Fractional space-time pseudo-differential operators and applications

Mark M. Meerschaert
Department of Statistics and Probability
Michigan State University

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Abstract: Continuous time random walks impose a random waiting time between jumps. The CTRW scaling limit is a Lévy process subordinated to a non-Markovian process, defined as the first passage time of a subordinator. The transition densities of the scaling limit solve a space-time pseudo-differential equation. In the case of power-law jumps and waiting times, we get a space-time fractional equation for anomalous diffusion. Exponentially tapered waiting times and/or jumps lead to transient anomalous diffusion, whose plumes gradually change back to the classical Gaussian shape. Particle tracking solutions to the variable coefficient equations require explicit construction of the underlying Markov process.

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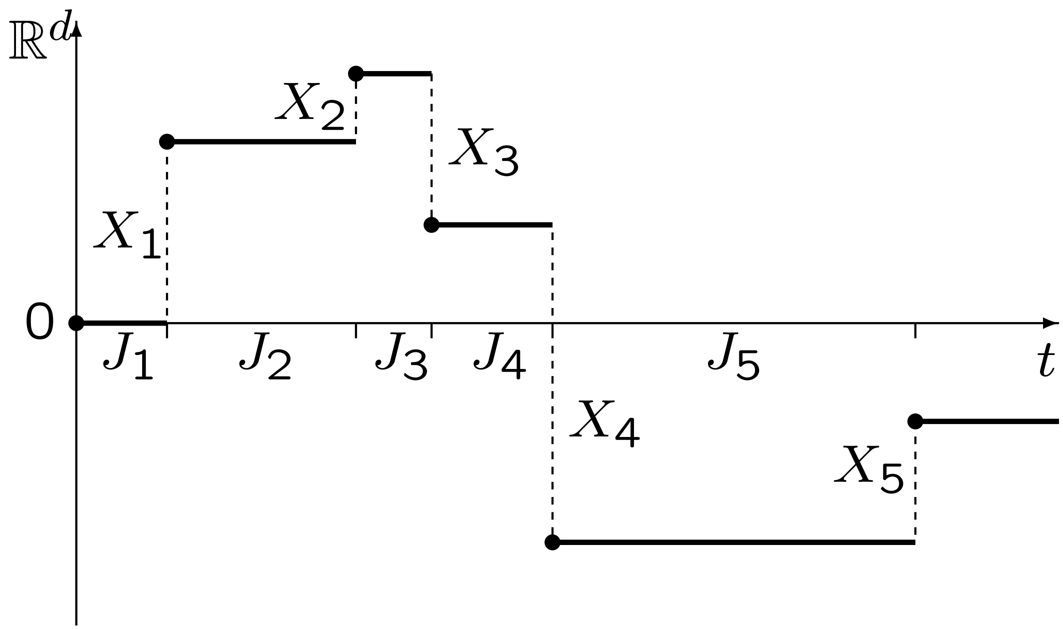
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

CTRW triangular arrays

Consider a sequence of CTRW at each scale $c > 0$

$S^{(c)}(n) = X_1^{(c)} + \cdots + X_n^{(c)}$ particle location after n jumps

$T^{(c)}(n) = J_1^{(c)} + \cdots + J_n^{(c)}$ time of the nth jump

$N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$ number of jumps by time $t > 0$

$S^{(c)}(N_t^{(c)})$ particle location at time $t > 0$ (CTRW)

Note $\{T^{(c)}(n) \leq t\} = \{N_t^{(c)} \geq n\}$ inverse processes

CTRW scaling limits

Assume $(S^{(c)}(cu), T^{(c)}(cu)) \Rightarrow (A(u), D(u))$ infinitely divisible

Write $\mathbb{E}(e^{-ik\cdot A(u)-sD(u)}) = e^{-u\psi(k,s)}$

$$\psi(k,s) = ia\cdot k + k\cdot Qk + \int \left(1 - e^{-ik\cdot x}e^{-st} - \frac{ik\cdot x}{1+\|x\|^2}\right)\phi(dx,dt)$$

$\mathbb{E}(e^{ik\cdot A(u)}) = e^{-u\psi_A(k)}$ with $\psi_A(k) = \psi(k,0)$
$\mathbb{E}(e^{-sD(u)}) = e^{-u\psi_D(s)}$ with $\psi_D(s) = \psi(0,s)$

Inverse mapping yields $N_{ct}^{(i)} \Rightarrow E(t)$
$E(t) = \inf\{u > 0 : D(u) > t\}$ inverse process.

CTRW scaling limit $S^{(c)}(N_t^{(c)}) \Rightarrow A(E(t))$

Semigroups and generators

The CTRW scaling limit defines a semigroup

$$T(u)f(x,t) = \int_0^t \int_{\mathbb{R}^d} f(x-y, t-r)\, P_{(A(u),D(u))}(dy, dr)$$

with generator

$$\psi(-iD_x, \partial_t) f(x,t) = a \cdot \nabla f(x,t) - \nabla \cdot Q \nabla f(x,t)$$
$$- \int \left(f(x-y, t-u) - f(x,t) + \frac{\nabla f(x,t) \cdot y}{1 + \|y\|^2} \right) \phi(dy, du)$$

The pseudoDO $\psi(-iD_x, \partial_t)$ has symbol $\psi(k, s)$

Inverse subordinators

Let $g(t,u)$ be Lebesgue density of $t = D(u)$

Assume $\phi_D(0,\infty) = \infty$ and $\int_0^1 y|\ln y|\,\phi_D(dy) < \infty$ (technical).

Theorem $E(t) = \inf\{u > 0 : D(u) > t\}$ has Lebesgue density

$$f(u,t) = \int_0^t \phi_D(t-y,\infty) g(y,u)\,dy.$$

Moreover, the mapping $(u,t) \mapsto f(u,t)$ is measurable.

Idea: $f(u,t) = \frac{d}{du} P(E(t) \le u) = \frac{d}{du} P(D(u) \ge t)$
Compute with Laplace tranforms

Space-time decomposition

Suppose A, D are independent $\psi(k,s) = \psi_A(k) + \psi_D(s)$

Suppose $x = A(u)$ has Lebesgue density $p(x,u)$

CTRW scaling limit $A(E(t))$ has density

$$m(x,t) = \int_0^\infty p(x,u) f(u,t)\, du$$

Governing equations:

$$\partial_u p(x,u) = -\psi_A(-iD_x)p(x,u); \quad p(x,0) = \delta(x)$$
$$\partial_u f(u,t) = -\psi_D(\partial_t)f(u,t) + \delta(u)\phi_D(t,\infty)$$
$$\psi_D(\partial_t)m(x,t) = -\psi_A(-iD_x)m(x,t) + \delta(x)\phi_D(t,\infty)$$

Fractional derivatives

The Fourier transform $\hat{f}(k) = \int e^{-ikx} f(x)\, dx$ for $x \in \mathbb{R}$

$D_x^\alpha f(x)$ has Fourier transform $(ik)^\alpha \hat{f}(k)$

In the simplest case $0 < \alpha < 1$

$$\int_0^\infty \left(e^{-iky} - 1\right) \alpha y^{-\alpha-1} dy = -\Gamma(1-\alpha)(ik)^\alpha$$

Since $f(x-y)$ has FT $e^{-iky}\hat{f}(k)$ we see that

$$D_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y)\Big) y^{-\alpha-1} dy$$

The pseudo-DO D_x^α has Fourier symbol $(ik)^\alpha$

Space-time fractional diffusion

Suppose $\psi_A(k) = (ik)^\alpha$ stable jump limit $0 < \alpha < 1$

$\psi_D(s) = s^\beta$ stable waiting time limit $0 < \beta < 1$

Then $\psi_A(-iD_x) = D_x^\alpha$ and $\psi_D(\partial_t) = \partial_t^\beta$ so

$$\partial_u p(x,u) = -D_x^\alpha p(x,u); \quad p(x,0) = \delta(x)$$

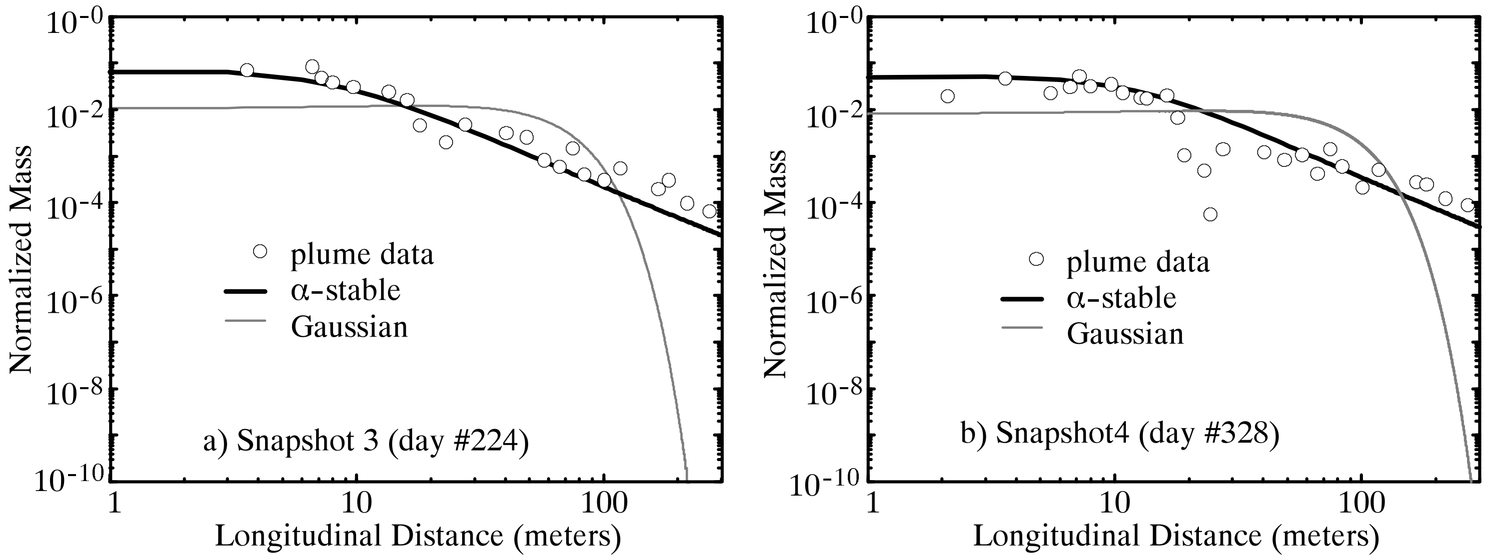
$$\partial_u f(u,t) = -\partial_t^\beta f(u,t) + \delta(u)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

$$\partial_t^\beta m(x,t) = -D_x^\alpha m(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

Coupled case: $\psi(-iD_x, \partial_t)m(x,t) = \delta(x)\phi_D(t,\infty)$

Tracer test in an underground aquifer

Space-fractional diffusion model captures early arrivals at the MADE experimental site.

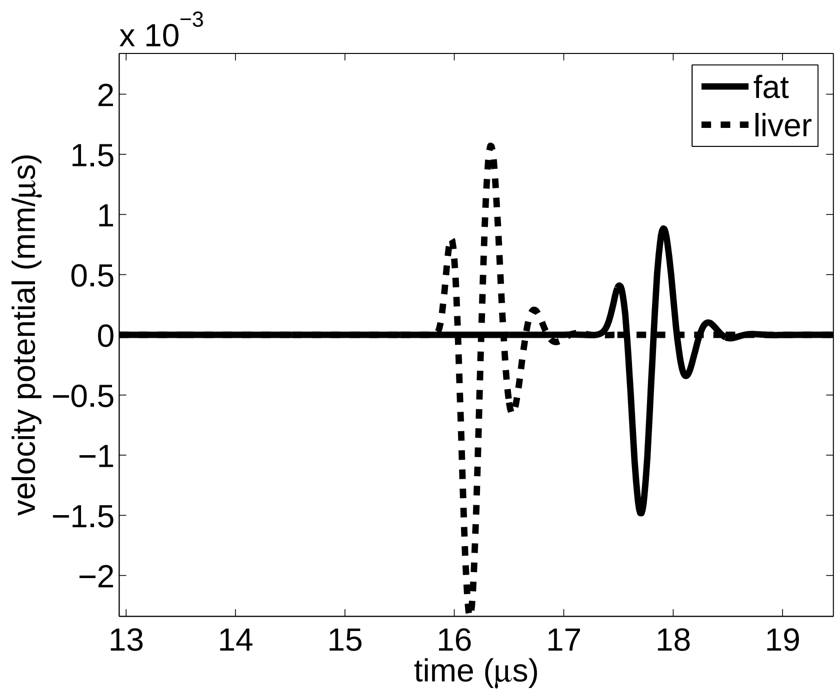


CTRW model has power-law jumps in the direction of flow

Sound wave propogation

We use $\beta = 2.5$ for human fat tissue and $\beta = 2.1$ for liver tissue.

$$\frac{\partial^2}{\partial t^2} c(t,x) + C \frac{\partial^\beta}{\partial t^\beta} c(t,x) = D \frac{\partial^2}{\partial x^2} c(t,x)$$



Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

Transient anomalous diffusion

Exponentially tapered Lévy measure $e^{-\lambda x}\phi_A(dx)$

Results in tapered density $e^{-\lambda x}p(x,t)$

CTRW scaling limits transition to Brownian motion at late time

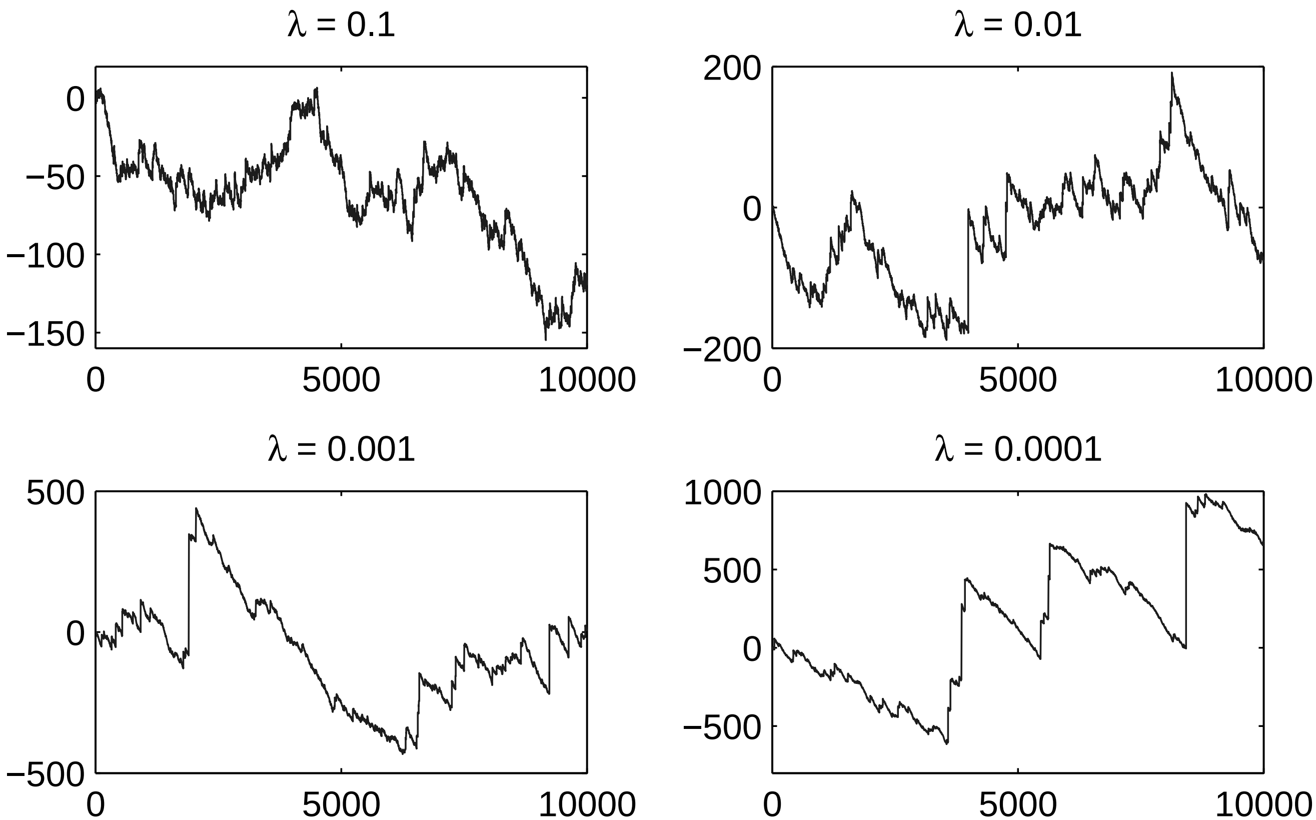
$D_x^\alpha p$ becomes $e^{-\lambda x}D_x^\alpha(e^{\lambda x}p)$

Time fractional derivative modification is similar.

Tempered model gives a superior fit in many cases.

Tempered Lévy motion

Tempered stable Lévy motion with $\alpha = 1.2$



Numerical methods

Variable coefficient fractional PDEs require numerical solution.

The Grünwald finite difference approximation

$$D_x^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(x - kh)$$

comes from the generator formula with a discrete Lévy measure.

Convergence proof uses Fourier methods:
Lévy representation, convergence criteria infinitely divisible laws

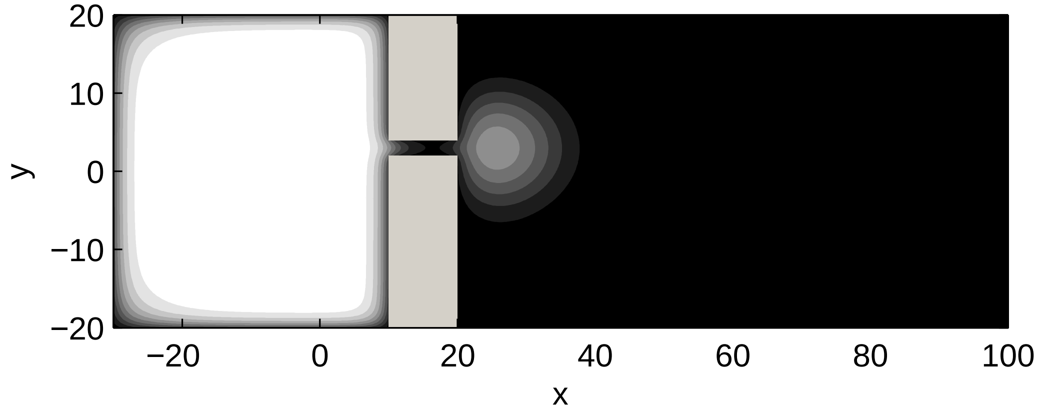
Proof of $O(h)$ convergence does not extend to bounded domains.
A new approach is required.

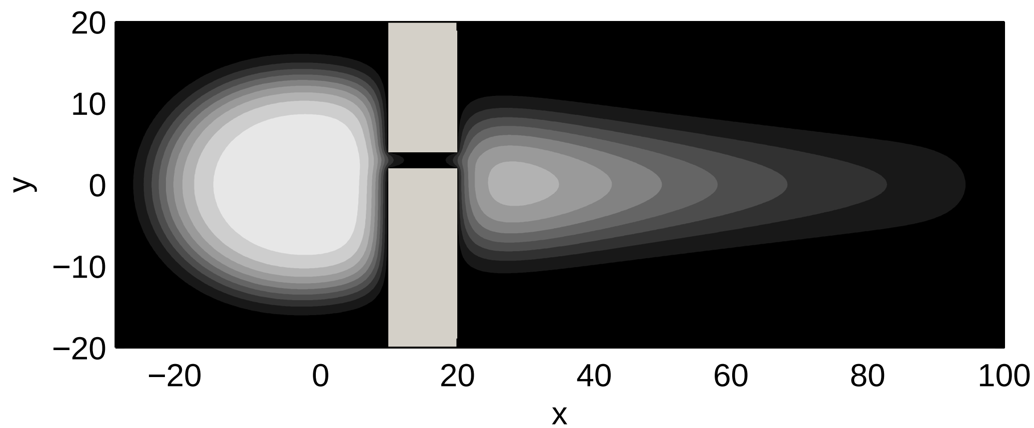
2-D numerical solution

Reaction-diffusion equation for species growth and movement.

$$\frac{\partial P}{\partial t} = C\frac{\partial^\alpha P}{\partial x^\alpha} + D\frac{\partial^2 P}{\partial y^2} + rP\left(1 - \frac{P}{K}\right)$$

Compare $\alpha = 2$ (top) to $\alpha = 1.7$ (bottom).





Numerical solutions by particle tracking

Compute adjoint to get backward equation and generator

Explicitly identify the underlying Markov process

Simulate many particles to estimate the transition density.

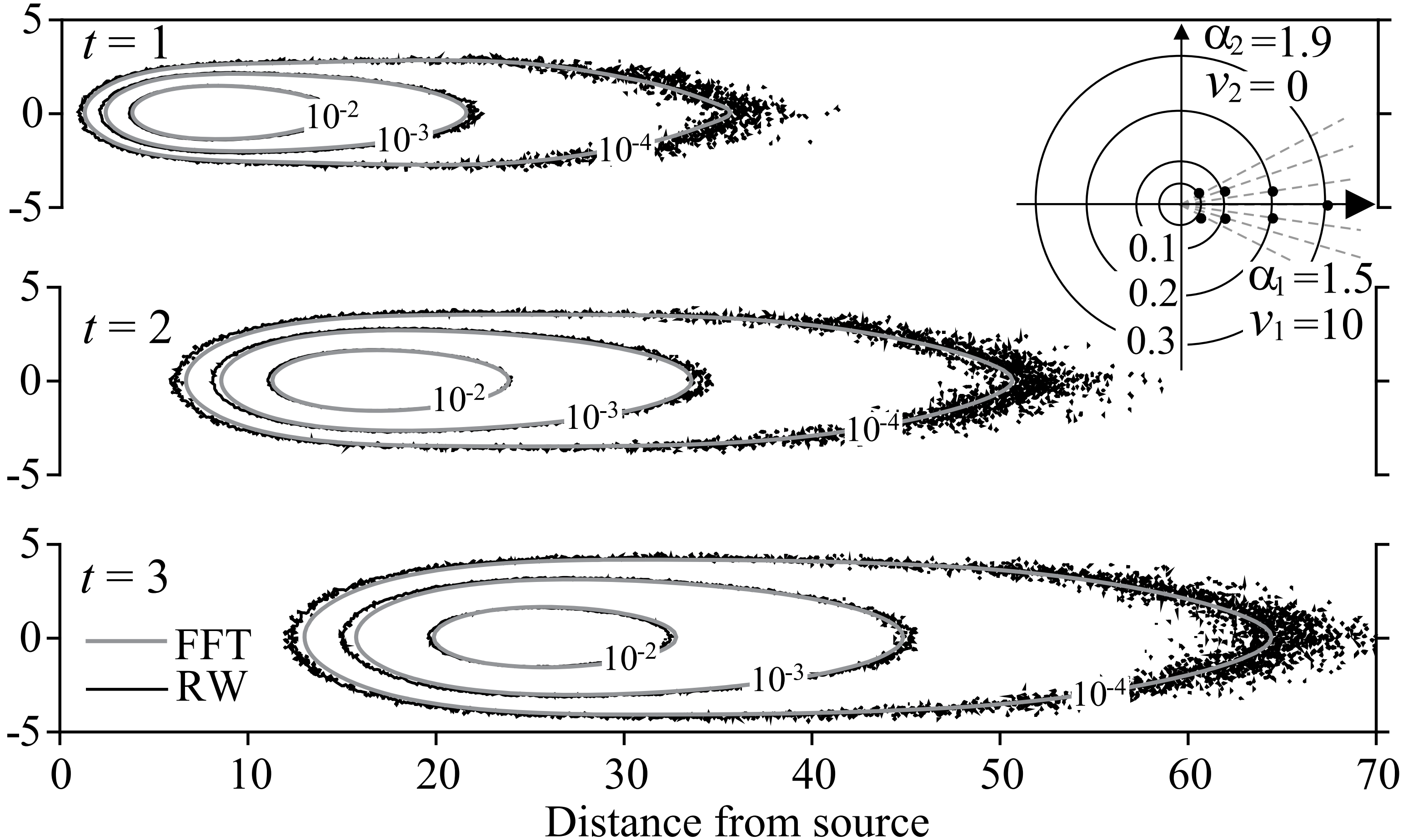
CTRW scaling limit $x = A(u)$ Markov but $u = E(t)$ is not:
Simulate Markov process $t = D(u)$ instead

Particle trajectories are (x_n, t_n) where $x_i = A(u_i)$, $t_i = D(u_i)$.

Theory recently established for space-fractional diffusion

2-D particle tracking solutions

Here $\partial_t p = -v \cdot \nabla p + a\partial_x^{1.5} p + b\partial_y^{1.9} p$



Iterated Brownian motion

IBM $A_{|B_t|}$ models diffusion in a crack. Essentially, the sample path A_t models the (fractal) crack. The density $c(x,t)$ of IBM is the Green's function solution to

$$\frac{\partial c(x,t)}{\partial t} = \frac{L\, f(x)}{\sqrt{\pi t}} + L^2 c(x,t); \qquad u(0,x) = f(x)$$

where $L = \Delta = \sum_j \partial^2/\partial x_j^2$ is the generator of the semigroup associated with the Brownian motion $A(t)$.

Taking $\beta = 1/2$ in the time-fractional diffusion equation yields exactly the same 1-D distributions $A_{|B_t|} \stackrel{d}{=} A_{E_t}$.

Proof: Laplace-Fourier transforms, local times

Fractional Cauchy problems on bounded domains

For $0 < \alpha < 1$ and $T_D(t)f(x) = E_x[f(X_t)I(t < \tau_D(X))]$, under some technical conditions

$$\begin{aligned} u(t,x) &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\ &= \frac{t}{\beta}\int_0^\infty T_D(l)f(x)g_\beta(tl^{-1/\beta})l^{-1/\beta-1}dl \end{aligned}$$

is the unique (classical) solution to

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D,\ t > 0 \\ u(t,x) &= 0,\ x \in \partial D,\ t > 0, \\ u(0,x) &= f(x),\ x \in D. \end{aligned}$$

PROOF: Eigenfunction expansion, Mittag-Leffler solutions

Some open problems

- Useful triangular arrays and infinitely divisible limits
- PseudoDO connection with fractional PDEs
- Particle tracking for time-fractional case
- Numerical methods for coupled case
- Boundary value problems
- Stochastic approximation schemes

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