Fractional Cauchy problems on bounded domains

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Abstract

Fractional Cauchy problems replace the usual first order time derivative by a fractional derivative. This paper develops classical solutions and stochastic analogues for fractional Cauchy problems in a bounded domain $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Stochastic solutions are constructed via an inverse stable subordinator whose scaling index corresponds to the order of the fractional time derivative. Dirichlet problems corresponding to iterated Brownian motion in a bounded domain are then solved by establishing a correspondence with the case of a half-derivative in time.

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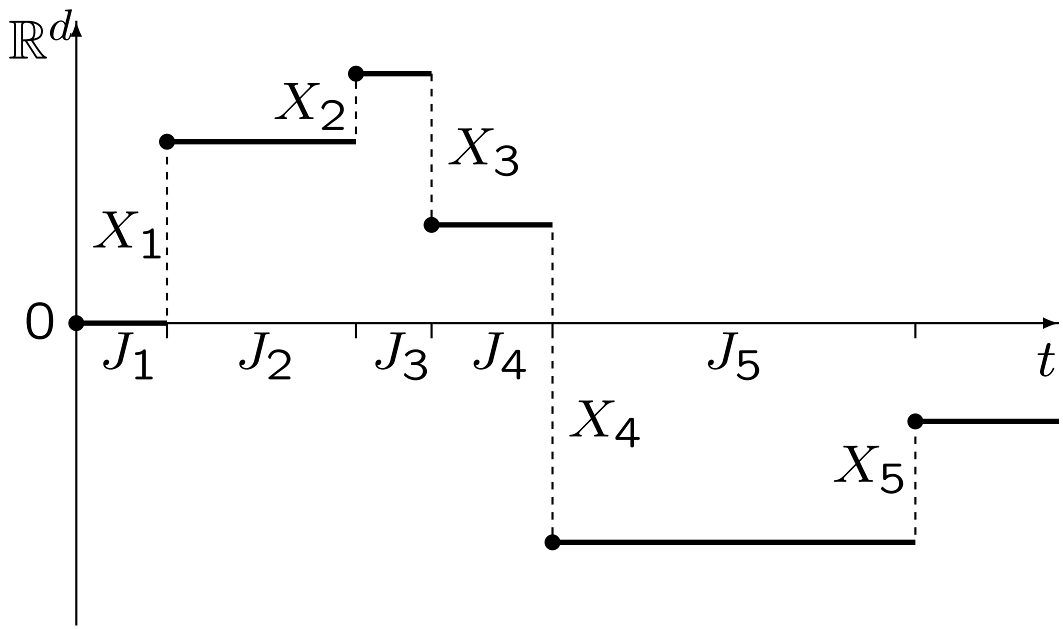
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Waiting time process

$T_n = J_1 + \cdots + J_n$ is time of nth jump.

$N_t = \max\{n \geq 0 : T_n \leq t\}$ is number of jumps by time $t > 0$.

Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$ (or RV tails)

Scaling limit $c^{-1/\beta} T_{[ct]} \Rightarrow D_t$ is a β-stable subordinator.

Inverse mapping yields $c^{-\beta} N_{ct} \Rightarrow E_t$ where the inverse or first passage time process $E_t = \inf\{u > 0 : D_u > t\}$.

The self-similar limit $E_{ct} \sim c^{\beta} E_t$ is non-Markovian.

CTRW scaling limit

$S(n) = X_1 + \cdots + X_n$ is particle location after n jumps.

$S(N_t)$ is particle location at time $t > 0$.

Suppose $c^{-1/2}S(ct) \Rightarrow B(t)$ a Brownian motion.

Then $c^{-\beta/2}S(N_{ct}) \Rightarrow A(E_t)$: Heuristically

$$\begin{aligned} c^{-\beta/2}S(N_{ct}) &= (c^\beta)^{-1/2}S(c^\beta \cdot c^{-\beta}N_{ct}) \\ &\approx (c^\beta)^{-1/2}S(c^\beta \cdot E_t) \\ &\Rightarrow B(E_t) \end{aligned}$$

Self-similar limit $B(E_{ct}) \sim c^{\beta/2}B(E_t)$ is non-Markovian.

Inverse subordinator density

Let $g_\beta(u)$ be the Lebesgue density of D_1 and write

$$\begin{aligned} P(E_t \le u) &= P(D_u \ge t) \\ &= P(u^{1/\beta} D_1 \ge t) \\ &= P(D_1 \ge t u^{-1/\beta}). \end{aligned}$$

Take derivatives to see that E_t has density

$$h(u,t) = \frac{t}{\beta} u^{-1-1/\beta} g_\beta(t u^{-1/\beta})$$

Fractional Cauchy problems

The density $p(x,t)$ of $B(t)$ solves the Cauchy problem

$$\frac{\partial}{\partial t}p(x,t) = \Delta p(x,t); \quad p(x,0) = \delta(x).$$

The process $B(E_t)$ has density (condition on $u = E_t$):

$$c(x,t) = \int_0^\infty p(x,u)h(u,t)\,du.$$

This density solves the fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta}c(x,t) = \Delta c(x,t); \quad c(x,0) = \delta(x)$$

where $\frac{\partial^\beta}{\partial t^\beta}$ denotes the Caputo fractional derivative.

Proof: Laplace transform arguments on $L^2(\mathbb{R}^d)$.

Caputo fractional derivatives

The Laplace transform $\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$.

Recall that $\frac{d}{dt} f(t)$ has Laplace transform $s\tilde{f}(s) - f(0)$.

The Caputo fractional derivative

$$\frac{\partial^\beta}{\partial t^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t f'(y) \frac{dy}{(t-y)^\beta}$$

has Laplace transform $s^\beta \tilde{f}(s) - s^{\beta-1} f(0)$.

Cauchy problems on bounded domains

Define the first exit time $\tau_D(X) = \inf\{t \geq 0 : \ X_t \notin D\}$. Then the Cauchy problem

$$\frac{\partial}{\partial t} p(x,t) = \Delta p(x,t); \quad x \in D, \ t > 0$$

with Dirichlet boundary conditions $p(x,0) = f(x)$ for $x \in D$ and $p(x,t) \equiv 0$ for $x \in \partial D$ has solution

$$p(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n u} \bar{f}(n) \phi_n(x) = E_x[f(B(t)) I(\tau_D(B) > t)]$$

where $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$ using the complete orthonormal Hilbert space basis of eigenfunctions $\Delta\phi_n = -\lambda_n \phi_n$.

Fractional Cauchy problems on bounded domains

Main result: The fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta} c(x,t) = \Delta c(x,t); \quad x \in D,\ t > 0$$

with Dirichlet boundary conditions has strong (classical) solution

$$c(x,t) = \int_0^\infty p(x,u) h(u,t)\, du$$
$$= E_x[f(B(E_t)) I(\tau_D(B) > E_t)]$$

when $p(x,t)$ solves the corresponding Cauchy problem.

Sketch of proof

Use Greens's second identity and the Dirichlet b.c. to write

$$\begin{aligned}\int_D \phi_n(x)\Delta c(x,t)dx &= \int_D c(x,t)\Delta\phi_n(x)dx \\ &= -\lambda_n \int_D c(x,t)\phi_n(x)dx \\ &= -\lambda_n \bar{c}(n,t)\end{aligned}$$

Apply to both sides of the fractional Cauchy problem to get

$$\frac{d^\beta}{dt^\beta}\bar{c}(n,t) = -\lambda_n \bar{c}(n,t)$$

Then take Laplace transforms to get

$$s^\beta \hat{c}(n,s) - s^{\beta-1}\bar{c}(n,0) = -\lambda_n \hat{c}(n,t)$$

Sketch of proof (page 2)

Rearrange to get

$$\hat{c}(n,s) = \frac{s^{\beta-1}\bar{c}(n,0)}{s^\beta + \lambda_n} = \frac{s^{\beta-1}\bar{f}(n)}{s^\beta + \lambda_n}.$$

Invert the Laplace transform to get

$$\bar{c}(n,t) = \bar{f}(n) E_\beta(-\lambda_n t)$$

where the Mittag-Leffler function

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\int_0^\infty e^{-\lambda u} h(u,t)\, du = E_\beta(-\lambda t).$$

Sketch of proof (page 3)

Expand in the complete orthonormal basis to get

$$
\begin{aligned}
c(x,t) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) \int_0^\infty e^{-\lambda_n u} h(u,t)\,du \\
&= \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\lambda_n u} \bar{f}(n)\phi_n(x) \right) h(u,t)\,du \\
&= \int_0^\infty p(x,u)h(u,t)\,du \\
&= \int_0^\infty E_x[f(B(u))I(\tau_D(B) > u)]\,h(u,t)\,du \\
&= E_x[f(B(E_t))I(\tau_D(B) > E_t)].
\end{aligned}
$$

Iterated Brownian motion

Iterated Brownian motion $I_t = B(|Y_t|)$, where Y_t is an independent Brownian motion on $\mathbb{R}^1$. The fourth order PDE

$$\frac{\partial}{\partial t} c(x,t) = \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 c(x,t), \quad x \in D, \ t > 0$$

with $c(x,t) = \Delta c(x,t) = 0$ on $x \in \partial D$ and $c(x,0) = f(x)$ has solution

$$\begin{aligned} c(x,t) &= 2 \int_0^\infty p(x,u) q(u,t)\, du \\ &= E_x[f(I_t) I(\tau_D(B) > |Y_t|)] \end{aligned}$$

where $p(x,t)$ solve the Cauchy problem $\partial p / \partial t = \Delta p$ on D and $q(x,t)$ is the density of Y_t.

IBM and fractional diffusion

The function $c(x,t)$ solves the fourth order PDE

$$\frac{\partial}{\partial t}c(x,t) = \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 c(x,t), \quad x \in D,\ t > 0$$

with $c(x,t) = \Delta c(x,t) = 0$ on $x \in \partial D$ and $c(x,0) = f(x)$ if and only if it solves the fractional Cauchy problem

$$\frac{\partial^{1/2}}{\partial t^{1/2}}c(x,t) = \Delta c(x,t); \quad x \in D,\ t > 0$$

with Dirichlet boundary conditions. Furthermore, $B(E_t) \sim B(|Y_t|)$.

Proof: Transform arguments, reflection principle.

Markov generators

The Markov process X_t with generator

$$Lu = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}$$

solves $dX_t = \sigma(X_t) dW_t + b(X_t) dt$ with $a = \sigma\sigma^T$. Then

$$p(x,t) = E_x[f(X_t) I(\tau_D(X) > t)]$$

solves the Cauchy problem

$$\frac{\partial}{\partial t} p(x,t) = Lp(x,t)$$

with Dirichlet boundary conditions.

Extension to Markov generators

The fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta} c(x,t) = Lc(x,t); \quad x \in D, \ t > 0$$

with Dirichlet boundary conditions has strong (classical) solution

$$\begin{aligned} c(x,t) &= \int_0^\infty p(x,u)h(u,t)\,du \\ &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \end{aligned}$$

when $p(x,t)$ solves the corresponding Cauchy problem.

Proof: Separation of variables, eigenfunctions as before.

Markov generators and IBM

The function $c(x,t)$ solves

$$\frac{\partial}{\partial t}c(x,t) = \frac{Lf(x)}{\sqrt{\pi t}} + L^2 c(x,t), \quad x \in D,\ t > 0$$

with $c(x,t) = Lc(x,t) = 0$ on $x \in \partial D$ and $c(x,0) = f(x)$ if and only if it solves the fractional Cauchy problem

$$\frac{\partial^{1/2}}{\partial t^{1/2}}c(x,t) = Lc(x,t); \quad x \in D,\ t > 0$$

with Dirichlet boundary conditions. Furthermore, $X(E_t) \sim X(|Y_t|)$.

Further research

- Extension to $\beta > 1$
- Jump diffusions
- Distributed order fractional derivatives
- Tempered stable waiting times
- Space-time duality

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