Tempered Fractional Calculus

2014 International Conference on
Fractional Differentiation and Its Applications

University of Catania, Italy
23–25 June 2014

Mark M. Meerschaert
Department of Statistics and Probability
Michigan State University

mcubed@stt.msu.edu
http://www.stt.msu.edu/users/mcubed

Abstract

Fractional derivatives and integrals are convolutions with a power law. Including an exponential term leads to tempered fractional derivatives and integrals. Tempered fractional Brownian motion, the tempered fractional integral or derivative of a Brownian motion, is a new stochastic process whose increments can exhibit semi-long range dependence. A tempered Grunwald-Letnikov formula provides the basis for finite difference methods to solve tempered fractional diffusion equations. The tempered finite difference operator is also useful in time series analysis, where it provides a useful new stochastic model for turbulent velocity data. Tempered stable processes are the limits of random walk models, where the power law probability of long jumps is tempered by an exponential factor. These random walks converge to tempered stable stochastic process limits, whose probability densities solve tempered fractional diffusion equations. Tempered power law waiting times lead to tempered fractional time derivatives. Applications include geophysics and finance.

Acknowledgments

Farzad Sabzikar, Statistics and Probability, Michigan State

Inmaculada Aban, Biostatistics, Univ Alabama Birmingham
Boris Baeumer, Maths & Stats, University of Otago, New Zealand
Jinghua Chen, School of Sciences, Jimei University, China
Peter Franz Kern, Math, Universität Düsseldorf, Germany
Anna K. Panorska, Mathematics and Statistics, U of Nevada
M.S. Phanikumar, Civil & Environ. Eng., Michigan State
Parthanil Roy, Indian Statistical Institute, Kolkata
Hans-Peter Scheffler, Math, Universität Siegen, Germany
Qin Shao, Mathematics, University of Toledo, Ohio
Alla Sikorskii, Statistics and Probability, Michigan State
Aklilu Zeleke, Lyman Briggs School, Michigan State
Yong Zhang, Department of Geological Sciences, U Alabama

Recent Books

Stochastic Models for Fractional Calculus
Mark M. Meerschaert and Alla Sikorskii
De Gruyter Studies in Mathematics **43**, 2012
Advanced graduate textbook
ISBN 978-3-11-025869-1

Mathematical Modeling, 4th Edition
Mark M. Meerschaert
Academic Press, Elsevier, 2013
Advanced undergraduate / beginning graduate textbook
(new sections on particle tracking and anomalous diffusion)
ISBN 978-0-12-386912-8

Fourier transform definition

Define the Fourier transform $\widehat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$.

Positive fractional derivatives and integrals of order $\alpha > 0$ satisfy

$$\mathbb{D}_x^\alpha f(x) \Longleftrightarrow (ik)^\alpha \widehat{f}(k) \quad \text{and} \quad \mathbb{I}_x^\alpha f(x) \Longleftrightarrow (ik)^{-\alpha} \widehat{f}(k)$$

Negative fractional derivatives and integrals of order $\alpha > 0$ satisfy

$$\mathbb{D}_{-x}^\alpha f(x) \Longleftrightarrow (-ik)^\alpha \widehat{f}(k) \quad \text{and} \quad \mathbb{I}_{-x}^\alpha f(x) \Longleftrightarrow (-ik)^{-\alpha} \widehat{f}(k)$$

Tempered fractional derivatives with $\alpha, \lambda > 0$ satisfy

$$\mathbb{D}_x^{\alpha,\lambda} f(x) \Longleftrightarrow (\lambda + ik)^\alpha \widehat{f}(k) \quad \text{and} \quad \mathbb{I}_x^{\alpha,\lambda} f(x) \Longleftrightarrow (\lambda + ik)^{-\alpha} \widehat{f}(k)$$

$$\mathbb{D}_{-x}^{\alpha,\lambda} f(x) \Longleftrightarrow (\lambda - ik)^\alpha \widehat{f}(k) \quad \text{and} \quad \mathbb{I}_{-x}^{\alpha,\lambda} f(x) \Longleftrightarrow (\lambda - ik)^{-\alpha} \widehat{f}(k)$$

Note the Fourier symbol $(\lambda \pm ik)^{-\alpha}$ no longer blows up at $k = 0$.

Riemann-Liouville definition

Tempered fractional integrals can also be defined by

$$\mathbb{I}_x^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(u)(x-u)^{\alpha-1} e^{-\lambda(x-u)} du$$

$$\mathbb{I}_{-x}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(u)(u-x)^{\alpha-1} e^{-\lambda(u-x)} du$$

for any $\alpha, \lambda > 0$. For $\lambda = 0$, these are Riemann-Liouville fractional integrals.

For $0 < \alpha < 1$ and $\lambda > 0$ we also have

$$\mathbb{D}_x^{\alpha,\lambda} f(x) = \lambda^\alpha f(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x) - f(u)}{(x-u)^{\alpha+1}} e^{-\lambda(x-u)} du$$

$$\mathbb{D}_{-x}^{\alpha,\lambda} f(x) = \lambda^\alpha f(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{+\infty} \frac{f(x) - f(u)}{(u-x)^{\alpha+1}} e^{-\lambda(u-x)} du$$

For $\lambda = 0$, these are Marchaud fractional derivatives.

Grünwald-Letnikov definition

For $\lambda > 0$ and $0 < \alpha < 1$ we have

$$\lim_{h \to 0} h^{-\alpha} \Delta_h^{\alpha,\lambda} f(x) = \mathbb{D}_x^{\alpha,\lambda} f(x)$$

where the tempered fractional difference

$$\Delta_h^{\alpha,\lambda} f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j h} f(x - jh) - (1 - e^{-\lambda h})^{\alpha} f(x).$$

using the fractional binomial coefficients

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{j!\Gamma(\alpha - j + 1)}$$

Can use to construct finite difference codes [**special session**].

These codes are mass-preserving since (by the Binomial formula)

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j h} = (1 - e^{-\lambda h})^{\alpha}$$

Tempered fractional Brownian motion

Yaglom noise is defined by

$$Y_{\alpha,\lambda}(t) = \int_{-\infty}^{t} (t-u)^{-\alpha} e^{-\lambda(t-u)} \; W(u)\,du = \mathbb{I}_t^{1-\alpha,\lambda} W(t)$$

where $W(u)$ is a white noise.

Tempered fractional Brownian motion (TFBM) is defined by

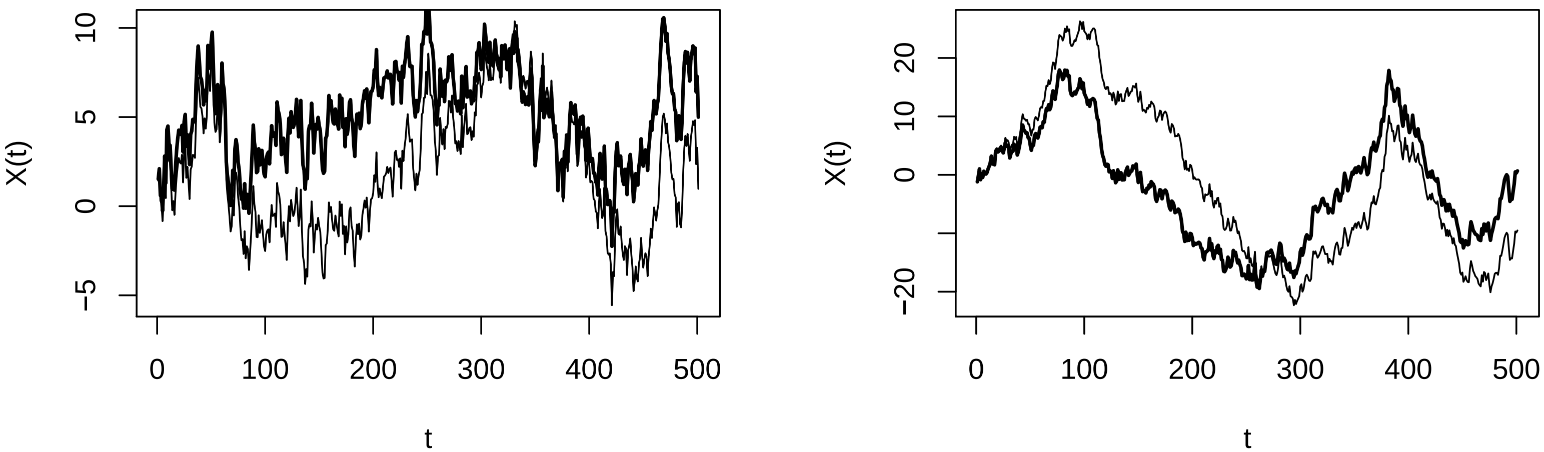
$$\begin{aligned} &B_{\alpha,\lambda}(t) = Y_{\alpha,\lambda}(t) - Y_{\alpha,\lambda}(0) \\ &= \int_{-\infty}^{\infty} \left[e^{-\lambda(t-u)_+}(t-u)_+^{-\alpha} - e^{-\lambda(0-u)_+}(0-u)_+^{-\alpha} \right] \; W(u)\,du \end{aligned}$$

where $(x)_+ = x$ for $x > 0$ and $(x)_+ = 0$ for $x \leq 0$.

When $\lambda = 0$, Yaglom noise is undefined, and TFBM = FBM.

FBM and tempered FBM

FBM (thin line) and TFBM (thick line) with same $W(u)$ for $H = 0.3, \lambda = 0.03$ (left) and $H = 0.7, \lambda = 0.01$ (right).



Sample paths are Hölder continuous of order $H = 1/2 - \alpha$.

Scaling: $B_{\alpha,\lambda}(ct) \overset{d}{=} c^H B_{\alpha,c\lambda}(t)$.

Tempered fractional time series

The ARMA(p, q) time series model is

$$Y_t - \sum_{j=1}^{p} \phi_j Y_{t-j} = Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i}$$

where $Z_t \sim \mathrm{IID}(0, \sigma^2)$.

We say that X_t follows an ARTFIMA(p, α, λ, q) model if

$$Y_t = \Delta_1^{\alpha,\lambda} X_t = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j} X_{t-j}$$

follows an ARMA(p, q) model.

Then $X_t = \Delta_1^{-\alpha,\lambda} Y_t$, a kind of tempered fractional integration.

Spectral approach

Kolmogorov argued that the spectral density of turbulence data follows $h(k) \approx k^{-5/3}$ in the inertial range of frequency k.

Tempered fractional Gaussian noise (TFGN) is defined by

$$X_j = B_{\alpha,\lambda}(j) - B_{\alpha,\lambda}(j-1) = Y_{\alpha,\lambda}(j) - Y_{\alpha,\lambda}(j-1).$$

Its spectral density (FT of the autocovariance function) is

$$h(k) = C\frac{|e^{-ik} - 1|^2}{(\lambda^2 + k^2)^{H+1/2}} \approx Ck^{1-2H}$$

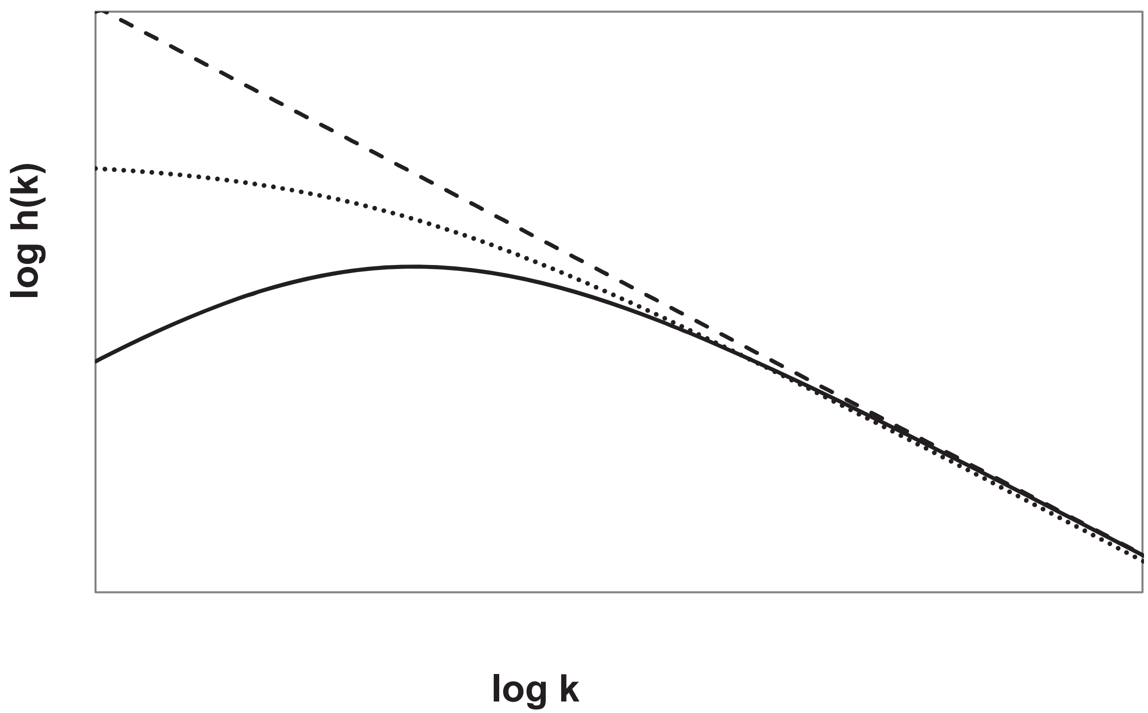
for k small and $\lambda \approx 0$. Set $H = 4/3$ for Kolmogorov scaling.

The ARTFIMA$(0, \alpha, \lambda, 0)$ model has spectral density

$$h(k) = C\left|1 - e^{-(\lambda+ik)}\right|^{-2\alpha} \approx Ck^{-2\alpha}$$

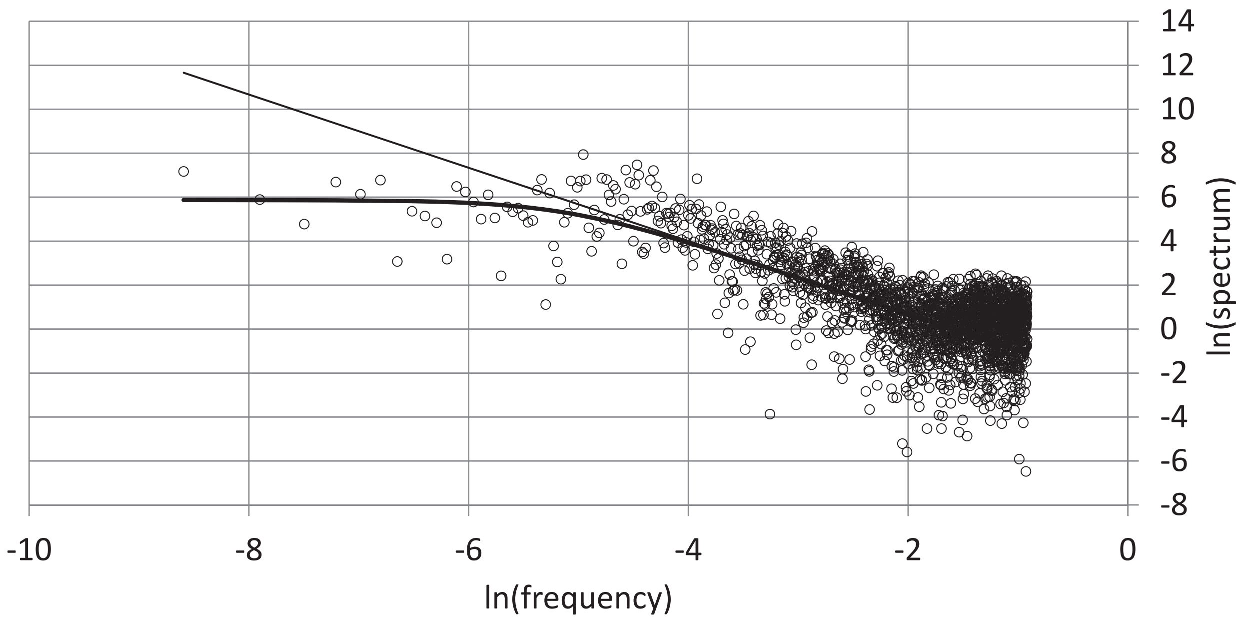
for k small and $\lambda \approx 0$. Set $\alpha = 5/6$ for Kolmogorov scaling.

Spectral density of TFGN and ARTFIMA



Log-log plot of spectral density for TFGN (solid), ARTFIMA (dotted), and Kolmogorov scaling (dashed). Here $\lambda = 0.03$.

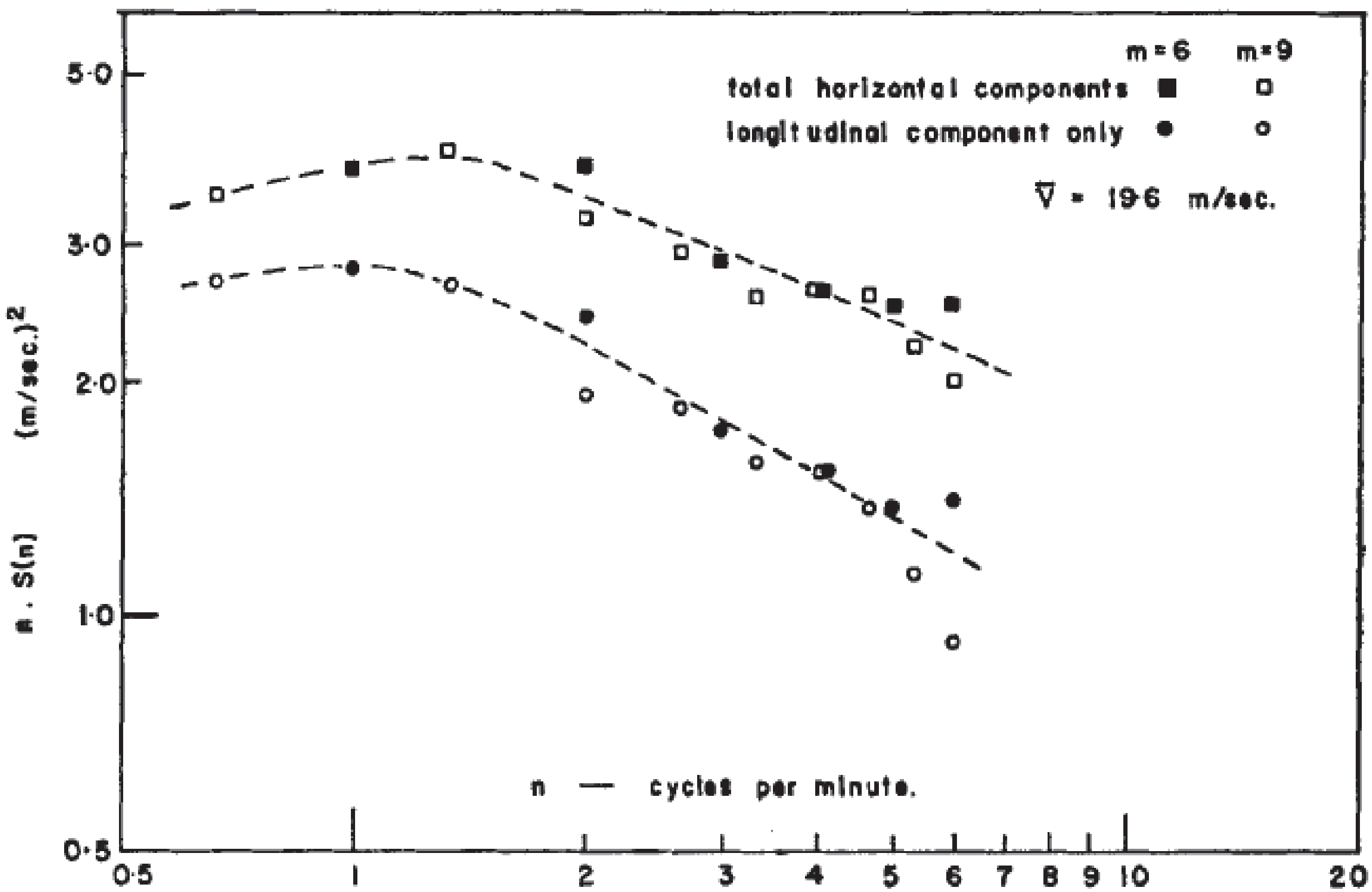
Lake Huron water velocity spectrum



Spectrum of Saginaw Bay velocity data (symbols), fitted ARTFIMA model (thick line) with $\alpha = 5/6$ and $\lambda = 0.006$, and Kolmogorov model (thin line).

Davenport spectrum for wind gusts

Spectral density of wind gustiness from Davenport (1961). The Davenport spectrum is the same as TFGN with $H = 5/6$.



Fractional diffusion and probability

The space-time fractional diffusion equation

$$\mathbb{D}_t^{\beta} p(x,t) = a\mathbb{D}_x^{\alpha} p(x,t) + b\mathbb{D}_{-x}^{\alpha} p(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

governs a stochastic process X_t with power law jumps in space

$$\mathbb{P}(|\Delta Y_u| > x) \approx Ax^{-\alpha}$$

and power law waiting times

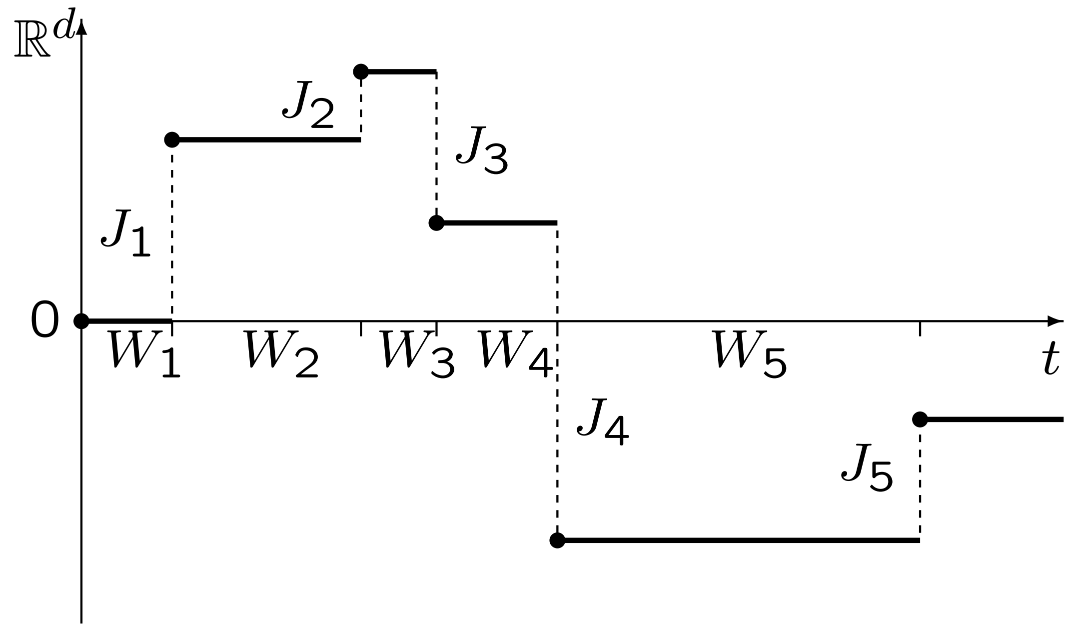
$$\mathbb{P}(\Delta D_u > t) \approx Bt^{-\beta}$$

for $0 < \alpha < 2$ and $0 < \beta < 1$.

Graph of sample paths (particle traces): $(t, X_t) = (D_u, Y_u)$.

Hence $\mathbb{D}_t^{\beta}$ codes power law waiting times, $\mathbb{D}_{\pm x}^{\alpha}$ power law jumps.

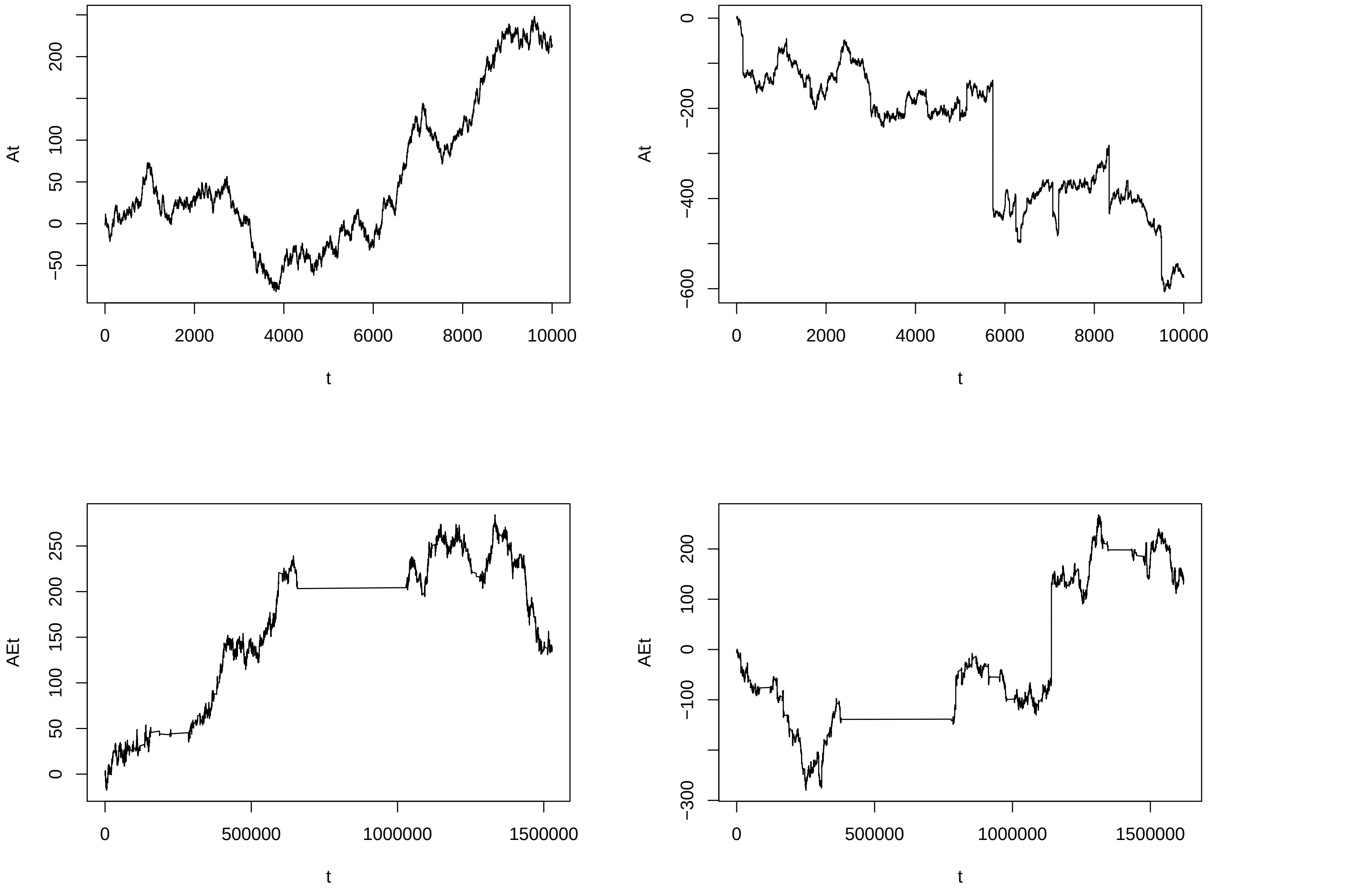
Continuous time random walks



For particle jumps $\mathbb{P}(J > x) \approx Ax^{-\alpha}$, $J_1 + \cdots + J_{[nu]} \approx n^{1/\alpha} Y_u$.
For waiting times $\mathbb{P}(W > t) \approx Bx^{-\beta}$, $W_1 + \cdots + W_{[nu]} \approx n^{1/\beta} D_u$.

Fractional diffusion particle tracking

Here $\alpha = 2.0, 1.7$ (left/right) and $\beta = 1.0, 0.9$ (top/bottom).



Tempered fractional diffusion

The tempered space-fractional diffusion equation

$$\mathbb{D}_t p(x,t) = a\partial_x^{\alpha,\lambda} p(x,t) + b\partial_{-x}^{\alpha,\lambda} p(x,t)$$

governs a stochastic process X_t with tempered power law jumps

$$\mathbb{P}(|\Delta X_u| > x) \approx A x^{-\alpha} e^{-\lambda x}$$

for $0 < \alpha < 2$ and $\lambda > 0$.

This *normalized* tempered fractional derivative is defined by

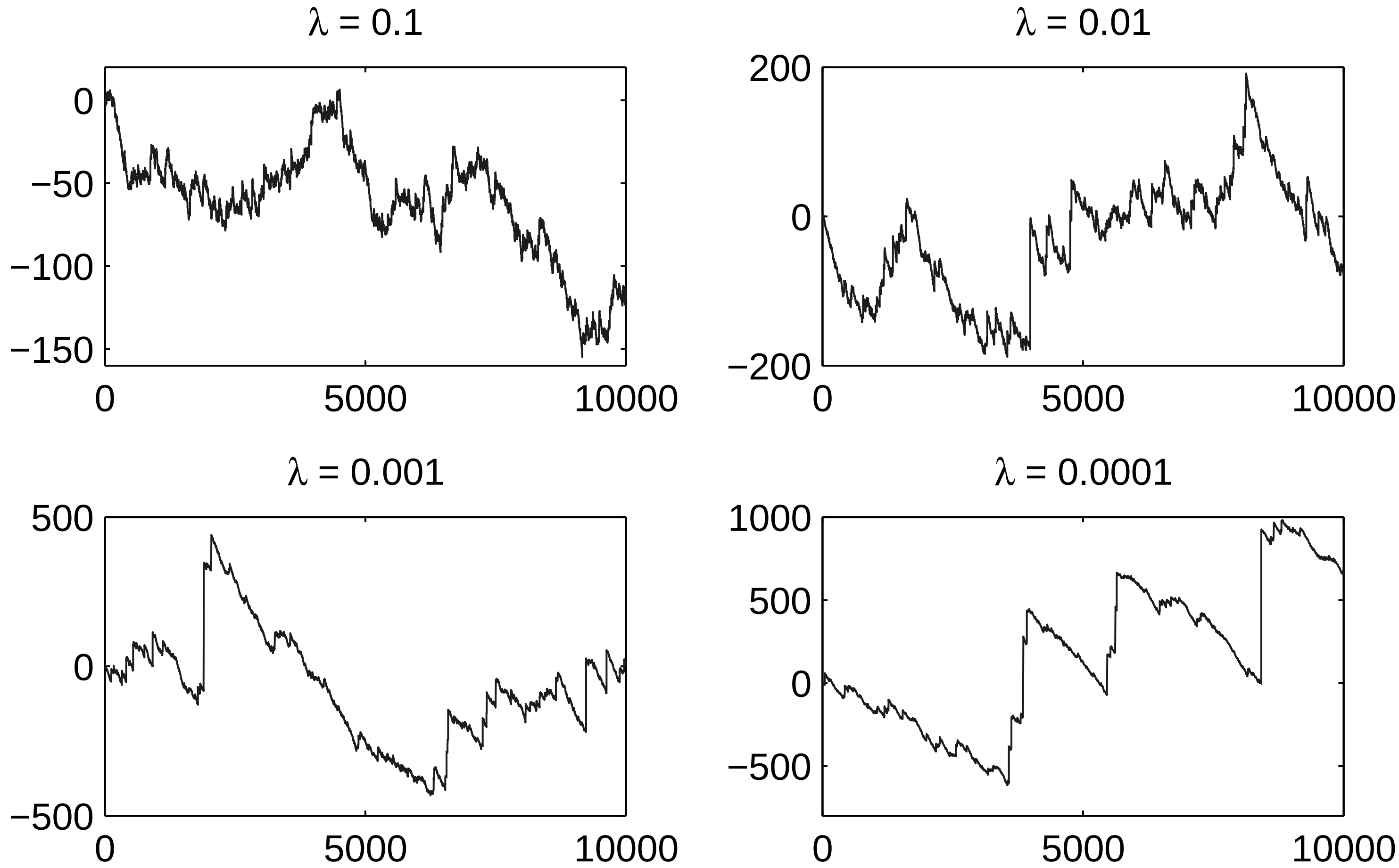
$$\begin{aligned} \partial_{\pm x}^{\alpha,\lambda} f(x) &\iff [(\lambda \pm ik)^\alpha - \lambda^\alpha]\hat{f}(k) && \text{for } 0 < \alpha < 1, \text{ and} \\ \partial_{\pm x}^{\alpha,\lambda} f(x) &\iff [(\lambda \pm ik)^\alpha - \lambda^\alpha \mp ik\alpha\lambda^{\alpha-1}]\hat{f}(k) && \text{for } 1 < \alpha < 2. \end{aligned}$$

The λ^α term makes $p(x,t)$ a PDF.

The $ik\alpha\lambda^{\alpha-1}$ term makes $p(x,t)$ have zero mean.

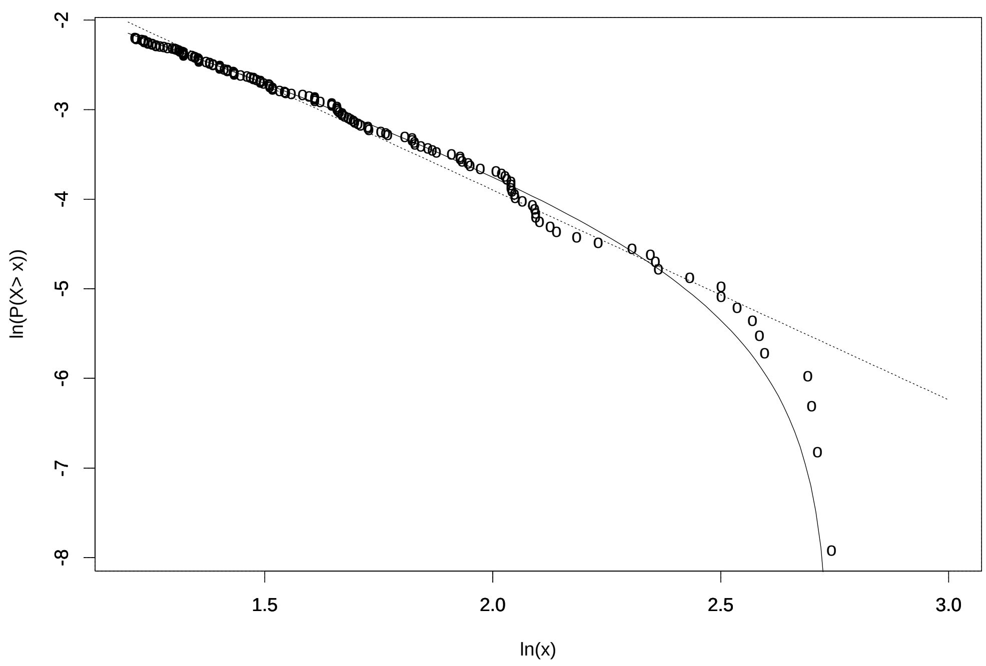
Tempered fractional diffusion particle tracking

Tempered stable process with $\alpha = 1.2$ transitions from Brownian motion to stable Lévy motion as λ decreases.



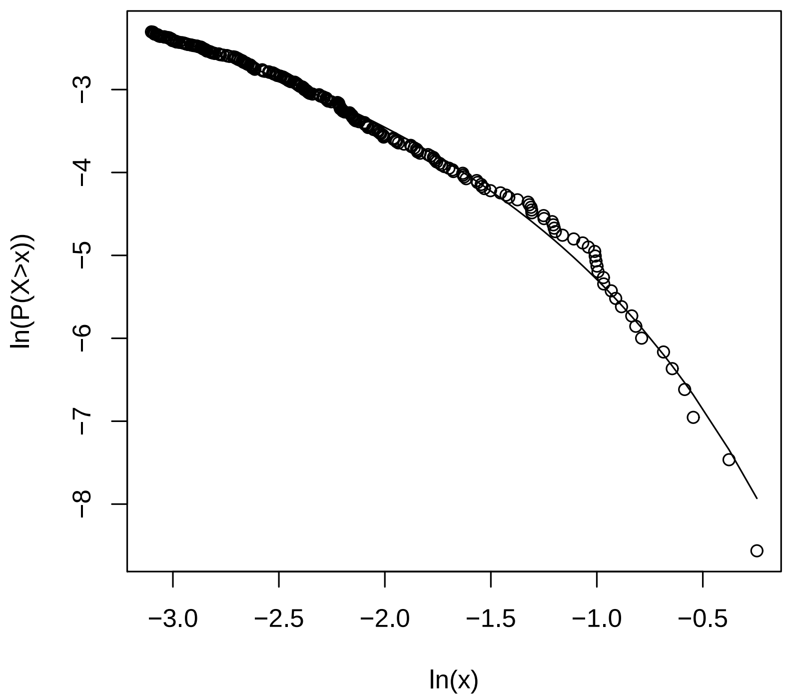
Tempered power laws in finance

AMZN stock price changes fit a tempered power law model $P(X > x) \approx x^{-0.6} e^{-0.3x}$ for x large



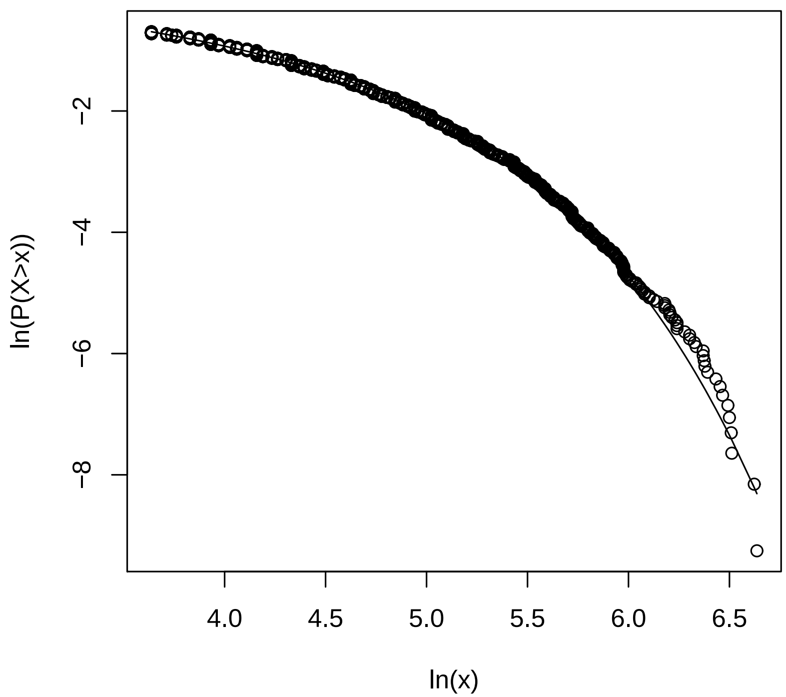
Tempered power laws in hydrology

Tempered power law model $P(X > x) \approx x^{-0.6}e^{-5.2x}$ for increments in hydraulic conductivity at the MADE site.

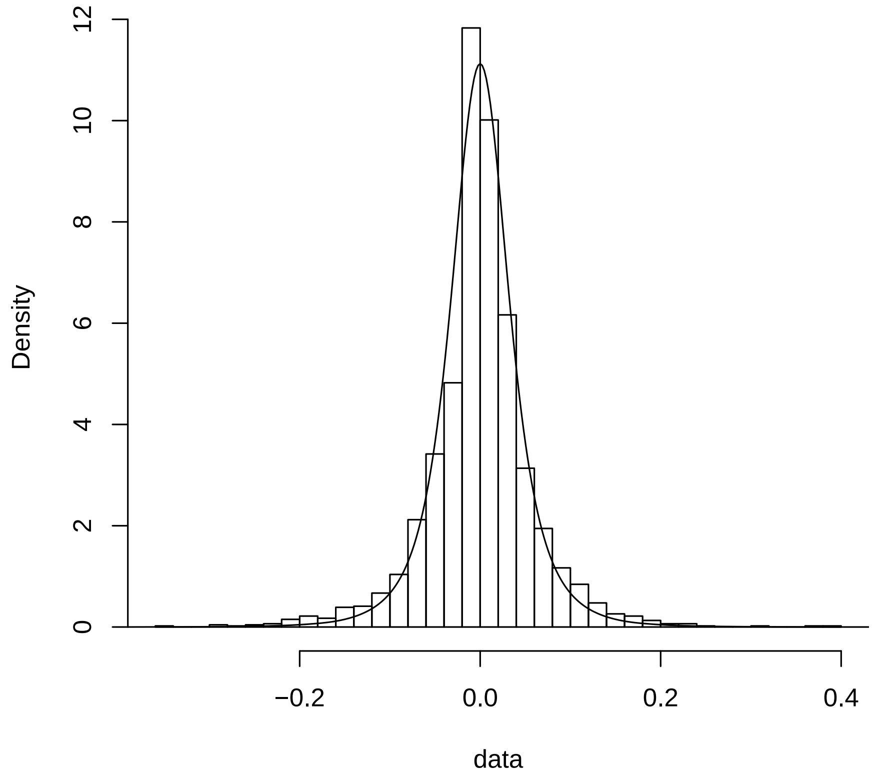


Tempered power laws in atmospheric science

Tempered power law model $P(X > x) \approx x^{-0.2} e^{-0.01x}$ for daily precipitation data at Tombstone AZ.

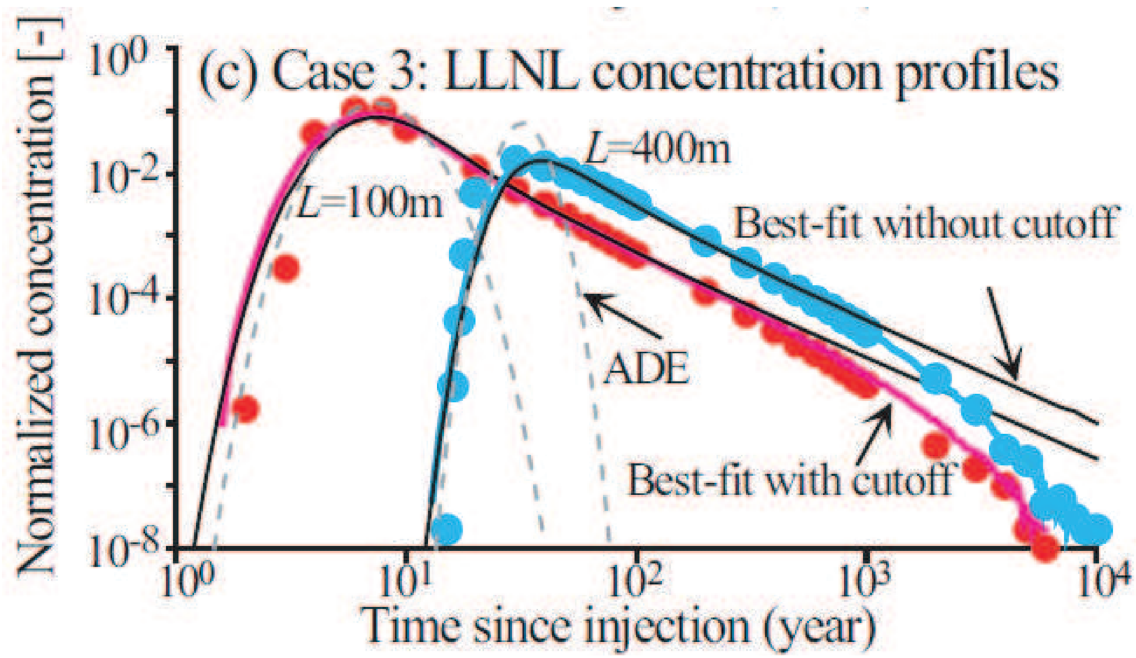


Tempered stable pdf in macroeconomics



One-step BARMA forecast errors for annual inflation rates fit a symmetric tempered stable with $\alpha = 1.1$ and $\lambda = 12$.

Tempered time-fractional diffusion model



Fitted concentration data from a 3-D supercomputer simulation. ADE fit uses $\alpha = 2, \beta = 1$. Without cutoff uses $\lambda = 0$.

Summary

- Tempered fractional derivatives
- Tempered fractional Brownian motion
- Applications to turbulence
- Tempered fractional diffusion
- Numerical methods [**special session**]
- Applications to finance and geophysics

References

1. I.B. Aban, M.M. Meerschaert, and A.K. Panorska (2006) Parameter Estimation for the Truncated Pareto Distribution. *Journal of the American Statistical Association: Theory and Methods*. **101**(473), 270–277.

2. B. Baeumer and M.M. Meerschaert (2010) Tempered stable Levy motion and transient super-diffusion. *Journal of Computational and Applied Mathematics* **233**, 2438–2448.

3. Á. Cartea and D. del Castillo-Negrete (2007) Fluid limit of the continuous-time random walk with general Lévy jump distribution functions. *Phys. Rev. E* **76**, 041105.

4. A. Chakrabarty and M. M. Meerschaert (2011) Tempered stable laws as random walk limits. *Statistics and Probability Letters 81*(8), 989–997.

5. A. V. Chechkin, V. Yu. Gonchar, J. Klafter and R. Metzler (2005) Natural cutoff in Lévy flights caused by dissipative nonlinearity. *Phys. Rev. E* **72**, 010101.

6. S. Cohen and J. Rosiński (2007) Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes, *Bernoulli* **13**, 195-210.

7. A. G. Davenport (1961) The spectrum of horizontal gustiness near the ground in high winds. *Quarterly Journal of the Royal Meteorological Society* **87**, 194–211.

8. I. Koponen (1995) Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, *Phys. Rev. E* **52**, 1197–1199.

9. M.M. Meerschaert and H.P. Scheffler (2001) *Limit Theorems for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Wiley Interscience, New York.

10. M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab.* **41**(3), 623–638.

11. Meerschaert, M.M., Scheffler, H.P. (2008) Triangular array limits for continuous time random walks. *Stoch. Proc. Appl.* **118**(9), 1606-1633.

12. Meerschaert, M.M., Y. Zhang and B. Baeumer (2008) Tempered anomalous diffusion in heterogeneous systems. *Geophys. Res. Lett.* **35**, L17403.

13. M.M. Meerschaert and A. Sikorskii (2012) *Stochastic Models For Fractional Calculus*. De Gruyter, Berlin/Boston.

14. Meerschaert, M.M., P. Roy and Q. Shao (2012) Parameter estimation for tempered power law distributions. *Communications in Statistics Theory and Methods* **41**(10), 1839–1856.

15. M.M. Meerschaert (2013) *Mathematical Modeling*. 4th Edition, Academic Press, Boston.

16. M.M. Meerschaert and F. Sabzikar (2013) Tempered fractional Brownian motion. *Statist. Prob. Lett.* **83** (2013), 2269–2275.

17. M.M. Meerschaert and F. Sabzikar (2014) Stochastic integration for tempered fractional Brownian motion. *Stoch. Proc. Appl.* **124**, 2363–2387.

18. J. Rosiński (2007), Tempering stable processes. *Stoch. Proc. Appl.* **117**, 677–707.

19. F. Sabzikar, M.M. Meerschaert, and Jinghua Chen (2014) Tempered Fractional Calculus. *J. Comput. Phys.*, to appear in the Special Issue on Fractional Partial Differential Equations, preprint at `www.stt.msu.edu/users/mcubed/TFC.pdf`

Simulating tempered stable laws (JCAP 2010)

Simulation codes for stable random variates are widely available.

If $X > 0$ has stable density density $f(x)$, TS density is

$$f_\lambda(x) = \frac{e^{-\lambda x} f(x)}{\int_0^\infty e^{-\lambda u} f(u)\, du}$$

Take $Y \sim \exp(\lambda)$ independent of X, (X_i, Y_i) IID with (X, Y).

Let $N = \min\{n : X_n \le Y_n\}$. Then $X_N \sim f_\lambda(x)$.

Proof: Compute $P(X_N \le x) = P(X \le x | X \le Y)$ by conditioning, then take d/dx to verify.

Triangular array scheme (SPL 2011)

Take $P(J > x) \approx Cx^{-\alpha}$ with $1 < \alpha < 2$. Triangular array limit

$$\sum_{k=1}^{[nt]} n^{-1/\alpha} J_k - b_t^{(n)} \Rightarrow Y_t$$

is stable. Define tempering variables:

$$P(Z > u) = u^{\alpha} \int_u^{\infty} r^{-\alpha-1} e^{-\lambda r} dr$$

Replace $n^{-1/\alpha} X_k$ by Z_k if $n^{-1/\alpha} X_k > Z_k$.

Triangular array limit is tempered stable.

Exponential tempering: sum of α and $\alpha - 1$ tempered stables.

Tail estimation (CIS 2012)

Hill-type estimator: Assume $P(X > x) \approx Cx^{-\alpha}e^{-\lambda x}$ for x large, use order statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.

Conditional MLE given $X_{(n-k+1)} > L \geq X_{(n-k)}$:

$$
\begin{aligned}
T_1 &:= \sum_{i=1}^{k} (\log X_{(n-i+1)} - \log L) \\
T_2 &:= \sum_{i=1}^{k} (X_{(n-i+1)} - L) \\
1 &= \sum_{i=1}^{k} \frac{x_{(n-i+1)}}{kx_{(n-i+1)} + \widehat{\alpha}(T_2 - T_1 x_{(n-i+1)})} \\
\widehat{\lambda} &= (k - \widehat{\alpha}T_1)/T_2 \\
\widehat{C} &= \frac{k}{n} L^{\widehat{\alpha}} e^{\widehat{\lambda} L}
\end{aligned}
$$

R code available at `www.stt.msu.edu/users/mcubed/TempParetoR.zip`

Testing for pure power law tail (JASA 06)

Null hypthesis $H_0 : P(X > x) = Cx^{-\alpha}$ Pareto for $x > L$.

Test based on extreme value theory rejects H_0 if

$$X_{(1)} < \left(\frac{nC}{-\ln q} \right)^{1/\alpha}$$

where α, C can be estimated using Hill's estimator

$$\widehat{\alpha}_H = \left[k^{-1} \sum_{i=1}^{k} \{ \ln X_{(n-i+1)} - \ln X_{(n-k)} \} \right]^{-1}$$
$$\widehat{C}_H = (k/n)(X_{(n-k)})^{\widehat{\alpha}_H}$$

Simple p-value formula $p = \exp\{-n\, C\, X_{(n)}^{-\alpha}\}$.

Tempered fractional calculus operators (SPA 2014)

$\mathbb{I}^{\alpha,\lambda}_{\pm x} f(x) \Longleftrightarrow (\lambda \pm ik)^{-\alpha} \hat{f}(k)$ for $f \in L^2(\mathbb{R}) = \{f : \int |f(x)|^2 dx < \infty\}$.

Define the fractional Sobolev space

$$W^{\alpha,2}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \int (\lambda^2 + k^2)^{\alpha} |\hat{f}(k)|^2 \, dk < \infty\}.$$

Then $\mathbb{D}^{\alpha,\lambda}_{\pm x} f(x) \Longleftrightarrow (\lambda \pm ik)^{\alpha} \hat{f}(k)$ for $f \in W^{\alpha,2}(\mathbb{R})$.

Semigroup property: $\mathbb{I}^{\alpha,\lambda}_{\pm} \mathbb{I}^{\beta,\lambda}_{\pm} f = \mathbb{I}^{\alpha+\beta,\lambda}_{\pm} f$ for $f \in L^2(\mathbb{R})$, and $\mathbb{D}^{\alpha,\lambda}_{\pm} \mathbb{D}^{\beta,\lambda}_{\pm} f = \mathbb{D}^{\alpha+\beta,\lambda}_{\pm} f$ for $f \in W^{\alpha+\beta,2}(\mathbb{R})$.

Adjoint property: $\langle f, \mathbb{I}^{\alpha,\lambda}_{+} g \rangle_2 = \langle \mathbb{I}^{\alpha,\lambda}_{-} f, g \rangle_2$ for $f, g \in L^2(\mathbb{R})$, and $\langle f, \mathbb{D}^{\alpha,\lambda}_{+} g \rangle_2 = \langle \mathbb{D}^{\alpha,\lambda}_{-} f, g \rangle_2$ for $f, g \in W^{\alpha,2}(\mathbb{R})$, using the L^2 inner product $\langle f, g \rangle_2 = \int f(x) g(x) \, dx$