Tempered stable models for anomalous diffusion

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Abstract

The fractional diffusion equation replaces the usual first order time derivative, and second order spatial derivative, by their fractional order analogues. Stable processes with non-Markovian inverse stable subordinators are governed by the fractional diffusion equation, in the same way that Brownian motion is governed by the classical diffusion equation. In certain applications to hydrology and finance, plume evolution is intermediate between the classical and fractional case. For example, daily price returns are heavy tailed, but annual returns are essentially Gaussian. Then a subordinated process involving tempered stable laws is effective at capturing the real world dynamics. The governing equation involves tapered fractional derivatives. Applications to ground water pollution migration illustrate the practical utility of the results.

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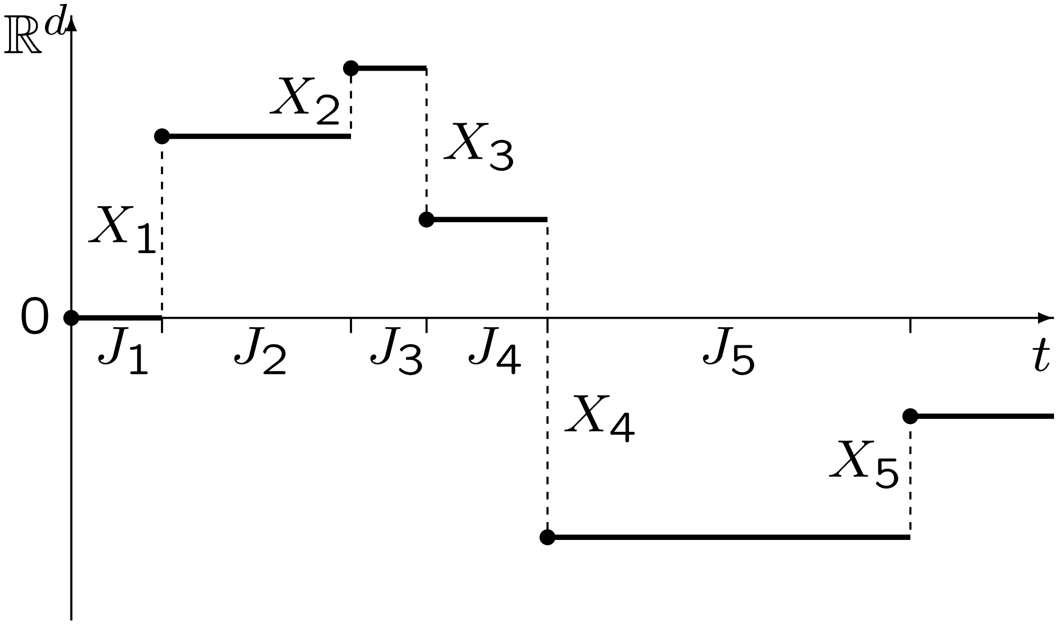
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

CTRW triangular arrays

Consider a sequence of CTRW at each scale $c > 0$

$S^{(c)}(n) = X_1^{(c)} + \cdots + X_n^{(c)}$ particle location after n jumps

$T^{(c)}(n) = J_1^{(c)} + \cdots + J_n^{(c)}$ time of the nth jump

$N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$ number of jumps by time $t > 0$

$S^{(c)}(N_t^{(c)})$ particle location at time $t > 0$ (CTRW)

Note $\{T^{(c)}(n) \leq t\} = \{N_t^{(c)} \geq n\}$ inverse processes

CTRW scaling limits

Assume $(S^{(c)}(cu), T^{(c)}(cu)) \Rightarrow (A(u), D(u))$ infinitely divisible

Write $\mathbb{E}(e^{-ik\cdot A(u)-sD(u)}) = e^{-u\psi(k,s)}$

$$\psi(k,s) = ia\cdot k + k\cdot Qk + \int \left(1 - e^{-ik\cdot x}e^{-st} - \frac{ik\cdot x}{1+\|x\|^2}\right) \phi(dx,dt)$$

$\mathbb{E}(e^{ik\cdot A(u)}) = e^{-u\psi_A(k)}$ with $\psi_A(k) = \psi(k,0)$
$\mathbb{E}(e^{-sD(u)}) = e^{-u\psi_D(s)}$ with $\psi_D(s) = \psi(0,s)$

Inverse mapping yields $N_{ct}^{(i)} \Rightarrow E(t)$
$E(t) = \inf\{u > 0 : D(u) > t\}$ inverse process.

CTRW scaling limit $S^{(c)}(N_t^{(c)}) \Rightarrow A(E(t))$

Semigroups and generators

The CTRW scaling limit defines a semigroup

$$T(u)f(x,t) = \int_0^t \int_{\mathbb{R}^d} f(x-y,t-r)\, P_{(A(u),D(u))}(dy,dr)$$

with generator

$$\begin{aligned}\psi(-iD_x,\partial_t)f(x,t) &= a\cdot\nabla f(x,t) - \nabla\cdot Q\nabla f(x,t)\\ &\quad - \int\left(f(x-y,t-u) - f(x,t) + \frac{\nabla f(x,t)\cdot y}{1+\|y\|^2}\right)\phi(dy,du)\end{aligned}$$

The pseudodifferential operator $\psi(-iD_x,\partial_t)$ has symbol $\psi(k,s)$

Inverse subordinators

Let $g(t,u)$ be Lebesgue density of $t = D(u)$

Assume $\phi_D(0,\infty) = \infty$ and $\int_0^1 y|\ln y|\,\phi_D(dy) < \infty$ (technical).

Theorem $E(t) = \inf\{u > 0 : D(u) > t\}$ has Lebesgue density

$$f(u,t) = \int_0^t \phi_D(t-y,\infty)g(y,u)\,dy.$$

Moreover, the mapping $(u,t) \mapsto f(u,t)$ is measurable.

Idea: $f(u,t) = \frac{d}{du}P(E(t) \le u) = \frac{d}{du}P(D(u) \ge t)$
Compute with Laplace tranforms

Space-time decomposition

Suppose A, D are independent $\psi(k,s) = \psi_A(k) + \psi_D(s)$

Suppose $x = A(u)$ has Lebesgue density $p(x,u)$

CTRW scaling limit $A(E(t))$ has density

$$m(x,t) = \int_0^\infty p(x,u) f(u,t)\, du$$

Governing equations:

$$\begin{aligned}\partial_u p(x,u) &= -\psi_A(-iD_x)p(x,u); \quad p(x,0) = \delta(x)\\ \partial_u f(u,t) &= -\psi_D(\partial_t)f(u,t) + \delta(u)\phi_D(t,\infty)\\ \psi_D(\partial_t)m(x,t) &= -\psi_A(-iD_x)m(x,t) + \delta(x)\phi_D(t,\infty)\end{aligned}$$

Fractional derivatives

The Fourier transform $\widehat{f}(k) = \int e^{-ikx} f(x)\, dx$ for $x \in \mathbb{R}$

$D_x^\alpha f(x)$ has Fourier transform $(ik)^\alpha \widehat{f}(k)$

In the simplest case $0 < \alpha < 1$

$$\int_0^\infty \left(e^{-iky} - 1\right) \alpha y^{-\alpha-1} dy = -\Gamma(1-\alpha)(ik)^\alpha$$

Since $f(x-y)$ has FT $e^{-iky}\widehat{f}(k)$ we see that

$$D_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y)\Big) y^{-\alpha-1} dy$$

The pseudodifferential operator D_x^α has Fourier symbol $(ik)^\alpha$

Space-time fractional diffusion

A. Piryatinska, A.I. Saichev, and W.A. Woyczynski (2005):

Suppose $\psi_A(k) = (ik)^\alpha$ stable jump limit $0 < \alpha < 1$

$\psi_D(s) = s^\beta$ stable waiting time limit $0 < \beta < 1$

Then $\psi_A(-iD_x) = D_x^\alpha$ and $\psi_D(\partial_t) = \partial_t^\beta$ so

$$\partial_u p(x,u) = -D_x^\alpha p(x,u); \quad p(x,0) = \delta(x)$$

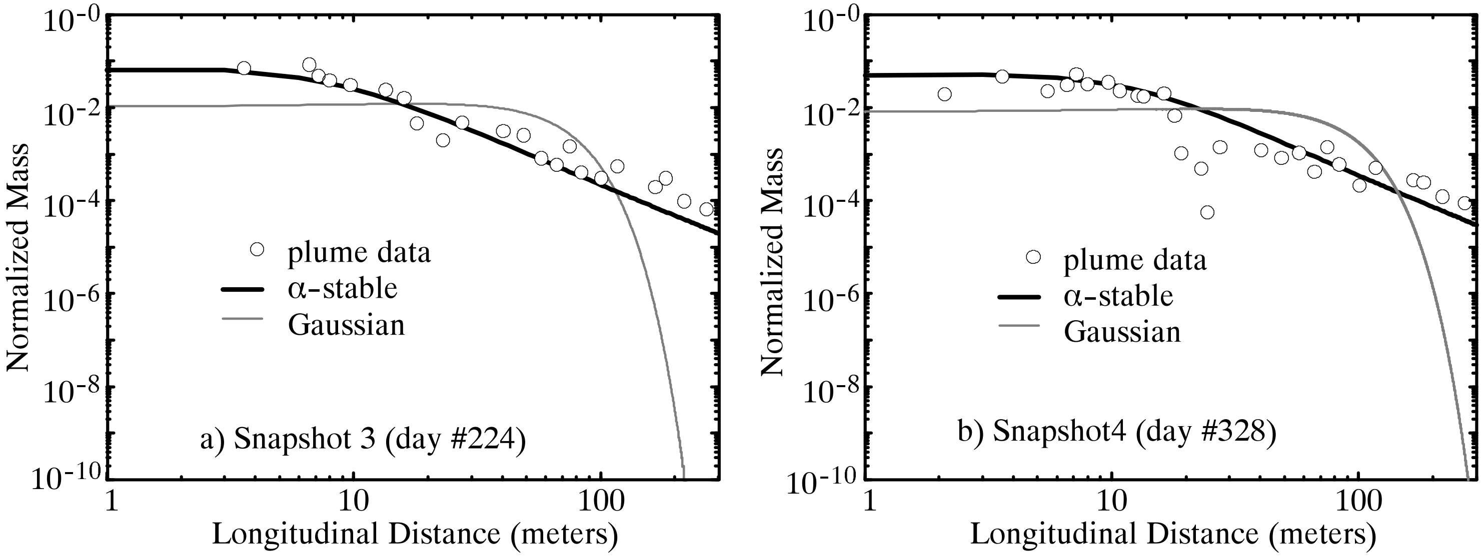
$$\partial_u f(u,t) = -\partial_t^\beta f(u,t) + \delta(u)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

$$\partial_t^\beta m(x,t) = -D_x^\alpha m(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

Coupled case: $\psi(-iD_x, \partial_t)m(x,t) = \delta(x)\phi_D(t,\infty)$

Tracer test in an underground aquifer

Space-fractional diffusion model captures early arrivals at the MADE experimental site.

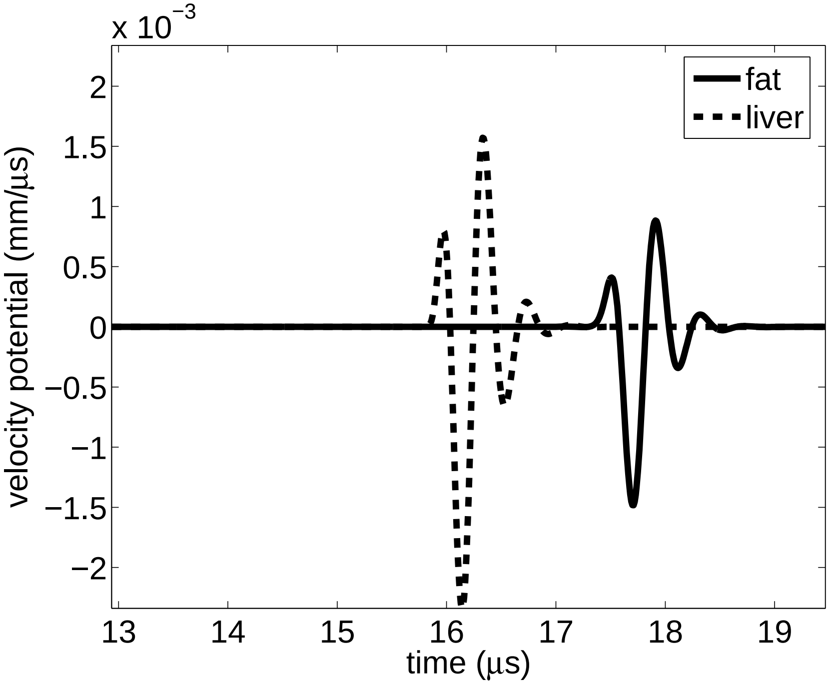


CTRW model has power-law jumps in the direction of flow

Sound wave propogation

We use $\beta = 2.5$ for human fat tissue and $\beta = 2.1$ for liver tissue.

$$\frac{\partial^2}{\partial t^2}c(t,x) + C\frac{\partial^\beta}{\partial t^\beta}c(t,x) = D\frac{\partial^2}{\partial x^2}c(t,x)$$



Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

Tempered anomalous diffusion

Terdik and Woyczynski (2006): Tempered stable laws

Lévy measure is modified to $e^{-\lambda x}\phi_A(dx)$

Results in tapered density $e^{-\lambda x}p(x,t)$

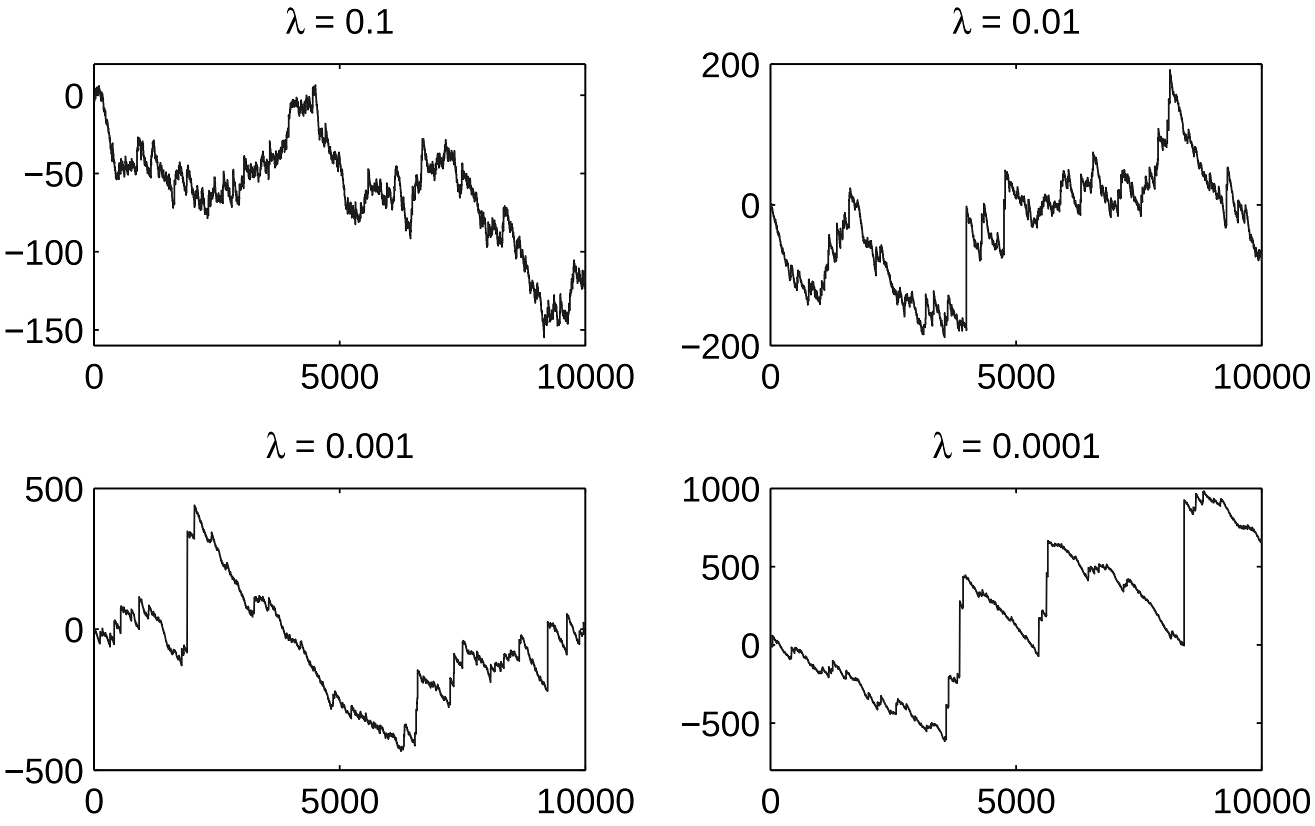
Scaling limits transition to Brownian motion at late time

$D_x^\alpha p$ becomes $D_x^{\alpha,\lambda}p = e^{-\lambda x}D_x^\alpha(e^{\lambda x}p) - \lambda^\alpha p$

$(ik)^\alpha$ becomes $(\lambda - ik)^\alpha - \lambda^\alpha$

Tempered Lévy motion

Tempered stable Lévy motion with $\alpha = 1.2$



Numerical methods

Variable coefficient fractional PDEs require numerical solution.

The Grünwald finite difference approximation

$$D_x^\alpha f(x) = \lim_{h\to 0} \frac{1}{h} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(x-kh)$$

comes from the generator formula with a discrete Lévy measure.

Convergence proof uses Fourier methods:
Lévy representation, convergence criteria infinitely divisible laws

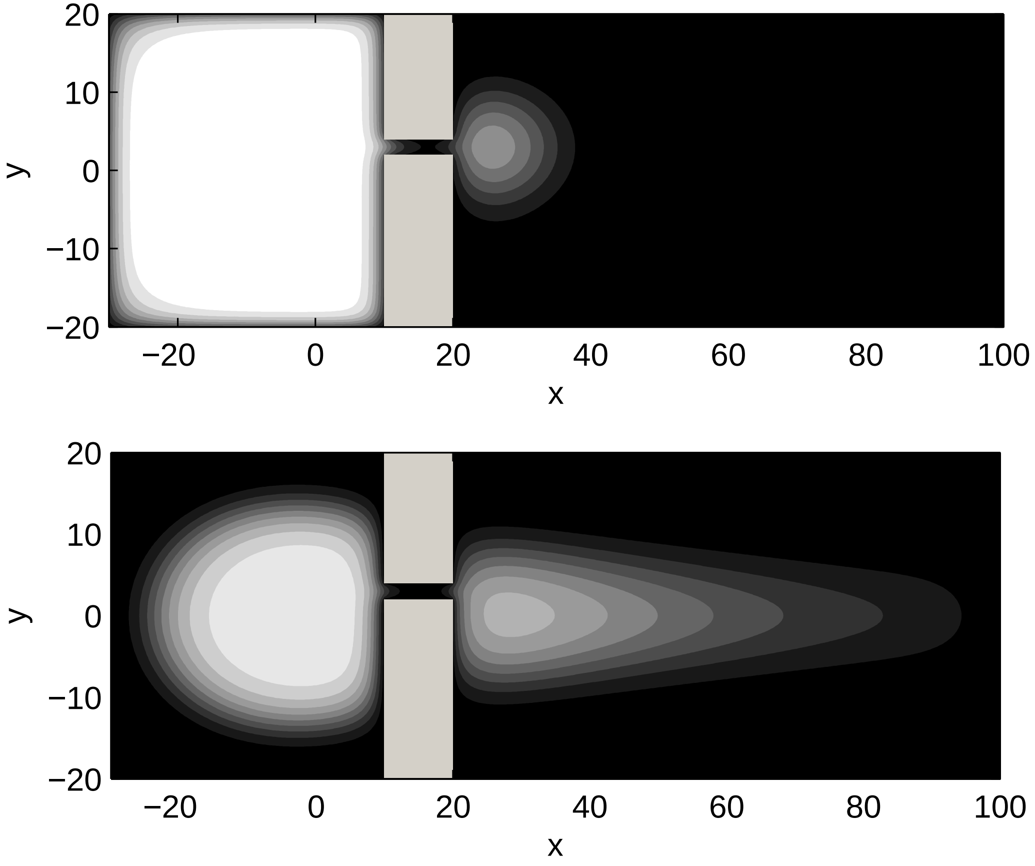
Proof of $O(h)$ convergence does not extend to bounded domains.
A new approach is required.

2-D numerical solution

Reaction-diffusion equation for species growth and movement.

$$\frac{\partial P}{\partial t} = C\frac{\partial^{\alpha} P}{\partial x^{\alpha}} + D\frac{\partial^2 P}{\partial y^2} + rP\left(1 - \frac{P}{K}\right)$$

Compare $\alpha = 2$ (top) to $\alpha = 1.7$ (bottom).



Numerical methods for tempered stable diffusion

Grünwald finite difference approximation

$$D_x^{\alpha,\lambda} f(x) \approx \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \Big(w_j e^{-(j-p)h\lambda} f(x-(j-p)h) \Big) - e^{ph\lambda} \frac{\left(1-e^{-h\lambda}\right)^\alpha}{h^\alpha} f(x)$$

where $w_j = \dfrac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = (-1)^j \binom{\alpha}{j}$ provides a convergent $O(h)$ estimate.

A stable and consistent 2nd order method:

Δt	Δx	Max Error	Error rate
1/10	1/50	2.8514×10^{-6}	–
1/20	1/100	7.2120×10^{-7}	3.95
1/40	1/200	1.8157×10^{-7}	3.97
1/80	1/400	4.5555×10^{-8}	3.99

Numerical solutions by particle tracking

Compute adjoint to get backward equation and generator

Explicitly identify the underlying Markov process

Simulate many particles to estimate the transition density.

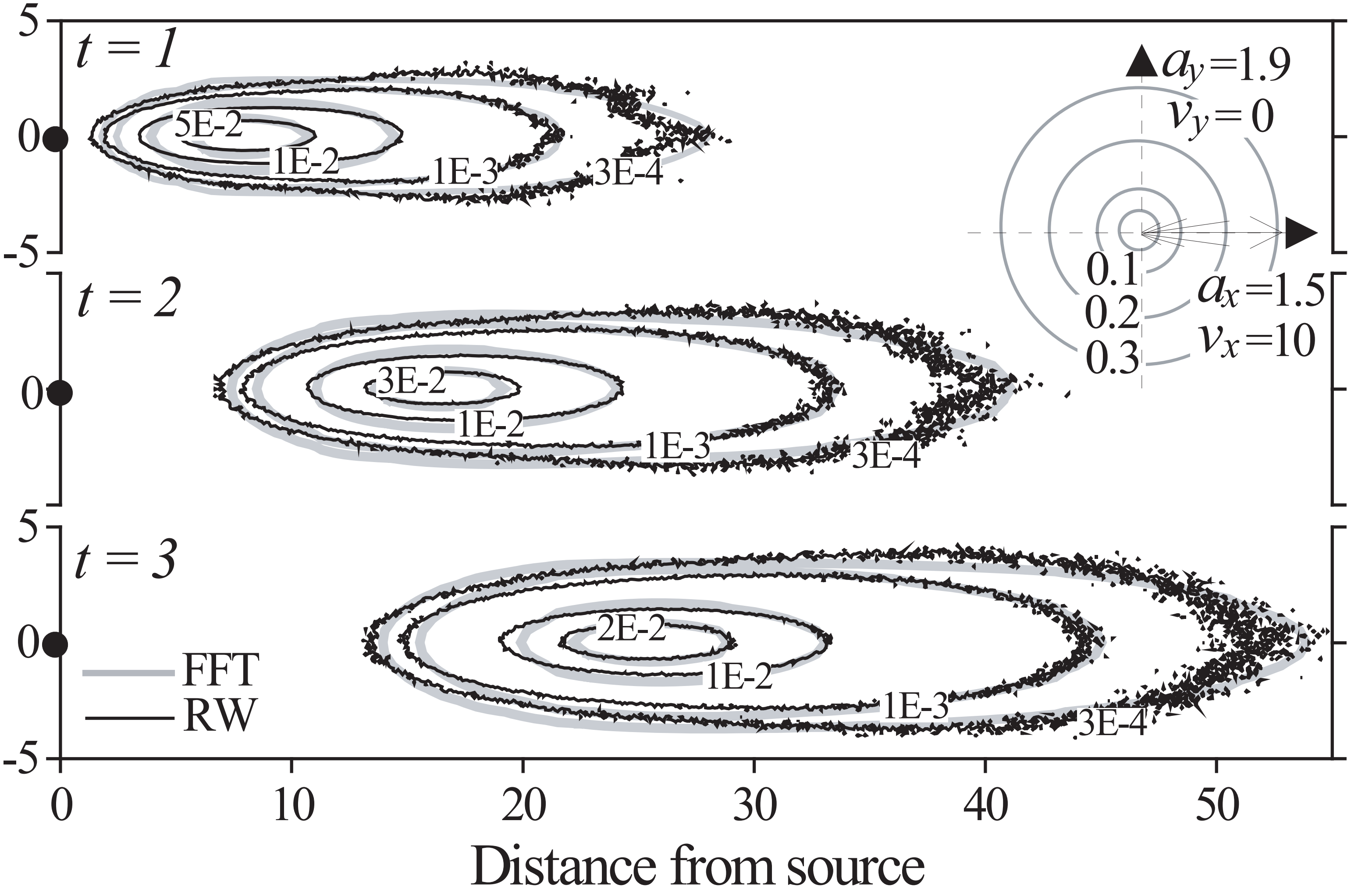
CTRW scaling limit $x = A(u)$ Markov but $u = E(t)$ is not:
Simulate Markov process $t = D(u)$ instead

Particle trajectories are (x_n, t_n) where $x_i = A(u_i)$, $t_i = D(u_i)$.

Theory recently established for space-fractional diffusion

2-D particle tracking solutions

Here $\partial_t p = -v \cdot \nabla p + a\partial_x^{1.5} p + b\partial_y^{1.9} p$



Simulating tempered stable laws

Simulation codes for stable random variates are widely available.

If $X > 0$ has stable density density $f(x)$, TS density is

$$f_\lambda(x) = \frac{e^{-\lambda x} f(x)}{\int_0^\infty e^{-\lambda u} f(u)\, du}$$

Take $Y \sim \exp(\lambda)$ independent of X, (X_i, Y_i) IID with (X, Y)

Let $N = \min\{n : X_n \leq Y_n\}$. Then $X_N \sim f_\lambda(x)$

Proof: Compute $P(X_N \leq x) = P(X \leq x | X \leq Y)$ by conditioning, then take d/dx to verify.

Fractal Burger's equation

Biler, Karch, and Woyczynski (2001): $x \in \mathbb{R}^d$ and $0 < \alpha < 2$

$-(-\Delta)^{\alpha/2}$ fractional Laplacian with symbol $\psi_A(k) = -\|k\|^\alpha$

Cauchy problem

$$\partial_t p = b \cdot \nabla(p|p|^{(\alpha-1)/d}) - (-\Delta)^{\alpha/2} p$$

has unique positive self-similar solution

$$p(x,t) = x^{-d/\alpha} p(xt^{-1/\alpha}, 1)$$

Open problem: Asymmetric/anisotropic extensions

Iterated Brownian motion

IBM $A_{|B_t|}$ models diffusion in a crack. Essentially, the sample path A_t models the (fractal) crack. The density $c(x,t)$ of IBM is the Green's function solution to

$$\frac{\partial c(x,t)}{\partial t} = \frac{L\,f(x)}{\sqrt{\pi t}} + L^2 c(x,t); \quad u(0,x) = f(x)$$

where $L = \Delta = \sum_j \partial^2/\partial x_j^2$ is the generator of the semigroup associated with the Brownian motion $A(t)$.

Taking $\beta = 1/2$ in the time-fractional diffusion equation yields exactly the same 1-D distributions $A_{|B_t|} \overset{d}{=} A_{E_t}$.

Proof: Laplace-Fourier transforms, local times

Fractional Cauchy problems on bounded domains

For $0 < \alpha < 1$ and $T_D(t)f(x) = E_x[f(X_t)I(t < \tau_D(X))]$, under some technical conditions

$$\begin{aligned} u(t,x) &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\ &= \frac{t}{\beta}\int_0^\infty T_D(l)f(x)g_\beta(tl^{-1/\beta})l^{-1/\beta-1}dl \end{aligned}$$

is the unique (classical) solution to

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D,\ t > 0 \\ u(t,x) &= 0,\ x \in \partial D,\ t > 0, \\ u(0,x) &= f(x),\ x \in D. \end{aligned}$$

PROOF: Eigenfunction expansion, Mittag-Leffler solutions

Some open problems

- Useful and tractable tempering schemes
- Triangular array models with TS limits
- Fitting concentration data, error bars
- Fractional boundary value problems
- Applications – interdisciplinary research

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