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A new stochastic model for fracture transmissivity assessment

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[1] A new stochastic model is proposed for fracture counts and transmissivities in a borehole interval. This new model incorporates several empirical observations, including: (i) Clustering of fractures; (ii) Exponential fracture spacings; (iii) Transmissivities extending over several orders of magnitude; (iv) Power law probability tail for transmissivities at finer scales; (v) Log-normal transmissivities at larger scales; and (vi) Dependence between fracture counts and transmissivities. Several example applications are provided using borehole data from Sweden.

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1. Introduction

[2] Many applications, such as modeling of groundwater flow and solute transport in fractured rock [*Dershowitz et al.*, 1998] or designing rock grouting [*Fransson*, 2001], require information on fractures and their transmissivities [*Gustafson and Fransson*, 2005]. Fractures in rock typically come in clusters [*Gillespie et al.*, 1993], with spacings following the exponential law [*Priest and Hudson*, 1976, 1981]. The transmissivities (denoted by the symbol T in the sequel) of certain intervals of a borehole can typically stretch over several orders of magnitude [*Gustafson and Fransson*, 2005]. Although approximate methods of obtaining the distribution of T from hydraulic packer tests are available [*Osnes et al.*, 1988; *Axelsson et al.*, 1990; *Fransson*, 2002], these are quite complicated. Therefore, simple, robust models based on few parameters that reproduce patterns of test data are strongly needed [*Gustafson and Fransson*, 2005].

[3] We propose a new stochastic model that accounts for fracture count across each borehole section and the value of T corresponding to that section. This simple model is in agreement with the above empirical findings, has only a few parameters that are straightforward to estimate, and was found to fit one typical borehole data set quite well.

2. Borehole Data

[4] We consider the cored borehole KLX01 between 106 and 691 m depth, discussed in *Gustafson and Fransson* [2005]. KLX01 is a deep subvertical borehole through granitic bedrock in Sweden. The data consists of fracture counts N and the corresponding transmissivities T for 3 m borehole sections, and the same data for 30 m sections. Since the 30 m measurements consist of relatively few data points, we first focus on the 3 m data, consisting of 195 pairs of values. The first variable N is discrete, and measures the number of fractures in successive 3 m intervals along the borehole. The second variable measures the corresponding transmissivity for that borehole section.

[5] Figure 1 shows that the fracture count data is highly skewed with a long right tail. The data have a sample mean of 8.5 and a sample variance of 36.3. This over-dispersion is not accounted for by the Poisson distribution, often used to model fracture counts, since the mean and the variance of the Poisson distribution are always equal. Instead we will employ the negative binomial distribution, a simple model that accommodates over-dispersion.

[6] A histogram of $\log(T)$ measured in 3 m intervals appears in Figure 2. The long right tail of the distribution is noticeable, and the normal density gives a poor fit. Hence, in this case, it is not reasonable to model T as log-normal. The probability plot gives further evidence that $\log(T)$ is not normally distributed. For data following a normal distribution, the points would scatter around the center line and 95% would lie between the error bars.

[7] The fracture count N and the $\log(T)$ data also exhibit significant statistical dependence. The sample correlation of $r = 0.21$ between N and $\log(T)$ is small but statistically significant (the chance of a correlation this large, if N and $\log(T)$ were independent, is given by the p-value of .003). For the KLX02 borehole reported in *Gustafson and Fransson* [2005] the same analysis yields an even larger correlation $r = 0.48$ ($p = .001$). We conclude that a coupled model of fracture count and $\log(T)$ is worth investigating.

3. Negative Binomial Model for Fracture Count

[8] We investigate a negative binomial (NB) model for the number N of fractures in each interval along a borehole. The NB model is a two parameter family with probability mass function (PMF)

$$f(k) = P(N = k) = \frac{\Gamma(t+k)}{\Gamma(t)k!} p^t (1-p)^k, k = 0, 1, 2, \ldots, \quad (1)$$

where $\Gamma(\cdot)$ is the standard gamma function and the range of the parameters is $t > 0$ and $0 < p < 1$ [see, e.g., *Johnson et*

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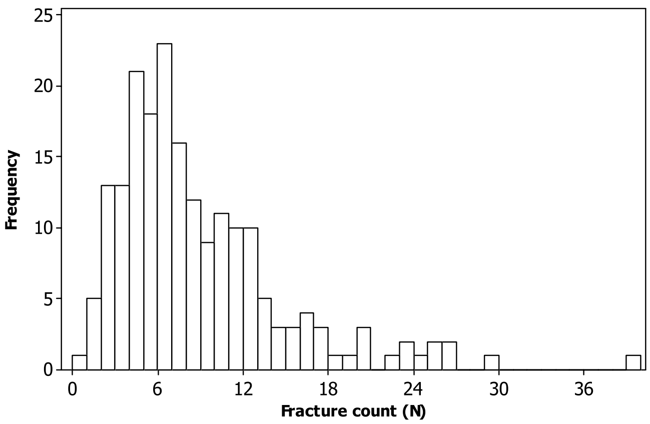

Figure 1. Fracture counts at 3 m intervals in the KLX 01 borehole.

al., 1993]. When t is an integer, the negative binomial variable represents a waiting time: It is the number of failures before the tth success in a sequence of independent random trials with success probability p. The mean and the variance of this distribution are $\mathrm{E}(N) = t(1 - p)/p$ and $\mathrm{Var}(N) = t(1 - p)/p^2$, so that the variance is always greater than the mean. The NB process $\{N(t), t \geq 0\}$ with PMF (1) counts the number of fractures in a borehole section whose length is proportional to t. The increments of the NB process $N_i = N(it) - N((i - 1)t)$, $i = 1, 2, 3, \ldots$, form a sequence of independent and identically distributed (IID) NB random variables representing the number of fractures in each borehole section. The parameter t is proportional to the length the borehole section, and we will use this fact to validate the model. This spatial NB model counts the number of fractures in any size interval, so that up-scaling is a simple matter.

[9] We fit the NB distribution (1) to the fracture count data along the borehole KLX 01 (3 m intervals with $n = 195$ data points). The standard statistical method of maximum likelihood estimation (MLE) is used to estimate the parameters in (1) (see Appendix for details). Finding the MLE for t involves maximizing the likelihood function $g(t)$ given by equation (A4) in the Appendix. Numerical optimization produces $t = 3.1$, which we round to 3 (the fact that t is approximately the length of the borehole interval in meters is a coincidence). A graph of the likelihood function (Figure 3), to validate the results of the optimization code, shows that the maximum value is indeed attained at this point. Substituting into $p = t/(t + \overline{N})$, we obtain $p = 0.263$. Figure 4 shows the empirical distribution function for this fracture count data along with fitted NB distribution function. A visual inspection indicates that the fit is quite good.

[10] To further check how well the fracture count data fits the NB model, we employ a standard chi-square test (with $m = 7$ cells chosen to make the cell probabilities approximately equal, see Appendix). The observed and expected cell frequencies under the NB model with $t = 3$ and $p = 0.263$ are shown in Table 1. The test statistic is $\chi^2 = \Sigma(O - E)^2/E = 4.91$. Since we are estimating $d = 2$ parameters, the 95th percentile of the chi-square distribution (with $m - 1 - d = 4$ degrees of freedom) is found from standard tables to be 9.488. Since the computed value of 4.91 is less than 9.488, there is no statistically significant evidence against the NB fit (the p-value is 0.3). We conclude that the fracture count data at 3 m intervals fits a NB distribution reasonably well.

[11] Next we fit the NB distribution to the 30 m sections along the same borehole, to validate the up-scaling properties of the model. For $p = 0.263$ (under the NB model, sections of any length must have the same value of p), the MLE of t becomes (see formula (A3) in the Appendix) $t = (84.150)(0.263)/(1 - 0.263) = 30.0291$, which we round to 30. Note that this t is 10 times the value for the 3 m intervals, in agreement with the NB process model, since in that model, the length of the borehole section is proportional to the parameter t. This calculation validates the NB model, since it shows we can up-scale to longer sections (the sum of ten independent NB random variables with $t = 3$ is NB with $t = 30$).

[12] Although with $n = 20$ values, our data set here is rather small, we nevertheless apply a standard chi-square test to check the fit. This time we use $m = 5$ cells chosen to make the cell probabilities approximately equal (to 0.2). The observed and expected cell frequencies are shown in Table 2 below. These lead to the value 3.834 of the test statistic, with the corresponding p-value of 0.15 (based on a chi-square distribution with 2 degrees of freedom). Hence

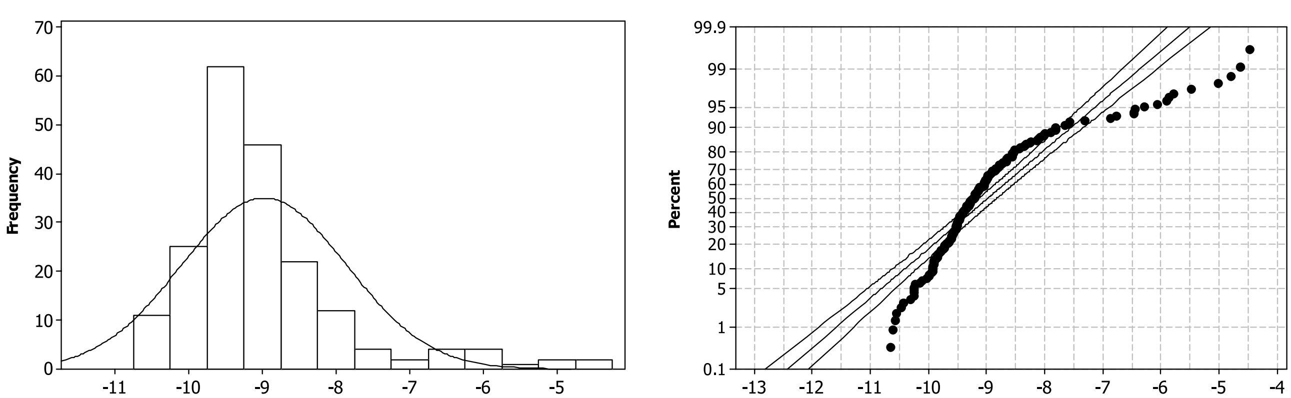

Figure 2. Fracture log(T) at 3 m intervals from the KLX 01 borehole, with the best fitting normal density (left). A probability plot (right) gives further evidence of a poor fit, showing that T is not log-normal at this scale.

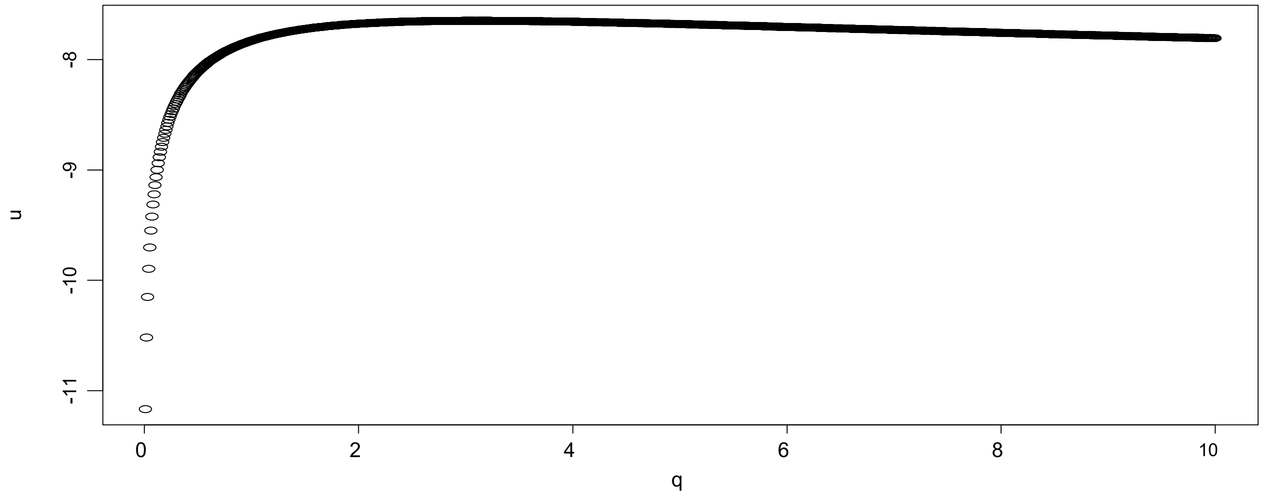

Figure 3. The objective function $g(t)$ (see (A4) in Appendix) used to find the maximum likelihood of the parameter t of the NB model (1).

we conclude that the fracture count data at 30 m intervals also fits a NB distribution reasonably well. The NB model is further validated by the fact that the scaling parameter t grows proportional to the borehole section length. In conclusion, we model the number of fractures in a borehole section of length t meters by a NB random variable $N(t)$ whose PMF (1) has parameters t and $p = 0.263$. The numbers of fractures in adjoining borehole sections are assumed independent.

[13] To investigate the general applicability of the NB model for fracture counts, we also fit the model to six additional borehole sections. Figure 5 illustrates the results of this exercise. KLX02 is another deep subvertical borehole through granitic bedrock, 3 m sections. KA3376B01 is another subhorizontal investigation borehole in the Äspö Hard Rock Laboratory. The Nygårds Tunnel dataset consists of stacked data from 4 boreholes collected in order to get a characteristic description of the fracturing of the rock. BHMSV01 is from the Hallandsås tunnel, is a railway tunnel under construction in Southwest Sweden. The project has been notorious because of the very fractured bedrock and high permeability of the rock. The KFM02A borehole is a subvertical investigation borehole of the site investigations for a future nuclear waste repository. It is situated in the Forsmark area close to the Baltic about 200 km north of Stockholm. The studied section is a bit extreme because of the high fracturing and high permeability. The studied section comprises a couple of very conductive fracture zones. KB971 is a cored subhorizontal investigation borehole for the Törnskogstunneln, a motorway tunnel recently built about 2 km North of Stockholm. The rock is typical Stockholm granite, and the section length is 3 m. We consider the NB fit satisfactory in all cases, except the KFM02A. In all cases, the Poisson model gives a poor fit due to over-dispersion. We conclude that the NB model is generally useful for characterizing fracture counts.

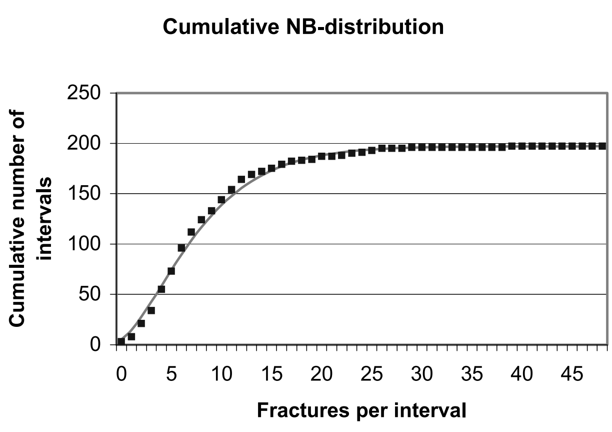

Figure 4. Empirical (squares) and fitted NB (line) cumulative distribution function $F(k) = P(N \leq k)$ corresponding to KLX 01 borehole fracture counts at 3 m intervals.

[14] We note that the NB process is a cluster Poisson process (see Appendix), with multiple events (i.e. fractures) occurring at each Poissonian arrival. Such clustering is one of the characteristics of spatial distributions of fractures [*Gillespie et al.*, 1993]. Moreover, the fracture spacings of this process are exponential, which is another characteristic of discontinuity spacings in rock often reported in the literature [*Priest and Hudson*, 1976, 1981]. A more complete physical argument for the NB model is included in the Appendix.

4. A New Stochastic Model for the Transmissivity T

[15] Since fracture number N and transmissivity T are statistically dependent, it is reasonable to investigate models

Table 1. Contingency Table for Chi-Square Goodness of Fit Test[a]

Cells	0–2	3–4	5–6	7–8	9–10	11–13	14+
Cell probabilities	0.118	0.153	0.165	0.148	0.121	0.128	0.167
Observed frequencies (O)	19	34	41	28	20	25	28
Expected frequencies (E)	23.01	29.84	32.18	28.86	23.60	24.96	32.57

[a]NB model for 3 m fracture count data from the KLX 01 borehole.

Table 2. Contingency Table for Chi-Square Goodness of Fit Test[a]

Cells	0–69	70–79	80–89	90–99	100+
Cell probabilities	0.213	0.209	0.219	0.171	0.189
Observed frequencies	7	3	2	3	5
Expected frequencies	4.26	4.18	4.38	3.42	3.78

[a]NB model for 30 m fracture count data from the KLX 01 borehole.

that capture this dependence. We begin by examining the way in which the (conditional) distribution of T varies with N. Hence, we group the log(T) data points according to the number of fractures N in each borehole section. Figure 6 shows normal probability plots of log(T) for several different values of N. While the normal fit is not perfect, especially for N between 4 and 7, the fit is much better than Figure 2. This shows that grouping the log(T) data

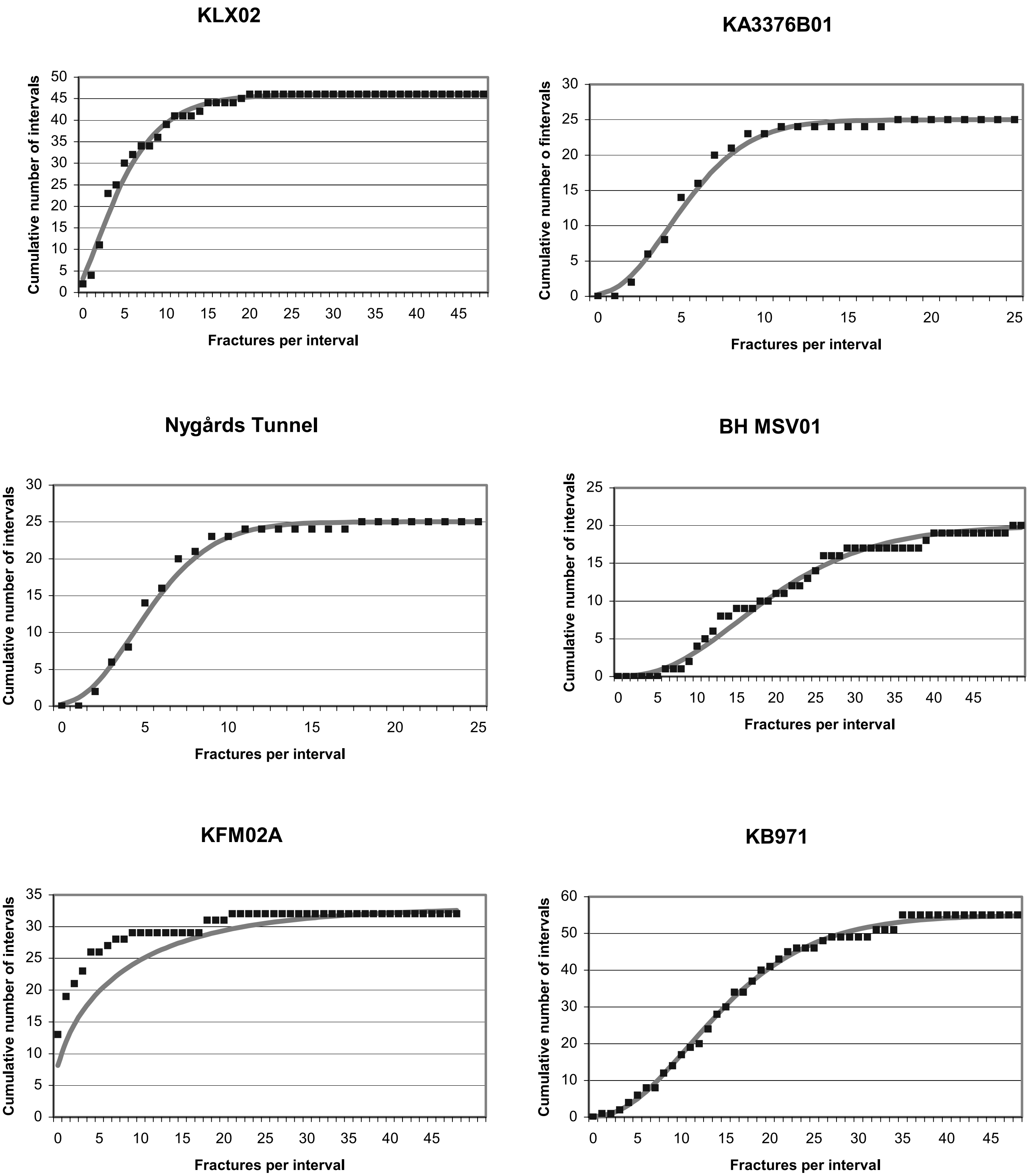

Figure 5. Empirical (squares) and fitted NB (line) cumulative distribution function $F(k) = P(N \leq k)$ corresponding to a variety of Swedish borehole fracture counts.

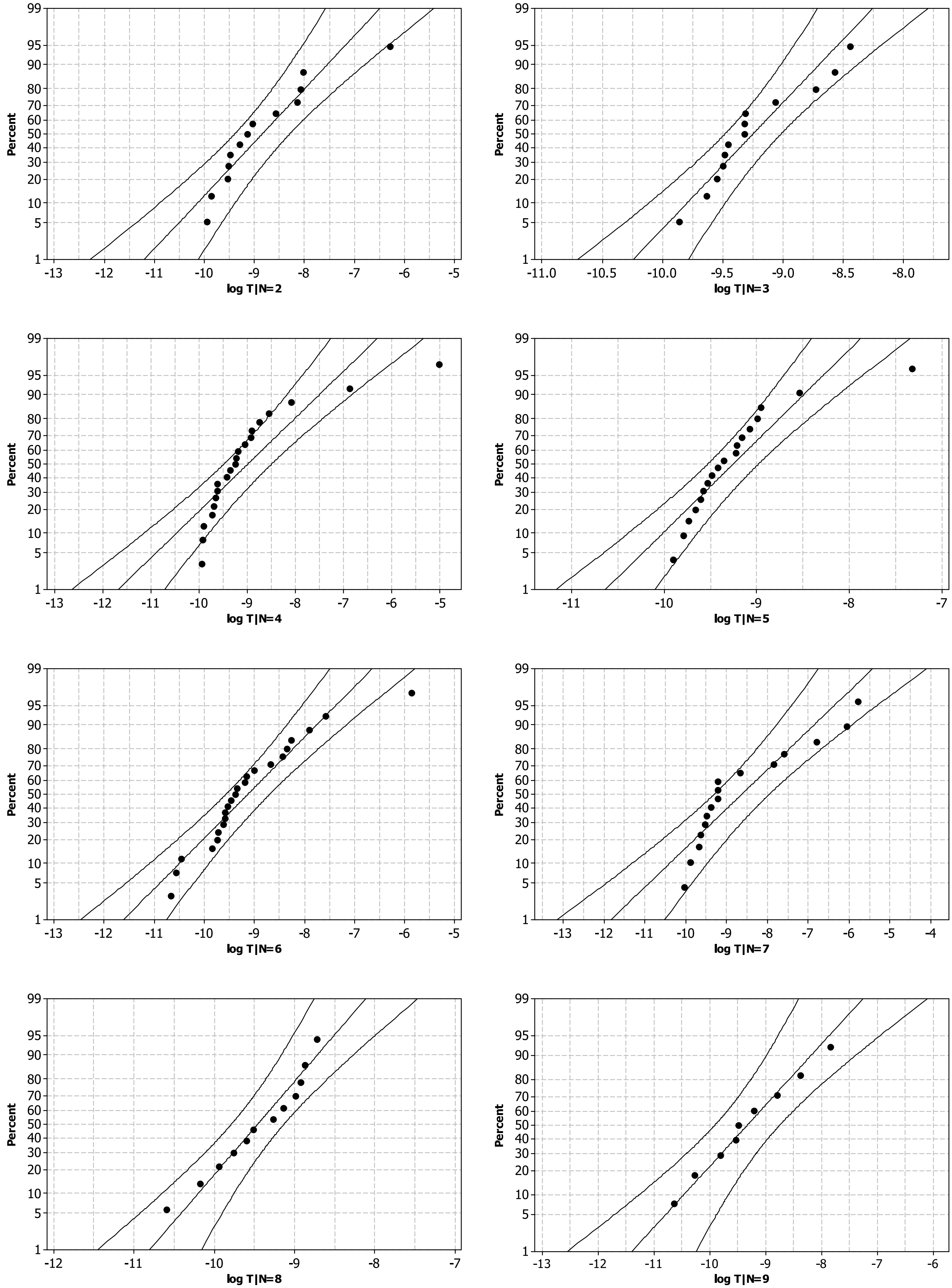

Figure 6. Probability plots illustrating the distribution of log(T) for different numbers N of fractures per 3 m borehole section in the KLX01 data set.

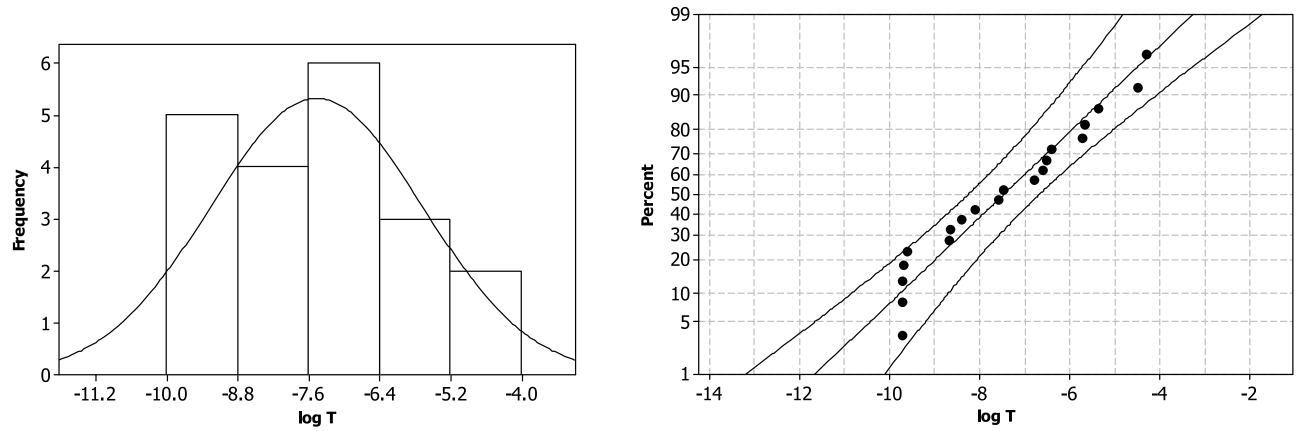

Figure 7. Fracture log(T) at 30 m intervals from the KLX 01 borehole, with the best fitting normal density (left). A probability plot (right) gives further evidence that the normal distribution provides a good fit, showing that T is log-normal at this scale.

values according to fracture count N gives a much better fit to the log-normal distribution. Then it is reasonable to consider a coupled model where T is (approximately) log-normal with a variance that depends on N.

[16] Next we examine the way that the variance of log(T) changes with the fracture count N. A simple linear regression reveals *variance* = $-0.917 + 0.261\,N$ with *R-squared* = 57.7% indicating a reasonable level of predictive power. The R-squared value measures the extent to which the variance can be predicted from the fracture count N. Since the intercept is not significantly different from zero (p = 0.190) we re-ran the regression with a zero intercept to obtain the final model *variance* = 0.205 N. Hence we can model log(T) as normal with mean zero and variance 0.205 N depending on the fracture count N. Since N is negative binomial, at some lag (borehole interval length) the random variable N will be geometric, a discrete analogue to the exponential distribution which is also a special case of the NB. Hence the distribution of log(T) will closely resemble a normal mixture with an exponential variance. (A normal mixture is a normal random variable whose variance is replaced by another non-negative random variable, resulting in a new random variable whose distribution is not normal.) This normal mixture turns out to be the Laplace (or double exponential) distribution, which was previously applied in *Meerschaert et al.* [2004] to model log-hydraulic conductivity in heterogeneous groundwater aquifers. The NB normal mixture model is completely consistent with the observation in *Gustafson and Fransson* [2005] that T has power-law (Pareto) probability tails, since taking logs converts the power law into an exponential.

[17] This NB normal mixture model for log(T) also explains the transition from non-normal to normal distribution as the scale increases, or in other words, the empirical fact that transmissivities for a long borehole section are often observed to fit a log-normal distribution. Figure 7 shows a histogram and probability plot of the KLX01 log(T) for the 30 m borehole sections. We note that the normal distribution fit is quite good, in marked contrast with the same analysis for the 3 m sections shown in Figure 2. When the scale (length of a borehole section) increases, the mean of the NB random variance is also increasing. Then it is not hard to show (using Slutsky's Theorem) that the NB normal mixture converges to a normal distribution as the scale t increases. Hence the NB normal mixture model explains both the power-law tails of T at finer scales, and the log-normal distribution of T at larger scales.

[18] The NB normal mixture model also allows us to impute the log(T) distribution for individual fractures from the data for borehole sections. This is accomplished by separating out the effect of the fracture count N for a borehole section. For the KLX01 data, we model log(T) for a single fracture as normal with mean zero and variance 0.205, obtained by setting N = 1 fracture in our variance model. Hence the distribution of log(T) for a single fracture is normal (i.e. T for a single fracture is log-normal). We conclude that the heavy power-law tail in the T distribution for a borehole section noted by *Gustafson and Fransson* [2005] is completely due to the random number of fractures in a section. Since a borehole section has a random number N of fractures, this randomizes the variance of log(T) in the section, and gives it a distribution with an exponential tail. Then reversing the log transform (exponentiating) yields a power law tail. Finally we note that the NB normal mixture model is just one of a large family of normal mixture models that share the same properties. Any model in which log(T) has a fixed distribution, conditional on the random fracture count N with a variance proportional to N will converge to a Gaussian (log-normal distribution for T) at larger scales, and if the conditional distribution of log(T) has exponential probability tails, then the T distribution will have power-law tails, in agreement with empirical observations. The NB model for fracture counts, in this framework, is merely the simplest descriptive model in agreement with observations.

5. Summary

[19] This paper presents a new model for fracture count N and transmissivity T in borehole sections in fractured rock.

The fracture counts follow a negative binomial model which can easily be fit to field data. To demonstrate the effectiveness of the model, we fit fracture count data from *Gustafson and Fransson* [2005] at 3 meter intervals, and show that the model correctly predicts the up-scaled fracture number distribution at 30 meter intervals. The model for log(T) is an NB normal mixture, based on the NB fracture count in each section, and hence provides a statistical link between fracture count and T, as well as a method for imputing the T distribution for individual fractures from that of borehole sections.

Appendix A

[20] For the convenience of the reader, we collect here some facts about negative binomial distributions and the corresponding Lévy process.

A1. Basic Properties

[21] The characteristic function (Fourier transform) of the negative binomial distribution (1) is

$$\phi_{NB}(u) = Ee^{iuN} = \left(\frac{p}{1-(1-p)e^{iu}}\right)^t, -\infty < u < \infty. \quad \text{(A1)}$$

This defines a continuous time stochastic process $\{N(t), t \geq 0\}$ with independent and homogenous increments (Lévy process), whose marginal distributions $N(t)$ are given by (A1). This is a pure jump process with both mean and variance linear in t, that is $\mathrm{E}(N(t)) = t(1-p)/p$ and $\mathrm{Var}(N(t)) = t(1-p)/p^2$, so that the variance is always greater than the mean (over-dispersion). This is actually a cluster Poisson process. The inter-arrival times are IID exponential variables with parameter $\lambda = -\log p$. The Poissonian arrivals come in clusters of size X_i, where the X_i's are IID integer valued variables with logarithmic distribution, $P(X_i = k) = -(1-p)^k/(k \log p)$, $k = 1, 2, 3 \ldots$. We thus have the following representation of a NB process, $N(t) = X_1 + \cdots + X_{\Lambda(t)}$, where $\Lambda(t)$ is a Poisson process with intensity λ above, independent of the X_i's. The NB process is also known as the gamma-Poisson process, since it can be defined as a Poisson process $\{\Lambda(t), t \geq 0\}$ with intensity $\lambda = (1-p)/p$, subordinated to a gamma process $\{W(t), t \geq 0\}$, $N(t) = \Lambda(W(t))$. The gamma process is another continuous time stochastic process with independent and homogenous increments, whose one dimensional distributions $W(t)$ are gamma distributions with shape parameter t and scale 1 (so that the mean of $W(t)$ is t). One can view this process as a random time deformation. More information on the NB process can be found in *Kozubowski and Podgórski* [2005, 2007].

A2. Estimation and Model Validation

[22] We now review the method of maximum likelihood estimation (MLE) for estimating the parameters t and p of the NB distribution. Suppose that $N_1, N_2, \ldots, N_n$ is a random sample of size n with PMF (1). First we compute the likelihood function by evaluating the right-hand side of (1) at each data point $N_i = k$, and taking the product of these n terms. The MLE is of the parameters t and p is the point (t, p) that maximizes the likelihood function or, equivalently, its logarithm. The log-likelihood function is

$$Q(t,p) = n\left(\frac{1}{n}\sum_{j=1}^{n}\log\frac{\Gamma(t+N_j)}{\Gamma(1+N_j)} - \log\Gamma(t) + t\log p + \overline{N}\log(1-p)\right), \quad \text{(A2)}$$

where $\overline{N}$ is the sample mean. Fixing $t > 0$ and maximizing the function Q in (A2) with respect to p results in

$$p = p(t) = \frac{t}{t+\overline{N}} \quad \text{or, equivalently} \quad t = \frac{p\overline{N}}{1-p}. \quad \text{(A3)}$$

When we now substitute $p = p(t)$ above back into (A2), we obtain the quantity $Q(t, p(t))$, which reduces the MLE calculation to the problem of maximizing

$$g(t) = \frac{1}{n}\sum_{j=1}^{n}\log\Gamma(t+N_j) - \log\Gamma(t) + t\log t - (t+\overline{N})\log(t+\overline{N}). \quad \text{(A4)}$$

This optimization can be performed numerically using standard codes. The optimum location yields the maximum likelihood estimator (MLE) of t, and the corresponding MLE of p is obtained by substituting back into (A3). Model validation to assess how well the data fits the NB model can include graphical evidence (e.g., comparing the PMF to a histogram of the data) or a more formal chi-square test for goodness of fit (see for example *D'Agostino and Stephens* [1986]). In the chi-square test, the possible values of the NB variable are subdivided into k intervals of roughly equal probability. Then the number O of data points in each of those intervals is compared to the expected number E under the NB model. The probability distribution of the resulting test statistic $\chi^2 = \Sigma(O-E)^2/E$ is available in standard tables. If the computed value is less than the 95th percentile of that distribution, then the model fit is judged adequate. The p-value, if also computed, measures the probability that a simulated NB random sample the same size as the data would produce an equally large chi-square statistic. The model is judged adequate on this basis if the p-value exceeds 0.05. Generally, the proper application of the chi-square test requires a minimum of five data points per interval.

A3. Derivation of the Negative Binomial Model

[23] Here we provide a constructive derivation of the negative binomial model for fracture numbers. We adapt an argument in *Hald* [1952], pp. 732–734. Let $P_x(l)$ denote the probability that there are exactly x fractures in the ith interval, which we define as $(0, l)$. Now we consider how this probability varies with x and l. In order that there are x fractures in the interval $(0, l + \Delta l)$, where Δl is an infinitesimal distance, we assume that there are either:

[24] 1. x fractures in $(0, l)$ and none in $(l, l + \Delta l)$; or

[25] 2. $x - 1$ fractures in $(0, l)$ and one in $(l, l + \Delta l)$.

[26] We assume that the probability of more than one fracture in $(l, l + \Delta l)$ is negligible, or more precisely, that this probability is $o(\Delta l)$. Let $p_x(l)$ denote the infinitesimal fracture intensity in the interval $(l, l + \Delta l)$ given that there are x fractures in the interval $(0, l)$. In other words, we assume

that the probability of having a fracture in $(l, l + \Delta l)$ is $p_x(l)\Delta l$. The probability of having x fractures in $(l, l + \Delta l)$ will then be

$$P_x(l + \Delta l) = P_x(l) \cdot (1 - p_x(l)\Delta l) + P_{x-1}(l) \cdot p_{x-1}(l)\Delta l + o(\Delta l), \quad \text{(A5)}$$

where $o(\Delta l)/\Delta l$ converges to zero as Δl converges to zero. This leads to the differential equations:

$$\frac{dP_x(l)}{dl} = \frac{P_x(l + \Delta l) - P_x(l)}{\Delta l} = P_{x-1}(l)p_{x-1}(l) - P_x(l)p_x(l) \quad \text{and:}$$
$$\frac{dP_0(l)}{dl} = -P_0(l)p_0(l) \qquad P_0(0) = 1 \qquad P_x(0) = 0 \quad \text{for} \quad x \geq 1 \quad \text{(A6)}$$

We assume that the infinitesimal fracture intensity

$$p_x(l) = \frac{r + x}{\beta + l} \quad \text{(A7)}$$

where r is the mean number of fractures in an interval of length β. Note that the infinitesimal intensity increases with the number x of fractures in the interval $(0, l)$ and decreases with the length of that interval. Next we define $\xi = r/\beta$ [fractures/m] the overall (or average) fracture intensity, normally named P_{10} in discrete fracture network modeling. Substituting and simplifying yields

$$p_x(l) = \frac{r + x}{r + \xi l}\xi \quad \text{(A8)}$$

Following *Hald* [1952], substituting the infinitesimal fracture intensity and solving the differential equation system gives:

$$P_x(l) = \left(\frac{r}{r + \xi l}\right)^r \left(\frac{\Gamma(x + r)}{\Gamma(r)x!}\right) \left(\frac{\xi l}{r + \xi l}\right)^x \quad \textit{for} \quad x = 0, 1, 2, 3, \ldots \quad \text{(A9)}$$

The mean or average number of fractures in the interval $(0, l)$ is ξl which agrees with the fact that ξ is the fracture intensity. The probabilities $P_x(l)$ form a NB-distribution (1) with parameters

$$t = r \qquad \text{and} \qquad p = \frac{r}{r + \xi l} \quad \text{(A10)}$$

The probabilities $P_x(l)$ were originally derived for the Pólya process, a (fundamentally different) Poisson process with a random, Gamma distributed rate parameter.

[27] **Acknowledgments.** The authors thank Swedish Nuclear Fuel and Waste Management Company (SKB) for making the borehole data available. Kozubowski was partially supported by NSF grant DMS-0139927. Meerschaert was partially supported by NSF grants DMS-0139927, DMS-0417869, DMS-0706440, and the Marsden Foundation in New Zealand. We also thank the Associate Editor and three anonymous referees for their comments.

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