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Fractional normal inverse Gaussian diffusion

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ABSTRACT

A fractional normal inverse Gaussian (FNIG) process is a fractional Brownian motion subordinated to an inverse Gaussian process. This paper shows how the FNIG process emerges naturally as the limit of a random walk with correlated jumps separated by i.i.d. waiting times. Similarly, we show that the NIG process, a Brownian motion subordinated to an inverse Gaussian process, is the limit of a random walk with uncorrelated jumps separated by i.i.d. waiting times. The FNIG process is also derived as the limit of a fractional ARIMA processes. Finally, the NIG densities are shown to solve the relativistic diffusion equation from statistical physics.

1. Introduction

The normal inverse Gaussian (NIG) process was introduced by Barndorff-Nielsen (1997) to model financial data. The NIG density exhibits "semi-heavy" tails typically seen in finance and geophysics, but as a Lévy process, it has independent increments. Kumar and Vellaisamy (2009) developed the fractional normal inverse Gaussian (FNIG) process, as a simple alternative to the NIG process with correlated increments. In this paper, we show how the NIG process emerges as the scaling limit of a random walk with i.i.d. jumps separated by i.i.d. waiting times. Then we show that, if the jumps are correlated, the FNIG limit can emerge, under a fairly general set of conditions. To connect these continuous time processes with time series models often used in finance, we also demonstrate how the FNIG process represents the continuum limit of a fractional ARIMA process. Finally, we apply fractional calculus theory to show that the probability densities of the NIG process solve a relativistic diffusion equation from statistical physics.

The inverse Gaussian (IG) process is defined by

$$G(t) = \inf\{s > 0;\, B(s) + \gamma s = \delta t\}, \tag{1.1}$$

where $B(t)$ is a standard Brownian motion, $\delta > 0$ and $\gamma > 0$ (e.g., see Applebaum (2009)). Since $B(s) + \gamma s$ has continuous sample paths, it follows from the strong Markov property that $G(t)$ is a subordinator, i.e., a non-decreasing Lévy process. Note also that $G(t) \sim \mathrm{IG}(\delta t, \gamma)$, the IG distribution (with parameters δt and γ) having density

$$h(x, t) = (2\pi)^{-1/2}(\delta t)x^{-3/2}e^{\delta\gamma t - \frac{1}{2}(\delta^2 t^2 x^{-1} + \gamma^2 x)}, \quad x > 0. \tag{1.2}$$

The NIG process is defined by

$$N(t) = B(G(t)) + \beta G(t) + \mu t, \tag{1.3}$$

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where $G(t)$ is an IG process and $B(t)$ is an independent Brownian motion. The NIG distribution tails are heavier than Gaussian, but lighter than power law, the so-called "semi-heavy" tails frequently used in finance (see Barndorff-Nielsen (1998)). Since the IG subordinator is a Lévy process, the NIG is also a Lévy process, with independent increments. However, many applications to finance require dependent increments (e.g., see Shephard (1995)). Semi-heavy tails also occur in geophysics. For example, Molz and Bowman (1993) show that increments in hydraulic conductivity in a borehole exhibit semi-heavy tails and long-range dependence.

To provide a simple alternative to the NIG process, but with correlated increments, Kumar and Vellaisamy (2009) proposed FNIG process defined by

$$X(t) = B_H(G(t)), \tag{1.4}$$

where $G(t)$ is an IG process and $B_H(t)$ is an independent fractional Brownian motion (FBM) with Hurst parameter $0 < H < 1$. Recall that $B_H(t)$ is a zero mean Gaussian process with covariance function

$$E[B_H(t)B_H(s)] = \frac{\sigma^2}{2}[t^{2H} + s^{2H} - |t-s|^{2H}] \tag{1.5}$$

for $t, s \geq 0$ (e.g., see Mandelbrot and Van Ness (1968)).

In this paper, we will show that the NIG process is the scaling limit of a random walk with i.i.d. jumps having zero mean and finite variance, separated by i.i.d. waiting times with positive mean (since the waiting times are positive) and finite variance. This justifies the NIG model as a late-time approximation for tick-by-tick financial data (see Scalas (2004) and the references therein for more applications of CTRWs), in the special case where price returns are i.i.d. Then we extend this model to allow correlations between jumps, leading to the FNIG scaling limit. We also show how the FNIG process can emerge as the limit of a fractional ARIMA time series. Finally, we use tools of fractional calculus to prove that the densities of the NIG process solve an anomalous diffusion equation that has a connection with statistical physics.

2. FNIG process as a random walk limit

Continuous time random walks (CTRW) were introduced as a model in statistical physics by Montroll and Weiss (1965) and Scher and Lax (1973). Given i.i.d. random variables Y_n representing the random jumps of a particle, the simple random walk $S_n = \sum_{i=1}^n Y_i$ gives the particle position after n steps. Impose i.i.d. waiting times $X_n \geq 0$ between particle jumps, so that another random walk $T_n = \sum_{i=1}^n X_i$ gives the time of the nth jump. The renewal process

$$N_t = \max\{n \geq 0 : T_n \leq t\} \tag{2.1}$$

counts the number of jumps by time t, and the CTRW (also called the renewal reward process)

$$S_{N_t} = \sum_{i=1}^{N_t} Y_i, \tag{2.2}$$

gives the particle position at time t. The CTRW is commonly used to model a wide variety of phenomena connected with anomalous diffusion (Metzler and Klafter, 2000, 2004). The densities of CTRW scaling limits solve fractional diffusion equations. Heavy tail jumps lead to superdiffusion, described by fractional derivatives in space, and heavy tail waiting times code anomalous subdiffusion, modeled by fractional time derivatives (Barkai et al. (2000); Meerschaert and Scheffler (2004)).

Given $EX_n = \mu > 0$, $\mathrm{Var}(X_n) = \sigma^2 < \infty$ and for any timescale $c > 0$, we define the rescaled random walk

$$T_n^{(c)} = \sum_{i=1}^n \left(\frac{1}{\sqrt{c}}(X_i - \mu) + \frac{\mu}{c} \right), \tag{2.3}$$

where $n \geq 1$. Let $\mathbf{D} = \mathbf{D}([0,\infty), \mathbb{R})$ denote the space of right continuous $\mathbb{R}$-valued functions on $[0,\infty)$ with left limits, endowed with the Skorokhod J_1 topology (see e.g., Billingsley (1968)). It follows easily from Donsker's Theorem (e.g., see Whitt (2002), Theorem 4.3.2) and continuous mapping that

$$T_{[ct]}^{(c)} \Rightarrow \sigma B(t) + \mu t \tag{2.4}$$

in this space. Note that the mapping $g : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ defined by $g(x, y) = x + y$ is continuous if at least one of x or y are continuous (e.g., see Jacod and Shiryaev (2002), Proposition 1.23). In our case, both components of the limit have continuous sample paths a.s. It is well known that Donsker's Theorem alone is insufficient to obtain Brownian motion with drift in the limit, since two scales are needed. The mean has to be rescaled by a linear factor, and the deviation from the mean follows square root scaling (e.g., see Ross (2003), Exercise 10.8).

Since the renewal process N_t in (2.1) is the process inverse of the random walk T_n of jump times, the CTRW scaling limit will depend on the first passage time or process inverse of $T_n^{(c)}$. Because of the two-scale setup in (2.3), it is possible that

$T_{n+1}^{(c)} \le T_n^{(c)}$. Since the first passage time of a stochastic process and that of the corresponding supremum process are same, it is convenient to consider the maximum (supremum) process of $T_n^{(c)}$, as in Becker-Kern et al. (2004). Let

$$L_n^{(c)} = \sup\{T_j^{(c)} : 0 \le j \le n\}.$$

Then

$$F_t^{(c)} = \min\{n \ge 0 : L_n^{(c)} \ge t\} \tag{2.5}$$

is the first passage time of the process $L_n^{(c)}$. Obviously $L_n^{(c)} \le L_{n+1}^{(c)}$, so $F_t^{(c)}$ has non-decreasing sample paths. Let $D(t) = \sigma B(t) + \mu t$. Although $D(t)$ is not monotone, $D(t) \to \infty$ a.s. as $t \to \infty$ by the strong law of large numbers for Lévy processes (e.g., see Sato (1999), Theorem 36.5). Hence, the supremum process $M(t) = \sup\{D(u) : 0 \le u \le t\}$ and the first passage time $G(t) = \inf\{x \ge 0 : M(x) \ge t\}$ are well defined. Note that almost all sample paths of $G(t)$ are strictly increasing, with jumps, since the sample paths of $M(t)$ are continuous and non-decreasing, with intervals of constancy. Clearly the hitting time of $D(t)$ and $M(t)$ are the same, i.e., we also have $G(t) = \inf\{x \ge 0 : D(x) \ge t\}$. From Lemma 13.6.3 of Whitt (2002), we have

$$\{G(t) \le x\} = \{M(x) \ge t\} \quad \text{and} \quad \{F_t^{(c)} \le x\} = \{L_{[x]}^{(c)} \ge t\}. \tag{2.6}$$

The next result establishes the scaling limit for the inner process in the FNIG model.

Proposition 2.1. *As $c \to \infty$, we have*

$$c^{-1} F_t^{(c)} \Rightarrow G(t) \tag{2.7}$$

in the J_1 topology on $\mathbf{D}$*, where $G(t)$ is IG.*

Proof. It follows from (2.4) along with Theorem 13.4.1 in Whitt (2002) that

$$L_{[ct]}^{(c)} = \sup_{0 \le s \le t} T_{[cs]}^{(c)} \Rightarrow \sup_{0 \le s \le t} D(s) = M(t) \tag{2.8}$$

in the space $\mathbf{D}$. Then

$$\{L_{[ct]}^{(c)}, t \ge 0\} \stackrel{fdd}{\Longrightarrow} \{M(t), t \ge 0\}, \tag{2.9}$$

as $c \to \infty$, where $\stackrel{fdd}{\Longrightarrow}$ means convergence of all finite dimensional distributions. Here we get convergence of all finite dimensional distributions because $M(t)$ has continuous sample paths, while in general it follows only for almost all t (see Proposition 3.14, Jacod and Shiryaev (2002), p. 349). Let $0 < t_1 < \cdots < t_m$ and $x_1, x_2, \ldots, x_m \ge 0$. Then, as $c \to \infty$,

$$\begin{aligned} P\{c^{-1} F_{t_i}^{(c)} \le x_i, i = 1, \ldots, m\} &= P\{F_{t_i}^{(c)} \le c x_i,\ i = 1, \ldots, m\} \\ &= P\{L_{[cx_i]}^{(c)} \ge t_i, i = 1, \ldots, m\} \quad \text{(by (2.6))} \\ &\to P\{M(x_i) \ge t_i, i = 1, \ldots, m\} \quad \text{(by (2.9))} \\ &= P\{G(t_i) \le x_i, i = 1, \ldots, m\}, \end{aligned}$$

using (2.6) again. Note that sample paths of $F_t^{(c)}$ are monotone, and $G(t)$ is a Lévy process, and hence continuous in probability. Then (2.7) follows by virtue of Theorem 3 in Bingham (1971). Since $D(t)$ is a Brownian motion with drift, its inverse process $G(t)$ is IG. □

Next we prove that the NIG process is the scaling limit of a CTRW. One special case was proven by Pacheco-Gonzalez (2009) using Bernoulli jumps. Here, we establish the NIG process limit for an arbitrary CTRW with finite second moments and mean zero jumps. This shows that the NIG provides a very flexible model for late-time behavior of tick-by-tick data with independent price returns.

Theorem 2.1. *Given $T_n^{(c)}$ as in (2.3), take Y_n i.i.d. independent of $\{X_n\}$ with $E(Y_i) = \beta$ and $\mathrm{Var}(Y_i) = 1$. Then the rescaled CTRW*

$$S_{F_t^{(c)}}^{(c)} + \sum_{i=1}^{[ct]} \frac{\rho}{c} \Rightarrow W(G(t)) + \rho t, \quad \rho \in \mathbb{R}, \tag{2.10}$$

in $\mathbf{D}$*, where $W(t) = B(t) + \beta t$ is a Brownian motion with drift, and $G(t)$ is IG, so that the limit $W(G(t))$ is NIG.*

Proof. By the same argument as (2.4), we have $S^{(c)}_{[ct]} \Rightarrow W(t)$ in $\mathbf{D}$. Proposition 2.1 shows that $c^{-1}F^{(c)}_t \Rightarrow G(t)$ in $\mathbf{D}$, where $G(t)$ is IG. Note that $W(t)$ has continuous sample paths, and that $G(t)$ is a Lévy subordinator. Since the sequences $\{X_n\}$ and $\{Y_n\}$ are independent, $(S^{(c)}_{[ct]}, c^{-1}F^{(c)}_t) \Rightarrow (B(t) + \beta t, G(t))$ in $\mathbf{D} \times \mathbf{D}$. Now the continuous mapping theorem, along with Theorem 13.2.2 of Whitt (2002), yields $S^{(c)}_{F^{(c)}_t} \Rightarrow W(G(t))$ in the J_1 topology. Theorem 13.2.2 of Whitt (2002) is applicable here since the sample paths of $W(t)$ are continuous. Also, $\sum_{i=1}^{[ct]} \rho/c \Rightarrow \rho t$ in the J_1 topology. The result now follows using Proposition 1.23 of Jacod and Shiryaev (2002) and the continuous mapping theorem. □

Theorem 2.1 shows that the NIG process provides a universal model for the long-time behavior of tick-by-tick financial data, in the case where the price returns are uncorrelated. More generally, any random walk with mean zero and finite variance jumps, separated by finite variance waiting times, can be approximated by the NIG process in the continuum limit. However, most financial data exhibits strongly correlated price jumps. Therefore, we now extend Theorem 2.1 to allow correlated jumps. This leads to a subordinated process limit in which the outer process exhibits long-range dependence. Some general results on correlated CTRW are contained in Meerschaert et al. (2009). However, they did not consider the case of finite variance waiting times presented here.

A stationary linear process is defined by

$$Y_n = \sum_{j=0}^{\infty} c_j Z_{n-j}, \tag{2.11}$$

where Z_n are i.i.d. and c_j are real constants such that $\sum_{j=0}^{\infty} c_j^2 < \infty$. When $\sum_{j=0}^{\infty} |c_j| = \infty$ we say that the sequence $\{Y_n\}$ exhibits long-range dependence (LRD). In this case, fractional Brownian motion (FBM) can emerge as the scaling limit of the correlated random walk $S_n = \sum_{i=1}^{n} Y_i$ (Davydov (1970); Whitt (2002)). The next result shows that a CTRW with LRD jumps converges to the FNIG in the scaling limit. Thus, the FNIG process provides a universal model for tick-by-tick financial with LRD price returns.

Theorem 2.2. *Let Y_n be a linear process with mean zero and finite variance, independent of the i.i.d. waiting times X_n. Suppose the variance σ_n^2 of the sum $S_n = \sum_{j=1}^{n} Y_j$ varies regularly at ∞ with index $2H$ for some $0 < H < 1$, and*

$$E(S_n^{2\rho}) \le K[E(S_n)^2]^{\rho} \tag{2.12}$$

for some constants $K > 0$ and $\rho > 1/H$. Then

$$\sigma_{[c]}^{-1} S_{F_t^{(c)}} \Rightarrow B_H(G(t)) \tag{2.13}$$

as $c \to \infty$ in $\mathbf{D}$, and the limit is an FNIG process.

Proof. Proposition 2.1 yields $c^{-1}F_t^{(c)} \Rightarrow G(t)$, and Theorem 4.6.1 in Whitt (2002) implies $\sigma_{[c]}^{-1}S_{[ct]} \Rightarrow B_H(t)$. Then $(\sigma_{[c]}^{-1}S_{[ct]},\ c^{-1}F_t^{(c)}) \Rightarrow (B_H(t),\ G(t))$ in $\mathbf{D} \times \mathbf{D}$, since the underlying sequences $\{X_n\}$ and $\{Z_n\}$ are independent. Then (2.13) follows using Theorem 13.2.2 of Whitt (2002) along with the continuous mapping theorem, using the fact that $B_H(t)$ has a modification with continuous sample paths (Embrechts and Maejima (2002), Theorem 4.1.1). Since G(t) is IG, the limit in (2.13) is an FNIG process. □

To provide a concrete example of the general situation in Theorem 2.2, we now explicitly construct a sequence $\{Y_n\}$ in (2.11) that satisfies those requirements. Let $B(a, b)$ denote the beta function.

Corollary 2.1. *Suppose $H \in (\frac{1}{2}, 1)$, $E|Z_n|^4 < \infty$, and $c_j = j^{-\gamma}$, where $\gamma = (\frac{3}{2} - H)$. Then*

$$[c]^{-H} S_{F_t^{(c)}} \Rightarrow \sqrt{A_H} B_H(G(t)),$$

in the J_1 topology, where A_H is a positive constant defined by

$$A_H = 8H^2 B\left(H - \frac{1}{2}, 2 - 2H\right). \tag{2.14}$$

Proof. Lemma 4 in Davydov (1970) implies that, if $E|Z_j|^{2k} < \infty$ for some integer k, then

$$E|S_n|^{2k} \le A(\mathrm{Var}\, S_n)^k \tag{2.15}$$

for some $A > 0$. For $c_j = j^{-\gamma}$, we have from Eq. (6.11) of Whitt (2002, p. 124) that

$$\mathrm{Var}(S_n) \sim Cn^{3-2\gamma}, \quad \text{as } n \to \infty \tag{2.16}$$

where $C = 2B(1-\gamma, 2\gamma-1)\,(3-2\gamma)^2$. For $\gamma = \frac{3}{2} - H$, (2.16) becomes $\mathrm{Var}(S_n) \sim A_H n^{2H}$, as $n \to \infty$. Also, when $E|Z_j|^4 < \infty$, (2.15) implies $E|S_n|^4 \le K(\mathrm{Var}(S_n))^2$, and hence (2.12) holds with $\rho = 2$, since $\frac{1}{2} < H < 1$. □

Next we show that the FNIG also emerges as the continuum limit of a time series with LRD. An ARIMA $(0, d, 0)$ process, the discrete-time analogue of fractional Gaussian noise, is defined by

$$(1-B)^d Y_n = Z_n, \tag{2.17}$$

where B is the backward shift operator, and $\{Z_n\}$ is a white noise (uncorrelated sequence) with mean 0 and finite variance (e.g., see Brockwell and Davis (2002)). The fractional difference operator

$$(1-B)^d = \sum_{j=0}^{\infty} \frac{\Gamma(d+1)}{\Gamma(j+1)\Gamma(d-j+1)}(-B)^j \tag{2.18}$$

so that Y_n is a stationary linear process.

Proposition 2.2. *Let $\{Y_n\}$ be a zero mean ARIMA $(0, d, 0)$ process (2.17) with $-\frac{1}{2} < d < \frac{1}{2}$ $(d \neq 0)$ and $E|Z_j|^{2k} < \infty$ for some integer $k > [1/H]$, where $H = d + \frac{1}{2}$. Let $S_n = \sum_{j=1}^{n} Y_j$ denote the correlated random walk with these jumps. Then*

$$[c]^{-H} S_{F_t^{(c)}} \Rightarrow \sqrt{b_H} B_H(G(t)), \tag{2.19}$$

*in **D** as $c \to \infty$, and the limit is an FNIG process.*

Proof. Since $E|Z_j|^{2k} < \infty$ with $k > [1/H]$, it follows from Lemma 4 in Davydov (1970), that (2.12) is satisfied. Hosking (1981) shows that

$$\gamma_k = \mathrm{Cov}(Y_n, Y_{n+k}) = (-1)^k \frac{\Gamma(1-2d)}{\Gamma(1+k-d)\Gamma(1-k-d)}, \tag{2.20}$$

and then it follows using Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and Stirling's approximation that $\gamma_k \sim a_H k^{2H-2}$, where $d = H - \frac{1}{2}$ and

$$a_H = \frac{\Gamma(2-2H)\sin\pi\left(H-\frac{1}{2}\right)}{\pi}, \tag{2.21}$$

so that $a_H < 0$ when $0 < H < 1/2$ (negative dependence) and $a_H > 0$ when $1/2 < H < 1$ (LRD). Then Lemma 4.6.1 of Whitt (2002) yields

$$\mathrm{Var}(S_n) \sim b_H n^{2H}, \tag{2.22}$$

where $b_H = |\frac{a_H}{H(1-2H)}|$, so that $\mathrm{Var}(S_n)$ is regularly varying at infinity with index $2H$. The result now follows from Theorem 2.2. □

3. NIG diffusion

The NIG densities solve anomalous diffusion equations that can be useful to describe the long-time behavior of CTRW models for tick-by-tick financial data. The governing equation for the NIG has also found applications in relativistic physics. The simplest explication of these ideas relies on transform theory. Let $\hat{h}(u, t) = \int_{\mathbb{R}} e^{-iux} h(x, t)\,dx$ denote the Fourier transform (FT) of a function h and $\bar{h}(u, s) = \int_0^\infty e^{-st}\hat{h}(u, t)\,dt$ its Fourier–Laplace transform (FLT). Our first result gives the governing equation for the IG.

Theorem 3.1. *The densities $h(x, t)$ of the IG process $G(t)$ in (1.1) solve*

$$\frac{\partial^2 h}{\partial t^2} - 2\delta\gamma\frac{\partial h}{\partial t} = 2\delta^2\frac{\partial h}{\partial x}. \tag{3.23}$$

Proof. The density (1.2) of $G(t)$ has FT

$$\hat{h}(u, t) = e^{-\delta t(\sqrt{\gamma^2 + 2iu} - \gamma)}$$

(e.g., see Applebaum (2009), p. 54). Then certainly

$$\frac{\partial \hat{h}}{\partial t} = -\delta(\sqrt{\gamma^2 + 2iu} - \gamma)\hat{h} \tag{3.24}$$

with $\hat{h}(u, 0) \equiv 1$. Recall that, if $F(s)$ is the LT of $f(t)$, then $sF(s) - f(0)$ is the LT of $f'(t)$. Now apply the Laplace transform (LT) to both sides of (3.24) to get

$$s\bar{h}(u, s) - 1 = -\delta(\sqrt{\gamma^2 + 2iu} - \gamma)\bar{h}(u, s).$$

Solve for $\bar{h}$ and manipulate to obtain

$$\begin{aligned}\bar{h}(u,s) &= \frac{1}{s+\delta(\sqrt{\gamma^2+2\mathrm{i}u}-\gamma)}\\ &= \frac{1}{(s-\delta\gamma)+\delta\sqrt{\gamma^2+2\mathrm{i}u}}\cdot\frac{(s-\delta\gamma)-\delta\sqrt{\gamma^2+2\mathrm{i}u}}{(s-\delta\gamma)-\delta\sqrt{\gamma^2+2\mathrm{i}u}}\\ &= \frac{(s-\delta\gamma)-\delta\sqrt{\gamma^2+2\mathrm{i}u}}{s^2-2s\delta\gamma+\delta^2\gamma^2-\delta^2(\gamma^2+2\mathrm{i}u)}.\end{aligned}$$

Rearrange to get

$$s^2\bar{h}(u,s)-s\hat{h}(u,0)+\delta(\sqrt{\gamma^2+2\mathrm{i}u}-\gamma)\hat{h}(u,0)-2\delta\gamma(s\bar{h}(u,s)-\hat{h}(u,0))=2\delta^2(\mathrm{i}u)\bar{h}(u,s)$$

and use (3.24) to reduce to

$$s^2\bar{h}(u,s)-s\hat{h}(u,0)-\frac{\partial\hat{h}}{\partial t}(u,0)-2\delta\gamma(s\bar{h}(u,s)-\hat{h}(u,0))=2\delta^2(\mathrm{i}u)\bar{h}(u,s). \tag{3.25}$$

Use the fact that $s^2F(s)-sf(0)-f'(0)$ is the LT of $f''(t)$, and invert the LT in (3.25) to get

$$\frac{\partial^2\hat{h}}{\partial t^2}-2\delta\gamma\frac{\partial\hat{h}}{\partial t}=2\delta^2(\mathrm{i}u)\hat{h}(u,t) \tag{3.26}$$

and finally invert the FT to arrive at (3.23), since $(\mathrm{i}u)\hat{h}$ is the FT of $\partial h/\partial x$. □

The IG governing equation (3.23) is closely related to the governing equation of the Brownian motion with drift in definition (1.1). Note that $G(t)$ is the first passage time or process inverse of the Brownian motion with drift $\delta^{-1}B(t)+\delta^{-1}\gamma t$, whose densities $g(x,t)$ solve the diffusion equation

$$\frac{\partial g}{\partial t}=-\frac{\gamma}{\delta}\frac{\partial g}{\partial x}+\frac{1}{2\delta^2}\frac{\partial^2 g}{\partial x^2}.$$

Exchange the roles of t and x to arrive back at (3.23). Next we derive the FNIG governing equation (the NIG is a special case).

Theorem 3.2. *The densities $m(x,t)$ of the NIG process $X(t)$ in (1.3) with $\beta=\mu=0$ solve*

$$\frac{\partial^2 m}{\partial t^2}+2\gamma\delta\frac{\partial m}{\partial t}=\sigma^2\delta^2\frac{\partial^2 m}{\partial x^2}. \tag{3.27}$$

Proof. Since the inner and outer process in (1.3) are independent, the density function of $B(G(t))$ is

$$m(x,t)=\int_0^\infty f(x,r)h(r,t)\,\mathrm{d}r \tag{3.28}$$

where $f(x,t)$ is the density of the BM $B(t)$, and $h(r,t)$ is the density of the IG subordinator $G(t)$. Since the FT $\hat{f}(u,t)=\exp(-\frac{\sigma^2}{2}tu^2)$, we have

$$\frac{\partial\hat{f}}{\partial t}=\frac{\sigma^2}{2}(\mathrm{i}u)^2\hat{f} \tag{3.29}$$

and hence

$$\frac{\partial f}{\partial t}=\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} \tag{3.30}$$

is the governing equation for BM, the usual difusion equation. Now write

$$\begin{aligned}\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}m(x,t) &= \int_0^\infty\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}f(x,r)h(r,t)\mathrm{d}r\\ &= \int_0^\infty\frac{\partial}{\partial r}f(x,r)h(r,t)\mathrm{d}r\\ &= -\int_0^\infty f(x,r)\frac{\partial}{\partial r}h(r,t)\mathrm{d}r\end{aligned}$$

$$= \int_0^\infty f(x,r)\left(-\frac{1}{2\delta^2}\frac{\partial^2}{\partial t^2} + \frac{\gamma}{\delta}\frac{\partial}{\partial t}\right) h(r,t)\mathrm{d}r$$
$$= \left(-\frac{1}{2\delta^2}\frac{\partial^2}{\partial t^2} + \frac{\gamma}{\delta}\frac{\partial}{\partial t}\right)\int_0^\infty f(x,r)h(r,t)\mathrm{d}r$$

which is equivalent to (3.27). □

The NIG governing equation (3.27) can be considered as a variation on the usual diffusion equation, in which the second time derivative term codes deviations from the mean waiting time. In this sense, the IG subordinator represents a second order correction, just as the central limit theorem is a second order correction to the law of large numbers.

The connection between the NIG and relativistic diffusion was discussed in Baeumer et al. (2010). Taking FLT in (3.27) and manipulating as in the proof of Theorem 3.1 shows that the NIG density has FT

$$\hat{m}(u,t) = \mathrm{e}^{t\delta(\gamma - \sqrt{\gamma^2+\sigma^2u^2})}$$

which solves

$$\frac{\partial}{\partial t}\hat{m}(u,t) = \delta(\gamma - \sqrt{\gamma^2+\sigma^2u^2})\,\hat{m}(u,t)$$

with boundary conditions $\hat{m}(u,0) = 1$ and $\hat{m}(u,\infty) = 0$. Inverting the FT reveals the relativistic diffusion equation

$$\frac{\partial}{\partial t}m(x,t) = \delta\left(\gamma - \sqrt{\gamma^2 - \sigma^2\frac{\partial^2}{\partial x^2}}\right)m(x,t) \tag{3.31}$$

which involves the fractional power of the shifted second derivative operator. The relativistic diffusion equation is derived from the Schroedinger wave equation by analytic continuation, using the equation $E^2 = p^2c^2 + m^2c^4$ for the relativistic total energy. Eq. (3.31) provides an alternative governing equation for the NIG that uses fractional calculus.

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