, D., Stefanski, L.A. and Crainiceanu, C.P. (2006). *Measurement Error in Nonlinear Models: A Modern Perspective. Second Edition.* Chapman & Hall/CRC, Bota Raton, FL.

Cheng, C.L. and Van Ness, J.W. (1999). *Statistical Regression with Measurement Error.* Wiley, New York (?)

Fuller, W.A. (1987). *Measurement Error Models.* Wiley, New York.

Giraitis, L., Koul, H.L. and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes.* Imperial College Press, London.

Granger, C.W.J. and Joyeux, R. (1980). An introduction to long-memory time series and fractional differencing. *J. Time Series Anal.*, **1**, 15–29.

Hall, P. and Heyde, C.C. (1980). *Martingale Limit Theory and Its Applications.* Academic Press, New York.

Horvath, L. and Kokoszka, P. (2008). Sample autocovariances of long-memory time series. *Bernoulli*, **14**, 405–418.

Hosking, J.R.M.(1981). Fractional differencing. *Biometrika*, **68**, 165–176.

Surgailis, D. (2004). Stable limits of sums of bounded functions of long memory moving averages with finite variance. *Bernoulli*, **10**, 327–355.