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Fractional Pearson diffusions

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ABSTRACT

Pearson diffusions are governed by diffusion equations with polynomial coefficients. Fractional Pearson diffusions are governed by the corresponding time-fractional diffusion equation. They are useful for modeling sub-diffusive phenomena, caused by particle sticking and trapping. This paper provides explicit strong solutions for fractional Pearson diffusions, using spectral methods. It also presents stochastic solutions, using a non-Markovian inverse stable time change.

1. Introduction

The diffusion equation with constant coefficients

$$\frac{\partial}{\partial t}p = -\frac{\partial}{\partial x}[\mu p] + \frac{\partial^2}{\partial x^2}[Dp] \tag{1.1}$$

governs a Brownian motion with drift, the scaling limit of a random walk with finite variance jumps. The famous paper of Einstein [13] details these connections. Sokolov and Klafter [40] discuss the modern theory, which involves fractional derivatives. Particle sticking and trapping is modeled using a fractional time derivative [19,28]. The resulting time-fractional diffusion equations have found many applications in science, engineering and finance [17,24,30,31,37,38,41].

When the coefficients μ, D vary in space, the diffusion equation (1.1) governs a Markov process. Proving the existence of strong solutions of time-fractional diffusion equations is a difficult problem [5,27] even on unbounded domains. If $\mu(x)$ and $D(x)$ are polynomials, the special structure allows explicit solutions. The normalized steady state solutions comprise a family of probability density functions classified by Pearson [32]. The study of these Pearson diffusions began with Kolmogorov [20] and Wong [44], and continued in [2,3,15,22,23,39]. The Pearson diffusion equation governs several useful classes of Markov processes, including the Ornstein–Uhlenbeck process [43], and the Cox–Ingersoll–Ross process [11], which are useful in finance.

This paper considers fractional Pearson diffusions, where the first time derivative in (1.1) is replaced by a Caputo fractional derivative [9] of order $0 < \alpha < 1$. Explicit strong solutions are developed, using spectral methods involving the

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Mittag-Leffler function. Stochastic solutions are then obtained, using a non-Markovian time change involving the inverse stable subordinator. These solutions can be useful for particle tracking codes [25,45].

Time-fractional diffusion equations are important for applications to hydrology, finance, and physics, to name a few, see [29] for full details. In hydrology, the fractional time derivative models sticking and trapping between mobile periods for contaminant particles in a porous medium [38] or a river flow [10]. In finance, it models delays between trades [37], and has been used to develop the Black–Scholes formalism in this context [24,41]. In statistical physics, the fractional time derivative appears in the limit equation for a continuous time random walk, characterized by random waiting times between particle jumps [28,30,31]. The methods of this paper provide concrete governing equations, explicit solutions, and a stochastic interpretation for such situations, when the underlying diffusion has variable coefficients to model spatial inhomogeneities. To the best of our knowledge, this is the only known case of a time-fractional diffusion with variable coefficients for which one can explicitly compute the transition densities.

2. Pearson diffusions and their classification

2.1. Pearson diffusions

Pearson diffusions satisfy a stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW(t), \qquad t \geq 0, \tag{2.1}$$

where $W(t)$, $t \geq 0$, is a standard Brownian motion, and the drift $\mu(x)$ and diffusion (volatility) $\sigma^2(x)$ are polynomials of at most first and second degree, respectively:

$$\mu(x) = a_0 + a_1 x, \qquad D(x) = \frac{\sigma^2(x)}{2} = d_0 + d_1 x + d_2 x^2. \tag{2.2}$$

Let (l, L) be an interval such that $D(x) > 0$ for all $x \in (l, L)$. Given a Markov process X, let $p = p(x, t; y, s)$ be the transition density, i.e., the conditional density of $x = X_t$ given $y = X_s$. We will only consider time-homogeneous processes, which means that $p(x, t; y, s) = p(x, t - s; y, 0)$ for $t > s$, and will write $p(x, t; y) = \frac{\partial}{\partial x} P(X_t \leq x | X_0 = y)$.

The Fokker–Planck operator $\mathcal{L}$ is defined as

$$\mathcal{L}g(x) = -\frac{\partial}{\partial x}\left[\mu(x)g(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)g(x)\right],$$

and the Kolmogorov forward or Fokker–Planck equation is

$$\frac{\partial p(x,t;y)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x)p(x,t;y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p(x,t;y)\right] \quad \text{or} \quad \frac{\partial p}{\partial t} = \mathcal{L}p. \tag{2.3}$$

The transition density satisfies this equation with the point source initial condition. The infinitesimal generator of the diffusion (2.1) is:

$$\mathcal{G}g(y) = \left[\mu(y)\frac{\partial}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2}{\partial y^2}\right] g(y).$$

This operator appears on the right-hand side of Kolmogorov backward equation:

$$\frac{\partial p(x,t;y)}{\partial t} = \mu(y)\frac{\partial p(x,t;y)}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p(x,t;y)}{\partial y^2} \quad \text{or} \quad \frac{\partial p}{\partial t} = \mathcal{G}p. \tag{2.4}$$

If the stationary (invariant) density $\mathbf{m}$ of the diffusion (2.1) exists, it satisfies a time-independent Fokker–Planck equation (2.3) with zero on the left-hand side [18]. For Pearson diffusions, this equation reduces to

$$\frac{\mathbf{m}'(x)}{\mathbf{m}(x)} = \frac{\mu(x) - D'(x)}{D(x)} = \frac{(a_0 - d_1) + (a_1 - 2d_2)x}{d_0 + d_1 x + d_2 x^2}. \tag{2.5}$$

Eq. (2.5) is the famous Pearson equation introduced by K. Pearson in 1914 [32] in order to unify some important classes of distributions. Because of this connection to the Pearson equation, the processes that satisfy Eq. (2.1) are known as Pearson diffusions.

A function h is an eigenfunction of $(-\mathcal{G})$ if there exists a complex number λ (the eigenvalue) such that the Sturm–Liouville equation holds:

$$\mathcal{G}g = -\lambda g.$$

In the Pearson case, this equation can be written as a differential equation of hypergeometric type:

$$(d_0 + d_1 x + d_2 x^2)g'' + (a_0 + a_1 x)g' + \lambda g = 0. \tag{2.6}$$

Consider a system of polynomials $\{Q_n(x),\ n \in \mathcal{N}\}$, where $Q_n(x)$ is a polynomial in x of degree at most n, and the index set is either $\mathcal{N} = \mathbb{N}$, the set of nonnegative integers (including zero), or $\mathcal{N} = \{0, 1, 2, \dots, N\}$ for a finite nonnegative N. We say that $\{Q_n\}$ is an orthogonal system, for some pdf (weight) $\mathbf{m}(x)$, if the following orthogonality relations hold:

$$\int_S Q_n(x)Q_m(x)\mathbf{m}(x)dx = \delta_n^m c_n^2, \qquad n, m \in \mathcal{N},$$

where S is the support of the density $\mathbf{m}$, c_n^2 are nonzero constants, and δ_n^m is the Kronecker tensor. For Pearson diffusions, the stationary density is the weight function $\mathbf{m}$ with respect to which the eigenfunctions are orthogonal [18, p. 331].

There are six basic types of solutions to (2.6), depending on whether the polynomial $D(x)$ is constant, linear, or quadratic and, in the last case, on whether the discriminant $\Delta = d_1^2 - 4d_0d_2$ is positive, negative, or zero [2]. These solutions are the classical orthogonal polynomials. Three types of solutions of (2.6) correspond to the case when the spectrum of the generator $\mathcal{G}$ is discrete. In this case, $\mathcal{N} = \mathbb{N}$. The remaining three types correspond to mixed spectrum of the generator, when only finitely many orthogonal polynomials exist ($\mathcal{N} = \{0, 1, 2, \dots, N\}$ for a finite nonnegative N), and the remaining part of the spectrum is continuous (see [2,3,22,23]). A useful summary of the six types of Pearson diffusions is given in [29]. In this paper, we consider fractional Pearson diffusions with purely discrete spectrum. When the spectrum of the generator $\mathcal{G}$ is discrete, the eigenvalues are given by the formula:

$$\lambda_n = -n\mu'(x) - \frac{1}{2}n(n-1)D''(x) = -n[a_1 + d_2(n-1)]. \tag{2.7}$$

The associated eigenfunctions are of the form $g(x) = Q_n(x)$, where Q_n is a polynomial of degree at most n. Next we briefly describe the three classes of Pearson diffusions with discrete spectrum.

2.2. Ornstein–Uhlenbeck (OU) process

When $D(x)$ in (2.2) is a constant, Eq. (2.1) takes the form

$$dX_t = -\theta\,(X_t - \mu)\ dt + \sqrt{2\theta\sigma^2}\,dW_t, \qquad \theta > 0, t \geq 0,$$

with an obvious change of notation. For a stationary OU process, θ is a correlation function parameter ($\mathrm{corr}(X_t, X_s) = e^{-\theta|t-s|}$, see [6]), and μ and σ are distribution parameters. The invariant distribution is normal:

$$\mathbf{m}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}. \tag{2.8}$$

When $\theta > 0$, the diffusion is a stationary OU process when the initial distribution has density $\mathbf{m}$. The eigenvalues are $\lambda_n = \theta n$, $n \geq 0$. The eigenfunctions of $(-\mathcal{G})$ are Hermite polynomials:

$$H_n(x) = (-1)^n\,[\mathbf{m}(x)]^{-1}\,\frac{d^n}{dx^n}\mathbf{m}(x), \qquad x \in \mathbb{R}, n = 0, 1, 2, \dots$$

and the normalized Hermite polynomials are

$$Q_n(s) = \frac{\sigma^n}{\sqrt{n!}}H_n(x).$$

2.3. Cox–Ingersoll–Ross (CIR) process

When $D(x) = d_1x + d_0$, we may suppose $d_0 = 0$ (after normalizing, which would change a_0 to $a_0 - a_1d_0/d_1$). If $d_1 > 0$ then the process is a CIR (square root Feller) diffusion on the interval $(0, \infty)$ [11]. If $d_1 < 0$, then the state space is $(-\infty, 0)$, where $\sigma^2(x)$ is positive. This can be reduced to the case $d_1 > 0$ by a simple reparametrization. Using the traditional parametrization of the CIR process, Eq. (2.1) takes the form:

$$dX_t = -\theta\left(X_t - \frac{b}{a}\right)\,dt + \sqrt{\frac{2\theta}{a}X_t}\,dW_t, \qquad \theta > 0, a > 0, b > 0, t \geq 0.$$

Then we have the invariant density:

$$\mathbf{m}(x) = \frac{a^b}{\Gamma(b)}\,x^{b-1}e^{-ax} \qquad x > 0. \tag{2.9}$$

With this parametrization, when the initial distribution has density $\mathbf{m}$, the stationary CIR process has the correlation function $\mathrm{corr}(X_t, X_s) = e^{-\theta|t-s|}$, see [6]. The eigenvalues are $\lambda_n = \theta n$, $n \geq 0$. The orthogonal polynomials are Laguerre polynomials given by the formula:

$$L_n^{(b-1)}(ax), \qquad x > 0, n \in \mathbb{N} \quad \text{with } L_n^{(\gamma)}(x) = \frac{1}{n!}x^{-\gamma}e^x\frac{d^n}{dx^n}x^{n+\gamma}e^{-x}, \qquad \gamma > -1.$$

The normalized Laguerre polynomials are

$$Q_n(x) = \frac{L_n^{(b-1)}(ax)}{\sqrt{\Gamma(b+n)/(\Gamma(b)n!)}}.$$

2.4. Jacobi diffusion

Suppose $D(x) = d_2(x-x_1)(x-x_2)$, and $d_2 < 0$. Then the state space is (x_1, x_2) with $x_1 < x_2$. After rescaling we may assume $d_2 = -1$, and after a linear change of variables $\tilde{x} = 2x - (x_1+x_2)/(x_2-x_1)$, we can take $D(x) = 1-x^2$, $\mu(x) = -(a+b+2)x+b-a$. To separate the correlation and distribution parameters, use factor of $\theta/(a+b+2)$, and write Eq. (2.1) in the form

$$dX_t = -\theta\left(X_t - \frac{b-a}{a+b+2}\right) + \sqrt{\frac{2\theta}{a+b+2}(1-X_t^2)}dW_t.$$

With this parametrization, the invariant density is (up to a normalizing constant)

$$\mathbf{m}(x) \propto (1-x)^a(1+x)^b$$

In the recurrent case $a, b > -1$, we obtain the Beta density:

$$\mathbf{m}(x) = (1-x)^a(1+x)^b\frac{\Gamma(a+b+2)}{\Gamma(b+1)\Gamma(a+1)2^{a+b+1}}, \qquad x \in (-1, 1). \tag{2.10}$$

When the initial distribution has density $\mathbf{m}$, the stationary Jacobi diffusion has the correlation function $\mathrm{corr}(X_t, X_s) = e^{-\theta|t-s|}$, see [6]. The eigenvalues are $\lambda_n = n\theta(n+a+b+1)/(a+b+2)$, $n \geq 0$. The orthogonal polynomials are Jacobi polynomials given by the formula:

$$2^n n! P_n^{(a,b)}(x) = (-1)^n(1-x)^{-a}(1+x)^{-b}\frac{d^n}{dx^n}\left\{(1-x)^{a+n}(1+x)^{b+n}\right\}.$$

The orthonormal Jacobi polynomials are

$$Q_n(x) = \frac{P_n^{(a,b)}(x)}{c_n}, \tag{2.11}$$

where

$$c_n^2 = \frac{2^{a+b+1}}{2n+a+b+1}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(n+1)\Gamma(n+a+b+1)}.$$

3. Fractional Pearson diffusions

The Caputo fractional derivative of order $0 < \alpha < 1$ is defined by

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha} u(\tau, x)\, d\tau - \frac{u(0,x)}{t^\alpha}\right]. \tag{3.1}$$

When $0 < \alpha < 1$, and u is differentiable in t (or even absolutely continuous), the fractional derivative can also be expressed as

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(\tau,x)}{\partial \tau}(t-\tau)^{-\alpha}\, d\tau, \tag{3.2}$$

see for example [12]. Let $\tilde{u}(s,x) = \int_0^\infty e^{-st}u(t,x)\,dt$ be the usual Laplace transform. It is not hard to check that $\partial^\alpha u(t,x)/\partial t^\alpha$ has Laplace transform $s^\alpha\tilde{u}(s,x) - s^{\alpha-1}u(0,x)$, which reduces to the familiar form when $\alpha = 1$.

For $0 < \alpha < 1$ and function $p = p(x,t;y)$, $t > 0$, $x \in (l, L)$, we consider a time-fractional Fokker–Planck equation of the form

$$\frac{\partial^\alpha p}{dt^\alpha} = \frac{\partial}{\partial x}[-\mu(x)p] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\sigma^2(x)p\right] \quad \text{or} \quad \frac{\partial^\alpha p}{dt^\alpha} = \mathcal{L}p \tag{3.3}$$

subject to point source initial condition. Note that, y is a constant in this equation. We also consider a fractional diffusion (backward Kolmogorov) equation:

$$\frac{\partial^\alpha p}{dt^\alpha} = \mathcal{G}p = \mu(y)\frac{\partial p}{\partial y} + \frac{1}{2}\sigma^2(y)\frac{\partial^2 p}{\partial y^2}. \tag{3.4}$$

Note that, x is a constant in this equation. We use separation of variables approach and seek a solution of the fractional backward equation (3.4) in the form $p_\alpha(x, t; y) = T(t)\varphi(y)$, where the functions T and φ may depend on x and α. Then

$$\frac{d^\alpha T(t)}{dt^\alpha}\varphi(y) = T(t)\mathcal{G}\varphi(y) \quad \text{or} \quad \frac{1}{T(t)}\frac{d^\alpha T(t)}{dt^\alpha} = \frac{\mathcal{G}\varphi(y)}{\varphi(y)}$$

if T and φ do not vanish. The last equation can hold only if both sides are equal to a constant. Denote this constant by $-\lambda$ (so that $\lambda > 0$) and consider two resulting equations:

$$\mathcal{G}\varphi = -\lambda\varphi \tag{3.5}$$

and

$$\frac{d^\alpha T(t)}{dt^\alpha} = -\lambda T(t). \tag{3.6}$$

For Eq. (3.5), the eigenvalues are given by (2.7), and the eigenfunctions are either Hermite, Laguerre, or Jacobi polynomials, as noted in Section 2. As for Eq. (3.6), it was shown in [26,27] that strong solutions (temporal eigenfunctions $T_n(t)$) have the form:

$$T_n(t) = E_\alpha\left(-\lambda_n t^\alpha\right) = \sum_{j=0}^{\infty} \frac{(-\lambda_n t^\alpha)^j}{\Gamma(1+\alpha j)} \tag{3.7}$$

where $T_n(0) = 1$, and $E_\alpha(\cdot)$ is the Mittag-Leffler function. For $\alpha = 1$, the standard exponential form of T_n is recovered: $T_n(t) = e^{-\lambda_n t}$. These considerations lead to a heuristic solution of the backward equation:

$$p_\alpha(x, t; y) = \sum_{n=0}^{\infty} b_n E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(y),$$

where $\{Q_n\}$ represents the orthonormal system of Hermite, Laguerre or Jacobi polynomials in the case of OU, CIR, or Jacobi diffusion, respectively (see Section 2 for details), and the constants b_n may depend on x. The goal of this section is to make this heuristic argument precise, and provide expressions for strong solutions for the fractional Cauchy problems associated with (3.3) and (3.4).

Lemma 3.1. *For the three classes of fractional Pearson diffusions (OU, CIR, Jacobi) whose invariant density* $\mathbf{m}$ *and system of orthonormal polynomials* $\{Q_n,\ n \in \mathbb{N}\}$ *were detailed in Section 2, for any* $0 < \alpha < 1$, *the series*

$$p_\alpha(x, t; y) = \mathbf{m}(x) \sum_{n=0}^{\infty} E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(y) Q_n(x) \tag{3.8}$$

with E_α *given by (3.7) converges for fixed* $t > 0$, $x,\ y \in (l, L)$.

Proof. For a Mittag-Leffler function with $0 < \alpha < 1$ (see [21], Eq. (3.5)):

$$E_\alpha(-\lambda_n t^\alpha) \le \frac{c}{1 + \lambda_n t^\alpha},$$

and from [26, Eq. (5.26)]

$$E_\alpha(-\lambda_n t^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\lambda_n t^\alpha}$$

as the argument $\lambda_n t^\alpha \to \infty$. Here $f(t) \sim g(t)$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$. The eigenvalues are $\lambda_n = \theta n$ in the Hermite and Laguerre cases, and $\lambda_n = n\theta(n + a + b + 1)/(a + b + 2)$ in the Jacobi case. In the rest of the proof, we will assume without loss of generality that, $\mu = 0$ and $\sigma = 1$ in the OU case, and $a = 1$ in the CIR case.

Let us first deal with the OU case. From [36, p. 369]

$$|Q_n(x)| \le K e^{x^2/4} n^{-1/4}\left(1 + |x/\sqrt{2}|^{5/2}\right), \tag{3.9}$$

where K is a constant that does not depend on x, and the convergence of the series (3.8) follows from

$$\left|E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(y) Q_n(x)\right| \le \frac{C(x, y, t, \alpha)}{n^{1+1/2}}.$$

Above and in the later parts of the paper, we use notation $C(x, y, t, \alpha)$ for constants not all equal, but not dependent on n. These constants may also depend on the parameters of the distributions (i.e. the coefficients of $\mu(x)$ and $\sigma(x)$ in (2.2)).

In the CIR case, the orthonormal Laguerre polynomials satisfy [36, p. 348]

$$|Q_n(x)| = O\left(\frac{e^{x/2}}{x^{(2b-1)/4}} n^{-1/4}\right),$$

uniformly for x in finite intervals $[x_1, x_2]$, and therefore

$$|E_\alpha(-\lambda_n t^\alpha) Q_n(y) Q_n(x)| \le \frac{C(x, y, t, \alpha)}{n^{1+1/2}}$$

in this case. Finally, for orthonormal Jacobi polynomials from [42, p. 196] and [14] we have

$$Q_n(x) = C(x, a, b) \cos(N\theta + \gamma) + O(n^{-1}), \tag{3.10}$$

where $x = \cos\theta$, $N = n + 1/2(a + b + 1)$, and $\gamma = -(a + 1/2)\pi/2$, and O holds uniformly in θ on $[\epsilon, \pi - \epsilon]$ for any $\epsilon > 0$. The convergence of the series (3.8) follows from

$$|E_\alpha(-\lambda_n t^\alpha) Q_n(y) Q_n(x)| \le \frac{C(x, y, t, \alpha) \cos(N\theta + \gamma)}{n^2}. \quad \square$$

The next result proves a strong solution for the fractional backward equation.

Theorem 3.2. *Suppose that, the function $g \in L_2(\mathbf{m}(x)dx)$ is such that $\sum_n g_n Q_n$ with $g_n = \int_l^L g(x) Q_n(x) \mathbf{m}(x) dx$ converges to g uniformly on finite intervals $[y_1, y_2] \subset (l, L)$. Then the fractional Cauchy problem*

$$\frac{\partial^\alpha u(t; y)}{\partial t^\alpha} = \mathcal{G} u(t; u) = \mu(y) \frac{\partial u(t; y)}{\partial y} + \frac{1}{2}\sigma^2(y) \frac{\partial^2 u(t; y)}{\partial y^2} \tag{3.11}$$

with initial condition $u(0; y) = g(y)$ has a strong solution $u = u(t; y)$ given by

$$u(t; y) = u_\alpha(t; y) = \int_l^L p_\alpha(x, t; y) g(x) dx = \sum_{n=0}^\infty E_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n. \tag{3.12}$$

The series in (3.12) converges absolutely for each fixed $t > 0$, $y \in (l, L)$, and (3.11) holds pointwise.

Proof. Each term under the sum in (3.12) satisfies (3.11) because

$$\mathcal{G} E_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n = -\lambda_n g_n E_\alpha(-\lambda_n t^\alpha) Q_n(y) = \frac{\partial^\alpha}{\partial t^\alpha} E_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n.$$

To prove that (3.12) satisfies (3.11) pointwise, we need to show that, the series in (3.12) can be differentiated term by term, and in view of standard results in analysis (e.g., see [35, Theorem 7.16, p. 151; Theorem 7.17, p. 152]), this would follow from absolute and uniform convergence on finite intervals of the series $\sum Q_n(y) g_n$ and the series that involve the derivatives:

$$\begin{cases} \sum_{n=0}^\infty \frac{\partial^\alpha}{\partial t^\alpha} E_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n, \\ \sum_{n=0}^\infty E_\alpha(-\lambda_n t^\alpha) g_n Q_n'(y), \\ \sum_{n=0}^\infty E_\alpha(-\lambda_n t^\alpha) g_n Q_n''(y). \end{cases}$$

For the series with the fractional time derivative,

$$\frac{\partial^\alpha}{\partial t^\alpha} E_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n = -\lambda_n E_\alpha(-\lambda_n t^\alpha) Q_n(y) g_n.$$

Since $E_\alpha(-\lambda_n t^\alpha) = O(\lambda_n^{-1} t^{-\alpha})$ when $t > 0$ and $n \to \infty$ (Eq. (5.26) in [26]), the series with a fractional derivative in time converges if $\sum_n Q_n(y) g_n$ converges. Under the conditions on the function g, the series $\sum_n Q_n(y) g_n$ converges (pointwise) to $g(y)$ uniformly on finite intervals $[y_1, y_2] \subset (l, L)$.

Without loss of generality, assume that, $\mu = 0$ and $\sigma = 1$ in the OU case, and $a = 1$ in the CIR case. In the OU case, we use the relation ([1, p. 783]):

$$\frac{d}{dx} H_n(x) = n H_{n-1}(x).$$

For orthonormal Hermite polynomials,

$$\frac{d}{dx}Q_n(x) = \sqrt{n}Q_{n-1}(x),$$

and

$$\left|E_\alpha(-\lambda_n t^\alpha)g_n Q_n'(y)\right| \le C(t, y, \alpha)n^{-3/4}|g_n|.$$

Convergence of the series $\sum_n n^{-3/4}|g_n|$ follows by the Cauchy–Schwarz inequality:

$$\sum_n n^{-3/4}|g_n| \le \left(\sum_n n^{-3/2}\right)^{1/2}\left(\sum_n |g_n|^2\right)^{1/2}.$$

The series $\sum |g_n|^2$ converges because $g \in L_2(\mathbf{m}(x)dx)$ and $\int |g(x)|^2\mathbf{m}(x)dx = \sum |g_n|^2$.

For the second derivative in space, use the differential equation (2.6):

$$Q_n''(y) = yQ_n'(y) - nQ_n(y).$$

The series involving the first derivative in space was treated above, and for the second term

$$E_\alpha(-\lambda_n t^\alpha)nQ_n(y)g_n$$

we again use the asymptotics of Mittag-Leffler function (Eq. (5.26) in [26]) $E_\alpha(-\lambda_n t^\alpha) \sim 1/(\Gamma(1-\alpha)\lambda_n t^\alpha)$ for $t > 0$ and $n \to \infty$, and the facts that $\lambda_n = \theta n$, and that $\sum_n Q_n(y)g_n$ converges uniformly on finite intervals.

In the CIR case, from [42, p. 102] we have

$$\frac{d}{dx}L_n^{(b-1)}(x) = -L_{n-1}^{(b)}(x),$$

and for orthonormal Laguerre polynomials

$$\frac{d}{dx}Q_n^{(b-1)}(x) = -\frac{(n-1)^{b/2}}{n^{(b-1)/2}}Q_{n-1}^{(b)}(x).$$

The last quantity behaves like $C(x, b)n^{1/4}$ uniformly on finite intervals (see [36, p. 348]). Therefore in this case

$$\left|E_\alpha(-\lambda_n t^\alpha)g_n Q_n'(y)\right| \le C(t, y, \alpha, b)n^{-3/4}|g_n|,$$

and the rest of the argument for the series involving the first derivative in space is the same as in the OU case. The same argument also applies to the second derivative in space because, for Laguerre polynomials, Eq. (2.6) has the form

$$y\frac{d^2}{dy^2}Q_n(y) = (y-b)\frac{d}{dy}Q_n(y) - nQ_n(y).$$

For Jacobi polynomials,

$$(2n+a+b)(1-x^2)\frac{d}{dx}P_n^{(a,b)}(x) = n(a-b-(2n+a+b)x)P_n^{(a,b)}(x) + 2(n+a)(n+b)P_{n-1}^{(a,b)}(x)$$

and for orthonormal Jacobi polynomials

$$\frac{d}{dx}Q_n(x) = \frac{n(a-b-(2n+a+b)x)}{(2n+a+b)(1-x^2)}Q_n(x) + \frac{2(n+a)(n+b)}{(2n+a+b)(1-x^2)}\sqrt{n/(n-1)}Q_{n-1}(x).$$

The first term in the last relation leads to the series

$$\sum_n nE_\alpha(-\lambda_n t^\alpha)Q_n(x)g_n$$

that converges, because it is dominated by the absolutely convergent series $C(t, x, \alpha, a, b)\sum_n g_n/n$. The latter can be seen from Cauchy–Schwarz inequality

$$\left(\sum_n |g_n|/n\right)^2 \le \left(\sum_n |g_n|^2\right)\left(\sum_n 1/n^2\right).$$

The second term in the expression for the derivative, in the case of Jacobi polynomials, behaves in the same way as the first, and finally, the expression for the second derivative from (2.6) is

$$(1-y^2)\frac{d^2}{dy^2}Q_n(y) = -((b-a)-(a+b-2)y)\frac{d}{dy}Q_n(y) - n(n+a+b+1)Q_n(y).$$

The term with the first derivative was treated above. The second term leads to the series

$$\sum_n E_\alpha(-\lambda_n t^\alpha) n(n+a+b+1) Q_n g_n,$$

which converges since the series $\sum_n Q_n g_n$ converges by assumption and $\lambda_n = n\theta(n+a+b+1)/(a+b+2)$ in this case.

Thus in all the three cases, (3.12) can be differentiated term by term, and it satisfies the fractional backward Kolmogorov equation (3.11). The initial condition is satisfied since $u(0; y) = \sum_{n=0}^\infty Q_n(y) g_n = g(y)$ pointwise for $y \in (l, L)$ for g that satisfies conditions of the theorem. $\square$

Now, we consider the fractional Fokker–Planck (forward) equation. Because of the prefactor $\mathbf{m}(x)$ in (3.14) below, here we use an eigenfunction expansion of $f(x)/\mathbf{m}(x)$ in $L_2(\mathbf{m}(\mathbf{x})dx)$. Note that, $f(x)/\mathbf{m}(x)$ is well defined because $\mathbf{m}$ does not vanish on the interval (l, L).

Theorem 3.3. *Suppose that, the function $f/\mathbf{m} \in L_2(\mathbf{m}(\mathbf{x})dx)$, and that $\sum_n f_n Q_n$ with $f_n = \int_l^L f(y) Q_n(y) dy$ converges to $f/\mathbf{m}$ uniformly on finite intervals $[y_1, y_2] \subset (l, L)$. Then the fractional Cauchy problem*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathcal{L} u(x,t) = -\frac{\partial}{\partial x}\left[\mu(x) u(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x) u(x,t)\right] \tag{3.13}$$

with the initial condition $u(x, 0) = f(x)$ has a strong solution $u = u(x, t)$ given by

$$u(x,t) = u_\alpha(x,t) = \int_l^L p_\alpha(x, t; y) f(y) dy = \mathbf{m}(x) \sum_{n=0}^\infty E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(x) f_n. \tag{3.14}$$

The series in (3.14) converges absolutely for each $t > 0$, $x \in (l, L)$, and Eq. (3.13) holds pointwise (u is a strong solution).

Proof. To demonstrate that each term in (3.14) satisfies (3.13), recall that $\mathbf{m}$ satisfies the time-independent Fokker–Planck equation (2.5), and so $\mathbf{m}(x)\mu(x) = \frac{d}{dx}(\sigma^2(x)\mathbf{m}(x)/2)$. We have

$$\begin{aligned}\mathcal{L}(\mathbf{m}(x) E_\alpha(-\lambda_n t^\alpha) f_n Q_n(x)) &= E_\alpha(-\lambda_n t^\alpha) f_n \left[-\frac{d}{dx}(\mu(x)\mathbf{m}(x) Q_n(x)) + \frac{d^2}{dx^2}(\sigma^2(x)\mathbf{m}(x) Q_n(x)/2)\right] \\ &= E_\alpha(-\lambda_n t^\alpha) f_n \mathbf{m}(x) \mathcal{G} Q_n(x) = -\lambda_n E_\alpha(-\lambda_n t^\alpha) f_n \mathbf{m}(x) Q_n(x).\end{aligned}$$

The same expression is obtained for the Caputo fractional derivative in time applied to each term in (3.14):

$$\frac{\partial^\alpha}{\partial t^\alpha}(\mathbf{m}(\mathbf{x}) E_\alpha(-\lambda_n t^\alpha) f_n Q_n(x)) = -\lambda_n E_\alpha(-\lambda_n t^\alpha) f_n \mathbf{m}(x) Q_n(x).$$

The proof that series in (3.14) can be differentiated term-by-term with respect to the time and space variables is similar to the proof in Theorem 3.2. The conditions on f ensure that the series

$$\begin{cases} \sum_{n=0}^\infty \frac{\partial^\alpha}{\partial t^\alpha} E_\alpha(-\lambda_n t^\alpha) Q_n(x) f_n, \\ \sum_{n=0}^\infty E_\alpha(-\lambda_n t^\alpha) Q_n'(x) f_n, \\ \sum_{n=0}^\infty E_\alpha(-\lambda_n t^\alpha) Q_n''(x) f_n, \end{cases}$$

converge absolutely and uniformly on finite intervals.

Finally, under the assumptions on f, substitution of $t = 0$ into (3.14) gives

$$u(x, 0) = \mathbf{m}(x) \sum f_n Q_n(x).$$

Since $\sum_n f_n Q_n(x) = f(x)/\mathbf{m}(x)$ pointwise for $x \in (l, L)$, we see that $u(x, 0) = f(x)$. $\square$

Remark 3.4. Here we provide some sufficient conditions for convergence of the eigenfunction expansions that were assumed in Theorems 3.2 and 3.3. Assume that, $g \in L_2(\mathbf{m}(x)dx)$ is continuous on (l, L) and has bounded variation on any finite interval $[y_1, y_2] \subset (l, L)$ (or that g is differentiable on (l, L)). Then the conditions of the equiconvergence theorems [42, pp. 245–248] are sufficient to ensure that $\sum_n g_n Q_n$ converge to g uniformly on finite intervals $[y_1, y_2] \subset (l, L)$: for the case of Hermite series, it suffices that, the function g is absolutely integrable on any finite interval: $\int_{-a}^a |g(y)| dy < \infty$ for every $a > 0$, and

$$\int_n^\infty e^{-x^2/4} x^{-5/3}(|g(x)| + |g(-x)|) dx = o(n^{-1}), \qquad n \to \infty.$$

Sansone [36, p. 381] gives a different sufficient condition: $\int_{-\infty}^{\infty} g(x)\sqrt{\mathbf{m}(x)}dx$ converges. For the Laguerre series, it suffices that the integrals $\int_0^1 x^{b-1}|g(x)|dx$ and $\int_0^1 x^{(b-1)/2-1/4}|g(x)|dx$ exist and

$$\int_n^{\infty} e^{-x/2}x^{(b-1)/2-13/12}|g(x)|dx = o(n^{-1/2}), \qquad n \to \infty.$$

For the Jacobi series, it suffices that, the integrals

$$\int_{-1}^{1}(1-x)^a(1+x)^b|g(x)|dx \quad \text{and} \quad \int_{-1}^{1}(1-x)^{a/2-1/4}(1+x)^{b/2-1/4}|g(x)|dx$$

exist. For a different sufficient condition in the Jacobi case, see [33]. For Theorem 3.3, these sufficient conditions need to apply to the function $f/\mathbf{m}$.

Remark 3.5. If $\alpha = 1$, then (3.8) becomes

$$p_1(x, t; y) = \mathbf{m}(x)\sum_{n=0}^{\infty} e^{-\lambda_n t}Q_n(x)Q_n(y). \tag{3.15}$$

The series (3.15) converges pointwise, can be differentiated term-by-term, and its sum satisfies both forward and backward Kolmogorov equations (Eqs. (2.2) and (2.3), respectively) with point source initial conditions. The functions defined in (3.12) and (3.14) with $\alpha = 1$ solve the respective Cauchy problems under the conditions of Theorems 3.2 and 3.3.

For OU and CIR diffusions, (3.15) can be written in closed form. The transition density of the OU process was given in [18, p. 332] for a special case. In general,

$$p_1(x, t; y) = \frac{1}{\sigma\sqrt{2\pi(1-e^{-2\theta t})}}\exp\left\{-\frac{\left(x-\mu-(y-\mu)e^{-\theta t}\right)^2}{2\sigma^2(1-e^{-2\theta t})}\right\}, \qquad t \geq 0.$$

The transition density for the CIR process is

$$p_1(x, t; y) = \left(\frac{x}{y}\right)^{\frac{b-1}{2}}\frac{a}{(1-e^{-\theta t})\Gamma(b)}\exp\left\{\frac{\theta(b-1)t}{2} - ax - \frac{a(y+x)}{e^{\theta t}-1}\right\}I_{(b-1)}\left(\frac{a\sqrt{yx}}{\sinh(0.5\theta t)}\right), \qquad t \geq 0,$$

where $I_{(b-1)}(\cdot)$ is the modified Bessel function of the first kind.

Remark 3.6. For the OU case (when $\sigma^2(x)$ is a constant), the physical interpretation of Eq. (3.8) was provided in [30, Eqs. (117) and (119)]. The expansion (3.8) in the CIR and Jacobi cases can be interpreted similarly.

Remark 3.7. It is easy to see that the solutions (3.12) in Theorem 3.2 are unique. If there are two solutions u_1, u_2 with the same initial condition $u_i(0; y) = g(y)$ that satisfies the conditions of the theorem, then $u = u_1 - u_2$ is a solution with initial condition $g(y) \equiv 0$. Then every $g_n = 0$, and hence it follows from (3.12) that $u(t; y) \equiv 0$, proving uniqueness. The proof for Theorem 3.3 is similar.

4. Stochastic model for fractional Pearson diffusion

The Pearson diffusions defined by (2.1) have transition densities that solve the Fokker–Planck equation (2.3). Theorem 3.3 provides the corresponding solution to the fractional Fokker–Planck equation (3.13). In this section, we construct the fractional Pearson diffusion process, whose transition densities solve (3.13). This construction involves a non-Markovian time change, by way of the inverse (or first passage time) of a stable subordinator. Since the negative generator of a stable subordinator with index $0 < \alpha < 1$ is a (Riemann–Liouville) fractional derivative of order α, the appearance of the stable subordinator here is somewhat natural. Our approach extends the results of [4,28] to the case of variable coefficient diffusions. Because the fractional Pearson diffusion is non-Markovian, the transition densities do not determine the process. Hence the stochastic solution provides additional information about the movement of particles that diffuse under this model.

Let D_t be a standard stable subordinator with Laplace transform

$$E[e^{-sD_t}] = \exp\{-ts^{\alpha}\}, \qquad s \geq 0 \tag{4.1}$$

and define the inverse (hitting time, first passage time) process

$$E_t = \inf\{x > 0 : D_x > t\}. \tag{4.2}$$

Let $X_1(t)$ be a Pearson diffusion given by Eq. (2.1) whose transition densities solve the forward Fokker–Planck equation (2.3) with the point source initial condition. We take $X_1(t)$ to be independent of the subordinator D_t, and define the *fractional Pearson diffusion* process

$$X_{\alpha}(t) = X_1(E_t), \qquad t \geq 0. \tag{4.3}$$

We say that, the non-Markovian process $X_\alpha(t)$ has a transition density $p_\alpha(x,t;y)$ if

$$P(X_\alpha(t)\in B|X_\alpha(0)=y)=\int_B p_\alpha(x,t;y)dx$$

for any Borel subset B of (l,L).

Lemma 4.1. *The fractional Pearson diffusion (4.3) of OU, CIR, or Jacobi type has transition density defined by Eq. (3.8) from Lemma 3.1.*

Proof. Let $p_1(x,t;y)$ be the transition density (3.15) for a Pearson diffusion of OU, CIR, or Jacobi type. Since $\{E_t\le x\}=\{D_x\ge t\}$, a straightforward argument [28, Corollary 3.1] shows that the density of E_t is

$$f_t(x)=\frac{t}{\alpha}x^{-1-\frac{1}{\alpha}}g_\alpha\left(tx^{-\frac{1}{\alpha}}\right), \tag{4.4}$$

where g_α is the density of $D(1)$. Bingham [7] and Bondesson, Kristiansen, and Steutel [8] show that E_t has a Mittag-Leffler distribution with

$$E(e^{-sE_t})=\int_0^\infty e^{-sx}f_t(x)\,dx=E_\alpha(-st^\alpha).$$

Since the Pearson diffusion $X_1(t)$ is independent of the time change E_t, a Fubini argument along with (3.15) yields

$$\begin{aligned}P(X_\alpha(t)\in B|X_\alpha(0)=y)&=\int_0^\infty P(X_1(\tau)\in B|X_1(0)=y)f_t(\tau)\,d\tau\\&=\int_0^\infty\int_B p_1(x,\tau;y)dx\,f_t(\tau)\,d\tau\\&=\int_B \mathbf{m}(x)\sum_{n=0}^\infty Q_n(x)Q_n(y)\int_0^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau\,dx\\&=\int_B \mathbf{m}(x)\sum_{n=0}^\infty Q_n(x)Q_n(y)E_\alpha(-\lambda_n t^\alpha)dx\end{aligned} \tag{4.5}$$

which shows that (3.8) is the transition density of $X_\alpha(t)$.

The justification for term-by-term integration in

$$\int_0^\infty\sum_{n=0}^\infty Q_n(x)Q_n(y)e^{-\lambda_n\tau}f_t(\tau)d\tau$$

is as follows. For $\epsilon>0$, consider

$$\int_\epsilon^\infty\sum_{n=0}^\infty Q_n(x)Q_n(y)e^{-\lambda_n\tau}f_t(\tau)d\tau.$$

The series $\sum_{n=0}^\infty Q_n(x)Q_n(y)e^{-\lambda_n\tau}$ converges absolutely and uniformly for $\tau\in[\epsilon,\infty)$, and

$$\int_\epsilon^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau<\int_0^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau=E_\alpha(-\lambda_n t^\alpha)$$

is finite for all n. Therefore

$$\int_\epsilon^\infty\sum_{n=0}^\infty Q_n(x)Q_n(y)e^{-\lambda_n\tau}f_t(\tau)d\tau=\sum_{n=0}^\infty Q_n(x)Q_n(y)\int_\epsilon^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau.$$

The last series converges because

$$\sum_{n=0}^\infty|Q_n(x)Q_n(y)|\int_\epsilon^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau\le\sum_{n=0}^\infty|Q_n(x)Q_n(y)|e^{-\lambda_n\epsilon},$$

and the behavior of orthonormal polynomials was detailed in the proof of Lemma 3.1. Now, we let $\epsilon\to 0$. On the right hand side, the limit as $\epsilon\to 0$ can be brought inside the summation by the dominated convergence argument:

$$\int_\epsilon^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau<\int_0^\infty e^{-\lambda_n\tau}f_t(\tau)d\tau=E_\alpha(-\lambda_n t^\alpha),$$

and the series

$$\sum_{n=0}^{\infty} E_\alpha(-\lambda_n t^\alpha) Q_n(x) Q_n(y)$$

converges absolutely for all $t > 0$ and $x, y \in (l, L)$ as was shown in Lemma 3.1. $\square$

The next result shows that, (4.3) gives a stochastic solution to the fractional backward equation (3.11).

Theorem 4.2. *For any function g that satisfies the conditions of Theorem 3.2 (see also Remark 3.4), the function*

$$u(t; y) = E[g(X_\alpha(t))|X_\alpha(0) = y] \tag{4.6}$$

solves the fractional Cauchy problem (3.11) with initial condition $u(0; y) = g(y)$.

Proof. The proof is an application of Theorem 3.1 in [4] along with some ideas from [28]. Let $p_1(x, t; y)$ be the transition density (3.15) for a Pearson diffusion of OU, CIR, or Jacobi type. For a $t > 0$, define the operator

$$T(t)g(y) = E[g(X_1(t))|X_1(0) = y] = \int_l^L p_1(x, t; y) g(x) dx$$

for bounded continuous functions g that vanish at infinity in the OU case, bounded continuous functions on $[0, +\infty)$ that vanish at infinity in the CIR case, or bounded and continuous functions on $[-1, 1]$ in the Jacobi case. The Chapman–Kolmogorov equation for the transition density p_1 implies that the operators $\{T(t),\ t \geq 0\}$ form a semigroup: $T(t)T(s) = T(t + s)$. In view of [16, Theorem 3.4, p. 112], the operators $T(t)$ form a uniformly bounded semigroup on the respective Banach space of continuous functions, with the supremum norm. In addition, for any fixed $y \in (l, L)$

$$\begin{aligned} T(t)g(y) - g(y) &= \int_l^L p_1(x, t; y)(g(x) - g(y)) dx \\ &= \int_{|x-y|\le\epsilon\cap(l,L)} p_1(x, t; y)(g(x) - g(y)) dx + \int_{|x-y|>\epsilon\cap(l,L)} p_1(x, t; y)(g(x) - g(y)) dx \\ &\le \sup_{|x-y|\le\epsilon\cap(l,L)} |g(x) - g(y)| \int_{|x-y|\le\epsilon\cap(l,L)} p_1(x, t; y) dx + C \int_{|x-y|>\epsilon\cap(l,L)} p_1(x, t; y) dx \end{aligned}$$

since function g is bounded. From the properties of the diffusion processes, $\int_{|x-y|>\epsilon\cap(l,L)} p_1(x, t; y)dx \to 0$ as $t \to 0$ for any $\epsilon > 0$ [18], therefore the second term in the above expression tends to zero as $t \to 0$. The first term is bounded by $\sup_{|x-y|\le\epsilon\cap(l,L)} |g(x) - g(y)|$, which tends to zero as $\epsilon \to 0$ because of the continuity of g. Therefore we have a pointwise continuity of the semigroup in the sense that, for every fixed $y \in (l, L)$, $T(t)f(y) \to f(y)$ as $t \to 0$. Lemma 6.7, p. 241 in [34] yields strong continuity of the semigroup: $\|T(t)g - g\| \to 0$ as $t \to 0$. Therefore, $\{T(t),\ t \geq 0\}$ is a Feller–Dynkin semigroup in the sense of [34, Definition 6.5, p. 241]: a strongly continuous and uniformly bounded contraction semigroup on the respective Banach space of continuous functions.

In the notation of Theorem 3.2, $T(t)g(y) = u_1(t; y)$ solves the Cauchy problem

$$\frac{\partial u}{\partial t} = \mathcal{G} u, \qquad u(0; y) = g(y). \tag{4.7}$$

Since $T(t)$ is a uniformly bounded and strongly continuous semigroup, Theorem 3.1 in [4] shows that

$$S_t g(y) = \int_0^\infty T((s/t)^\alpha) g(y)\, g_\alpha(s) ds = u_\alpha(t; y) \tag{4.8}$$

solves the corresponding fractional Cauchy problem:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \mathcal{G} u, \qquad u(0; y) = g(y). \tag{4.9}$$

Then a change of variables $(t/s)^\alpha = \tau$ in (4.8), together with the density formula (4.4) for the time change E_t, yields

$$\begin{aligned} S_t g(y) &= \frac{t}{\alpha} \int_0^\infty T(\tau) g(y) g_\alpha(t\tau^{-1/\alpha}) \tau^{-1-1/\alpha} d\tau \\ &= \int_0^\infty T(\tau) g(y) f_t(\tau) d\tau \\ &= \int_0^\infty E[g(X_1(\tau))|X_1(0) = y] f_t(\tau) d\tau \\ &= E[g(X_1(E_t))|X_1(0) = y] \end{aligned}$$

which is equivalent to (4.6), since $E_0 = 0$ almost surely. $\square$

Remark 4.3. An alternative proof of Theorem 4.2 uses Theorem 3.2 along with the fact that

$$E[g(X_\alpha(t))|X_\alpha(0)=y]=\int_l^L p_\alpha(x,t;y)g(x)dx$$

in view of Lemma 4.1.

Next, we apply this transition density to solve the fractional forward equation (3.13).

Corollary 4.4. *Let $p_\alpha(x,t;y)$ be the transition density (3.8) of a fractional Pearson diffusion of OU, CIR, or Jacobi type. For any function f that satisfies the conditions of Theorem 3.3 (see also Remark 3.4), the function*

$$p_\alpha(x,t)=\int_l^L p_\alpha(x,t;y)f(y)dy \tag{4.10}$$

solves the fractional Cauchy problem (3.13) with initial condition $p_\alpha(x,0)=f(x)$.

Proof. This follows immediately from Theorem 3.2 and Lemma 4.1. □

Remark 4.5. If the initial function f in Corollary 4.4 is the probability density of $X_\alpha(0)$, then the solution (4.10) of the forward (Fokker–Planck) equation gives the probability density of $X_\alpha(t)$. Furthermore, the function

$$f_\alpha(t,x,y)=p_\alpha(x,t;y)f(y)=\mathbf{m}(x)\sum_{n=0}^{\infty}E_\alpha(-\lambda_n t^\alpha)Q_n(x)Q_n(y)f(y)$$

gives the joint density of $x=X_\alpha(t)$ and $y=X_\alpha(0)$. For instance, when $f=\mathbf{m}$, the conditions of Theorem 3.3 are satisfied (see Remark 3.4).

The fractional OU process is obtained when the drift is linear and the diffusivity is constant:

$$\begin{cases}\mu(x)=\theta\mu-\theta x,\theta>0,\\ \sigma^2(x)=2\theta\sigma^2.\end{cases} \tag{4.11}$$

The stationary density (2.8) is normal with mean μ and variance σ^2. The process X_α is not Gaussian, even if the initial function is Gaussian, due to the random time change. However, the next result shows that, the stationary distribution of the fractional OU process is the same as the traditional OU process.

Theorem 4.6. *Let $X_1(t)$ be an Ornstein–Uhlenbeck process, the solution of Eq. (2.1) with coefficients given by Eq. (4.11). Let E_t be the inverse (4.2) of a standard stable subordinator given by Eq. (4.1) independent of the process X_1. Given any initial density $f(x)$ for $X_\alpha(0)$ that satisfies the conditions of Theorem 3.3 (see also Remark 3.4), the density (4.10) of the fractional Ornstein–Uhlenbeck process $X_\alpha(t)=X_1(E_t)$ is asymptotically normal:*

$$p_\alpha(x,t)\to\ \mathbf{m}(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\quad \text{as } t\to\infty. \tag{4.12}$$

Proof. In the OU case, the eigenvalues are $\lambda_n=\theta n$, and the orthonormal polynomials are normalized orthogonal Hermite polynomials given by the Rodrigues formula

$$\bar{H}_n(x)=(-1)^n\frac{\sigma^n}{\sqrt{n!}}e^{\frac{(x-\mu)^2}{2\sigma^2}}\frac{d^n}{dx^n}\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right). \tag{4.13}$$

For example, the first four normalized Hermite polynomials are:

$$\bar{H}_0(x)=1,$$
$$\bar{H}_1(x)=-\frac{1}{\sigma}(-x+\mu),$$
$$\bar{H}_2(x)=\frac{1}{\sigma^2\sqrt{2}}\left((x-\mu)^2-\sigma^2\right),$$
$$\bar{H}_3(x)=-\frac{1}{\sigma^3\sqrt{6}}\left(-(x-\mu)^3+3\sigma^2(x-\mu)\right).$$

The fractional Fokker–Planck equation (2.3) becomes

$$\frac{\partial^\alpha p}{dt^\alpha}=\frac{\partial}{\partial x}[\theta(x-\mu)p]+\frac{\partial^2}{\partial x^2}\left[\sigma^2\theta p\right],\qquad t>0,0<\alpha<1,x\in\mathbb{R}, \tag{4.14}$$

and its solution satisfying the initial condition f has the form

$$\begin{aligned} p_\alpha(x,t) &= \int_{-\infty}^{\infty} p_\alpha(x,t;y)f(y)dy = \mathbf{m}(x)\sum_{n=0}^{\infty} E_\alpha(-\theta n t^\alpha)\bar{H}_n(x)f_n \\ &= \mathbf{m}(x)f_0\bar{H}_0(x) + \mathbf{m}(x)\sum_{n=1}^{\infty} E_\alpha(-\theta n t^\alpha)\bar{H}_n(x)f_n. \end{aligned} \tag{4.15}$$

Since $\bar{H}_0(x) = 1$, we have $f_0 = \int_0^\infty f(x)dx = 1$, so the first term in (4.15) is $\mathbf{m}(x)$. For $n \geq 1$, use the asymptotic behavior of the Mittag-Leffler function (Eq. (5.26) in [26]) and Hermite polynomials [36, p. 369] to get

$$|E_\alpha(-\theta n t^\alpha)\bar{H}_n(x)f_n| \leq C(x,\alpha)\frac{1}{t^\alpha}|f_n|/n^{5/4} \to 0 \quad \text{as } t \to \infty.$$

Then a dominated convergence argument yields (4.12), since $\sum_{n=1}^{\infty}|f_n|/n^{5/4} < \infty$. □

The drift and the squared diffusion parameters of the fractional CIR process are given by

$$\mu(x) = -\theta\left(x - \frac{b}{a}\right) \quad \text{and} \quad \sigma^2(x) = \frac{2\theta}{a}x. \tag{4.16}$$

The invariant density (2.9) is a gamma.

Theorem 4.7. *Let $X_1(t)$ be a CIR process, the solution of Eq. (2.1) with coefficients given by Eq. (4.16). Let E_t be the inverse (4.2) of a standard stable subordinator given by Eq. (4.1) independent of the process X_1. Given any initial density f for $X_\alpha(0)$ that satisfies the conditions of Theorem 3.3, the density (4.10) of the fractional CIR process $X_\alpha(t) = X_1(E_t)$ satisfies*

$$p_\alpha(x,t) \to \mathbf{m}(x) = \frac{a^b}{\Gamma(b)}x^{b-1}e^{-ax}, \qquad x > 0 \quad \text{as } t \to \infty. \tag{4.17}$$

Proof. The eigenvalues are $\lambda_n = \theta n$, $\theta > 0$, and the orthonormal Laguerre polynomials are given by the Rodrigues formula:

$$\bar{L}_n^{(b-1)}(x) = (-1)^n\sqrt{\frac{\Gamma(b)}{n!\Gamma(b+n)}}x^{1-b}e^{ax}\frac{d^n}{dx^n}\left(x^{n+b-1}e^{-ax}\right), \qquad n \geq 0. \tag{4.18}$$

The first three normalized Laguerre polynomials are:

$$\begin{aligned} \bar{L}_0^{(b-1)}(x) &= 1, \\ \bar{L}_1^{(b-1)}(x) &= \sqrt{\frac{1}{b}}(ax - b), \\ \bar{L}_2^{(b-1)}(x) &= \sqrt{\frac{1}{2b(b+1)}}\left(a^2x^2 - 2a(b+1)x + b(b+1)\right). \end{aligned}$$

The solution of the fractional Fokker–Planck equation

$$\frac{\partial^\alpha p}{dt^\alpha} = \frac{\partial}{\partial x}\left[\theta\left(x - \frac{b}{a}\right)p\right] + \frac{\partial^2}{\partial x^2}\left[\frac{x\theta}{a}p\right], \qquad t > 0,\ x > 0,\ 0 < \alpha < 1$$

with the initial condition f takes the form

$$p_\alpha(x,t) = \mathbf{m}(x)\sum_{n=0}^{\infty} E_\alpha(-\theta n t^\alpha)\bar{L}_n^{(b-1)}(x)f_n.$$

Then (4.17) follows as in the proof of Theorem 4.6. □

The drift and squared diffusion parameters for the Jacobi diffusion are

$$\mu(x) = \frac{\theta}{a+b+2}(-(a+b+2)x + b - a) \quad \text{and} \quad \sigma^2(x) = \frac{2\theta}{a+b+2}\left(1 - x^2\right), \tag{4.19}$$

and the stationary density (2.10) is a beta.

Theorem 4.8. *Let $X_1(t)$ be a Jacobi process, the solution of Eq. (2.1) with coefficients given by Eq. (4.19). Let E_t be the inverse (4.2) of a standard stable subordinator given by Eq. (4.1) independent of the process X_1. Given any initial density f for $X_\alpha(0)$ that satisfies the conditions of Theorem 3.3, the density (4.10) of the fractional Jacobi process $X_\alpha(t) = X_1(E_t)$ satisfies*

$$p_\alpha(x,t) \to \mathbf{m}(x) = (1-x)^a(1+x)^b \frac{\Gamma(a+b+2)}{\Gamma(b+1)\Gamma(a+1)2^{a+b+1}}, \qquad x \in (-1,1) \tag{4.20}$$

as $t \to \infty$.

Proof. Here the eigenvalues are $\lambda_n = n\theta(n+a+b+1)/(a+b+2)$ and the orthonormal polynomials are Jacobi polynomials given by the formula (2.11). The fractional Fokker–Planck equation has the form

$$\frac{\partial^\alpha p}{dt^\alpha} = \frac{\partial}{\partial x}\left[\theta\left(x - \frac{b-a}{a+b+2}\right)p\right] + \frac{\partial^2}{\partial x^2}\left[\frac{\theta(1-x^2)}{a+b+2}p\right], \tag{4.21}$$

for $t > 0$, $-1 < x < 1$, $0 < \alpha < 1$, and its solutions with the initial condition f can be represented as

$$p_\alpha(x,t) = \mathbf{m}(x)\sum_{n=0}^{\infty} E_\alpha(-t^\alpha n\theta(n+a+b+1)/(a+b+2))\bar{P}_n^{(a,b)}(x)f_n. \tag{4.22}$$

Then (4.20) follows as in the proof of Theorem 4.6. □

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References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[2] F. Avram, N.N. Leonenko, L. Rabehasaina, Series expansions for the first passage distribution of Wong–Pearson jump-diffusions, Stoch. Anal. Appl. 27 (4) (2009) 770–796.
[3] F. Avram, N.N. Leonenko, N. Šuvak, Parameter estimation for Fisher–Snedecor diffusion, Statistics 45 (1) (2011) 1–16.
[4] B. Baeumer, M.M. Meerschaert, Stochastic solutions for fractional Cauchy problems, Fract. Calc. Appl. Anal. 4 (2001) 4811–5000.
[5] B. Baeumer, M.M. Meerschaert, E. Nane, Brownian subordinators and fractional Cauchy problems, Trans. Amer. Math. Soc. 361 (2009) 3915–3930.
[6] B.M. Bibby, M.I. Skovgaard, M. Sørensen, Diffusion-type models with given marginal distribution and autocorrelation function, Bernoulli 11 (2) (2005) 191–220.
[7] N.H. Bingham, Limit theorems for occupation times of Markov processes, Z. Wahrscheinlichkeit. 17 (1971) 1–22.
[8] L. Bondesson, G. Kristiansen, F. Steutel, Infinite divisibility of random variables and their integer parts, Statist. Probab. Lett. 28 (1996) 271–278.
[9] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, Geophys. J. R. Astron. Soc. 13 (1967) 529–539.
[10] P. Chakraborty, M.M. Meerschaert, C.Y. Lim, Parameter estimation for fractional transport: a particle tracking approach, Water Resour. Res. 45 (2009) W10415.
[11] J.C. Cox, J.E. Ingersoll Jr., S.A. Ross, A theory of the term structure of interest rates, Econometrica 53 (1985) 385–407.
[12] S.D. Eidelman, S.D. Ivasyshen, A.N. Kochubei, Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type, Birkhäuser, Basel, 2004.
[13] A. Einstein, On the theory of the Brownian movement, Ann. Phys. 4 (19) (1906) 371–381.
[14] T. Erdélyi, A.P. Magnus, P. Nevai, Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994) 602–614.
[15] J.L. Forman, M. Sørensen, The Pearson diffusions: a class of statistically tractable diffusion processes, Scand. J. Stat. 35 (2008) 438–465.
[16] A. Friedman, Stochastic Differential Equations and Applications, vol. 1, Academic Press, New York, 1975.
[17] R. Gorenflo, F. Mainardi, Fractional diffusion processes: probability distribution and continuous time random walk, Lecture Notes in Phys. 621 (2003) 148–166.
[18] S. Karlin, H.M. Taylor, A Second Course in Stochastic Processes, Academic Press, 1981.
[19] A.N. Kochubei, A Cauchy problem for evolution equations of fractional order, Differential Equations 25 (1989) 967–974.
[20] A.N. Kolmogorov, On analytical methods in probability theory, in: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104 (1931) 415–458.
[21] A.M. Krägeloh, Two families of functions related to the fractional powers of generators of strongly continuous contraction semigroups, J. Math. Anal. Appl. 283 (2003) 459–467.
[22] N.N. Leonenko, N. Šuvak, Statistical inference for reciprocal gamma diffusion processes, J. Statist. Plann. Inference 140 (2010) 30–51.
[23] N.N. Leonenko, N. Šuvak, Statistical inference for Student diffusion process, Stoch. Anal. Appl. 28 (6) (2010) 972–1002.
[24] M. Magdziarz, Black–Scholes formula in subdiffusive regime, J. Stat. Phys. 136 (2009) 553–564.
[25] M. Magdziarz, A. Weron, K. Weron, Fractional Fokker–Planck dynamics: stochastic representation and computer simulation, Phys. Rev. E 75 (2007) 016708.
[26] F. Mainardi, R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes, J. Comput. Appl. Math. 118 (1–2) (2000) 283–299.
[27] M.M. Meerschaert, E. Nane, P. Vellaisamy, Fractional Cauchy problems on bounded domains, Ann. Probab. 37 (3) (2009) 979–1007.
[28] M.M. Meerschaert, H.-P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, J. Appl. Probab. 41 (2004) 623–638.
[29] M.M. Meerschaert, A. Sikorskii, Stochastic Models for Fractional Calculus, in: De Gruyter Studies in Mathematics, vol. 43, De Gruyter, Berlin, ISBN: 978-3-11-025869-1, 2012.
[30] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.

[31] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A 37 (2004) R161–R208.
[32] K. Pearson, Tables for Statisticians and Biometricians, Cambridge University Press, Cambridge, UK, 1914.
[33] M.A. Pinsky, Pointwise Fourier inversion and related Jacobi polynomial expansions, in: V. Bergelson, P. March, J. Rosenblatt (Eds.), Convergence in Ergodic Theory and Probability, Walter de Gruyter, 1996.
[34] L.C.G. Rogers, D. Williams, Diffusions, Markov Processes, and Martingales, vol. 1, John Willey & Sons, 1994.
[35] W. Rudin, Principles of Mathematical Analysis, third ed., McGraw-Hill, 1976.
[36] G. Sansone, Orthogonal Functions, Interscience Publishers, New York, 1959.
[37] E. Scalas, Five years of continuous-time random walks in econophysics, Complex Netw. Econ. Interactions 567 (2006) 3–16.
[38] R. Schumer, D.A. Benson, M.M. Meerschaert, B. Baeumer, Fractal mobile/immobile solute transport, Water Resour. Res. 39 (2003) 1269.
[39] W.T. Shaw, A. Manir, Dependency without copulas and ellipticity, Eur. J. Finance 15 (7–8) (2009) 661–674.
[40] I.M. Sokolov, J. Klafter, From diffusion to anomalous diffusion: a century after Einstein's Brownian motion, Chaos 15 (2005) 026103.
[41] A.A. Stanislavsky, Black–Scholes model under subordination, Physica A 318 (2009) 469–474.
[42] G. Szego, Orthogonal Polynomials, Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[43] G.E. Uhlenbeck, L.S. Ornstein, On the theory of Brownian motion, Phys. Rev. 36 (1930) 823–841.
[44] E. Wong, The construction of a class of stationary Markov processes, in: R. Bellman (Ed.), Sixteenth Symposium in Applied Mathematics — Stochastic Processes in Mathematical Physics and Engineering, vol. 16, American Mathematical Society, 1964, pp. 264–276.
[45] Y. Zhang, M.M. Meerschaert, B. Baeumer, Particle tracking for time-fractional diffusion, Phys. Rev. E 78 (2008) 036705.