CTRW Model for Fractional Wave Equations

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Abstract

Fractional wave equations are important in many areas of science and engineering. Fujita (1990) proposed a stochastic solution to the time-fractional wave equation

$$\frac{\partial^\gamma}{\partial t^\gamma} p = \Delta_x p,$$

using the supremum of a negatively skewed stable process. The index $\alpha \in (1,2)$ of that stable process corresponds to a fractional time derivative of order $\gamma = 2/\alpha$. In this talk, we present a continuous time random walk model for that same fractional wave equation. In the long time limit, this model leads to a stochastic solution involving the inverse or hitting time of a stable subordinator with index $1/\alpha$. The subordinator in our model is the first passage time of the stable process in Fujita. It is also related to the solution in Mainardi (2010) via the Zolotarev duality formula for stable densities. The continuous time random walk model can be useful for particle tracking solutions to the fractional wave equation.

New Book

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Solution to the wave equation

The traditional wave equation

$$\frac{\partial^2}{\partial t^2} p(x,t) = \Delta_x p(x,t)$$

models wave propagation in an ideal conducting medium.

For initial conditions $p(x,0) = \phi(x)$ and $p'(x,0) = \psi(x)$, the traditional wave equation has the (d'Alembert) solution

$$p(x,t) = \frac{1}{2}\left[\phi(x+t) + \phi(x-t)\right] + \int_{x-t}^{x+t} \psi(y)\, dy.$$

This is proven by considering the equivalent integral equation

$$p(x,t) = \phi(x) + t\psi(x) + \int_0^t \Delta_x p(x,s)\, ds$$

for any continuous and exponentially bounded functions ϕ, ψ.

Fractional wave equation

The fractional wave equation (integral form) with $1 < \gamma < 2$ is

$$p(x,t) = \phi(x) + \frac{t^{\gamma/2}}{\Gamma(1+\gamma/2)}\psi(x) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\Delta_x p(x,s)\,ds.$$

For $\psi = 0$, the equivalent differential form is

$$\frac{\partial^\gamma}{\partial t^\gamma} p(x,t) = \Delta_x p(x,t)$$

using the Caputo fractional derivative

$$\frac{\partial^\gamma}{\partial t^\gamma} p(x,t) = \frac{1}{\Gamma(2-\gamma)}\int_0^t p''(x,u)(t-u)^{1-\gamma} du$$

of order $1 < \gamma < 2$.

This equation models wave conduction in a heterogeneous medium.

Fujita's solution

Take $X_\gamma(t)$ a negatively skewed stable with index $2/\gamma$ and

$$\mathbb{E}\left[e^{ikX_\gamma(t)}\right] = \exp\left[-t|k|^{2/\gamma}e^{-i(\pi/2)(2-2/\gamma)\,\mathrm{sgn}(k)}\right]$$

and define its supremum process

$$Y_\gamma(t) = \sup_{0\le u\le t} X_\gamma(u).$$

Fujita (1990) gives the fractional wave equation solution

$$p(x,t) = \frac{1}{2}\,\mathbb{E}\left[\phi(x+Y_\gamma(t)) + \phi(x-Y_\gamma(t))\right] + \frac{1}{2}\,\mathbb{E}\int_{x-Y_\gamma(t)}^{x+Y_\gamma(t)} \psi(y)\,dy.$$

for continuous, exponentially bounded initial conditions ϕ, ψ.

Solution via inverse stable process

Bertoin (1996) Theorem VII.1 implies that the first passage time

$$D_u = \text{inf}\{t \geq 0 : X_\gamma(t) > u\}$$

is a stable subordinator with index $\beta = \gamma/2$ and Laplace transform $\mathbb{E}\left[e^{-sD_u}\right] = e^{-us^\beta}$ for all $u \geq 0$ and $s \geq 0$.

But the inverse β-stable subordinator

$$E_t = \text{inf}\{u \geq 0 : D_u > t\}$$

equals the supremum $Y_\gamma(t)$ of $X_\gamma(t)$ (inverse of the inverse).

Then we can set $E_t = Y_\gamma(t)$ in the Fujita solution.

Meaning of the E_t solution

The integral form of the fractional wave equation with $\gamma = 2\beta$ is

$$p(x,t) = \phi(x) + \frac{t^\beta}{\Gamma(1+\beta)}\psi(x) + \frac{1}{\Gamma(2\beta)}\int_0^t (t-s)^{2\beta-1}\Delta_x p(x,s)\,ds.$$

The unique solution for continuous, exponentially bdd ϕ, ψ is

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+E_t) + \phi(x-E_t)\right] + \frac{1}{2}\mathbb{E}\int_{x-E_t}^{x+E_t}\psi(y)\,dy.$$

The time randomization $t \mapsto E_t$ accounts for delays in wave propagation in a heterogeneous medium.

Generally E_t grows like t^β, so it slows the wave propagation.

Mainardi solution

Mainardi (2010): Solution to the FWE with $p'(x,0)=0$ is

$$p(x,t) = \int_0^\infty \frac{1}{2}\left[\phi(x-u)+\phi(x+u)\right] q(u,t)\,du.$$

where $q(u,t)$ is the pdf of $X_\gamma(t)$. That is,

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x-X_\gamma(t))+\phi(x+X_\gamma(t))\Big|X_\gamma(t)>0\right].$$

Baeumer et al. (2009) prove that the conditional pdf of $X_\gamma(t)$ given $X_\gamma(t)>0$ equals the pdf of the inverse stable E_t.

Then the Mainardi solution is equivalent to the E_t solution.

Random walk model for wave equation

Let X_0 be a random variable with density $\phi(x)$.

Take X_1 independent of X_0 with $\mathbb{P}[X_1 = \pm 1] = 1/2$.

Set $X_n = X_1$ for $n > 1$, and let $S(t) = X_1 + \cdots + X_{[t]}$.

Then $X_0 + c^{-1}S(ct) \Rightarrow U_t$ as $c \to \infty$ and U_t has a pdf

$$p(x,t) = \frac{1}{2}\left[\phi(x+t) + \phi(x-t)\right]$$

that solves the traditional wave equation.

Particle tracking solution: simulate this random walk

CTRW model for fractional wave equation

Take $S(t) = X_1 + \cdots + X_{[t]}$ as before.

Take $\mathbb{P}[W_n > t] = t^{-\beta}/\Gamma(1-\beta)$ iid independent of X_0, X_1.

Let $T_n = W_1 + \cdots + W_n$ and $N_t = \max\{n \geq 0 : T_n \leq t\}$.

Then $X_0 + c^{-\beta} S(N_{ct}) \Rightarrow U_t$ as $c \to \infty$ and U_t has a pdf

$$p(x,t) = \frac{1}{2}\mathbb{E}\left[\phi(x+E_t) + \phi(x-E_t)\right]$$

that solves the fractional wave equation.

Particle tracking solution: simulate this CTRW

Extension: Bounded domains

Given an open subset D of $\mathbb{R}^d$, consider the Laplacian operator $L_x = \Delta_x$ on $L^2(D)$ with Dirichlet boundary conditions.

For any $\phi \in \text{Dom}(L_x)$ there exists a unique solution $p(x,t)$ to the wave equation

$$\frac{\partial^2}{\partial t^2} p(x,t) = \Delta_x p(x,t); \quad p(x,0) = \phi(x);\ p'(x,0) = 0;\ p(x,t) = 0\ \forall x \notin D.$$

Then we show that

$$p_\gamma(x,t) = \mathbb{E}[p(x,E_t)]$$

solves the corresponding fractional wave equation for $1 < \gamma < 2$

$$\frac{\partial^\gamma}{\partial t^\gamma} p(x,t) = \Delta_x p(x,t); \quad p(x,0) = \phi(x);\ p'(x,0) = 0;\ p(x,t) = 0\ \forall x \notin D$$

where E_t is the inverse stable subordinator with index $\beta = \gamma/2$.

Remark: Caputo or Riemann-Liouville?

The Riemann-Liouville fractional derivative of order $1 < \gamma < 2$ is

$$\mathbb{D}_t^\gamma p(x,t) = \frac{1}{\Gamma(2-\gamma)}\frac{d^2}{dt^2}\int_0^\infty p(x,u)(t-u)^{1-\gamma}\,du.$$

The Caputo and Riemann-Liouville derivatives are related by

$$\frac{\partial^\gamma}{\partial t^\gamma}f(t) = \mathbb{D}_t^\gamma f(t) - f(0)\frac{t^{-\gamma}}{\Gamma(1-\gamma)} - f'(0)\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}.$$

The fractional wave equation in Riemann-Liouville form is:

$$\mathbb{D}_t^\gamma p(x,t) = \phi(x)\frac{t^{-\gamma}}{\Gamma(1-\gamma)} + \psi(x)\frac{t^{-\gamma/2}}{\Gamma(1-\gamma/2)} + \Delta_x p(x,t).$$

Since we have $t^{-\gamma/2}$ and not $t^{1-\gamma}$, Caputo is not as useful.

Remark: Power law wave equation

The power law wave equation from Kelly et al. (2008)

$$\frac{\partial^2}{\partial t^2}p + \frac{2\alpha_0}{b}\mathbb{D}^{y+1}p + \frac{\alpha_0^{2y}}{b^2}\mathbb{D}^{2y}p = \Delta_x p$$

with $b = \cos(\pi y/2)$ and $1 < y < 2$ is a model for ultrasound.

Here the Riemann-Liouville fractional derivatives are used.

Again, the solution comes from replacing t by E_t in the solution to the traditional wave equation, and taking expectations.

Now E_t is the inverse of a stable with drift with index y.

The CTRW model is similar, see Straka et al. (2013).

Summary

- Stochastic solution for fractional wave equation

- Randomize time via inverse stable subordinator

- Leads to CTRW model for particle tracking

- Riemann-Liouville derivative seems most appropriate

- Extends to more general equations

- What about space-time fractional wave equations?

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