Stochastic Approach to Fractional Diffusion

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Abstract

Fractional diffusion models replace the integer order derivatives in the classical diffusion model by their fractional analogues. Stable stochastic processes can be used for particle tracking, like a Gaussian process is used for classical diffusion. Fractional derivatives in space relate to long particle jumps, in one or more dimensions. Fractional time derivatives relate to long waiting times between jumps. Particle tracking uses a non-Markovian inverse stable subordinator. If waiting times and subsequent particle jumps are correlated, the subordinator is no longer independent of the outer process. This talk reviews the essential theoretical ideas, particle tracking codes, and applications to biology, finance, geophysics, and medicine.

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Classical random walk

$$S(t) = Y_1 + \cdots + Y_{[t]}$$

A particle takes a random jump Y_n at time $t = n$. Particle location at time t is a simple random walk $S(t)$ and scaling limit is a Brownian motion.

$$c^{-1/2} S(ct) \Rightarrow W(t) \approx \underbrace{N(0, \sigma^2 t)} \qquad (c \to \infty)$$

Contract
spatial scale

Expand
time scale

Normal limit
density

Add an advective drift: $L(t) = vt + W(t) \approx N(vt, \sigma^2 t)$

Classical advection and diffusion/dispersion

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2}$$

Fourier transform

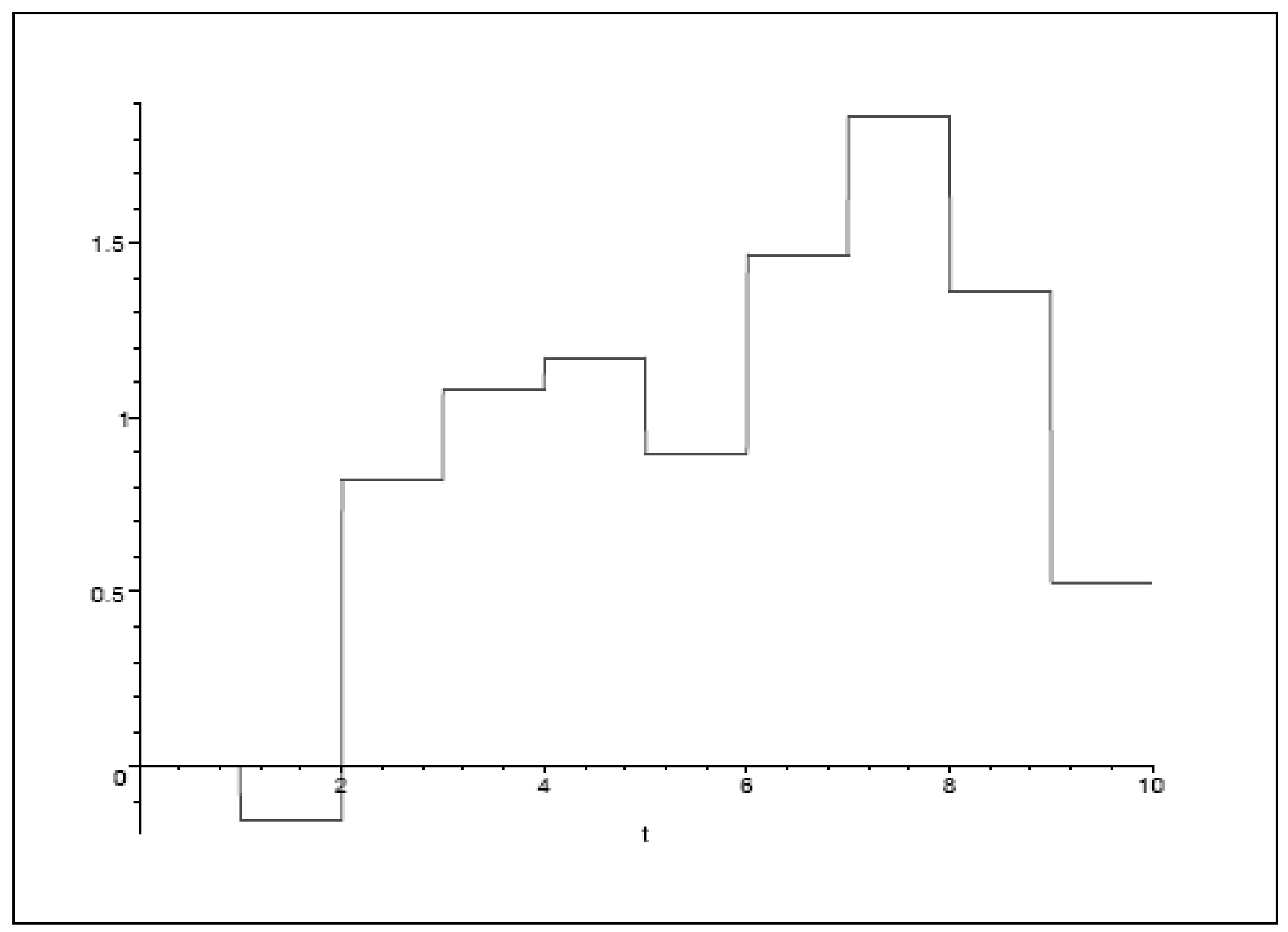
$$\boxed{\frac{d\hat{C}}{dt} = -v(ik)\hat{C} + D(ik)^2\hat{C}}$$

$$\hat{C} = \exp\left(-v(ik)t + D(ik)^2 t\right)$$

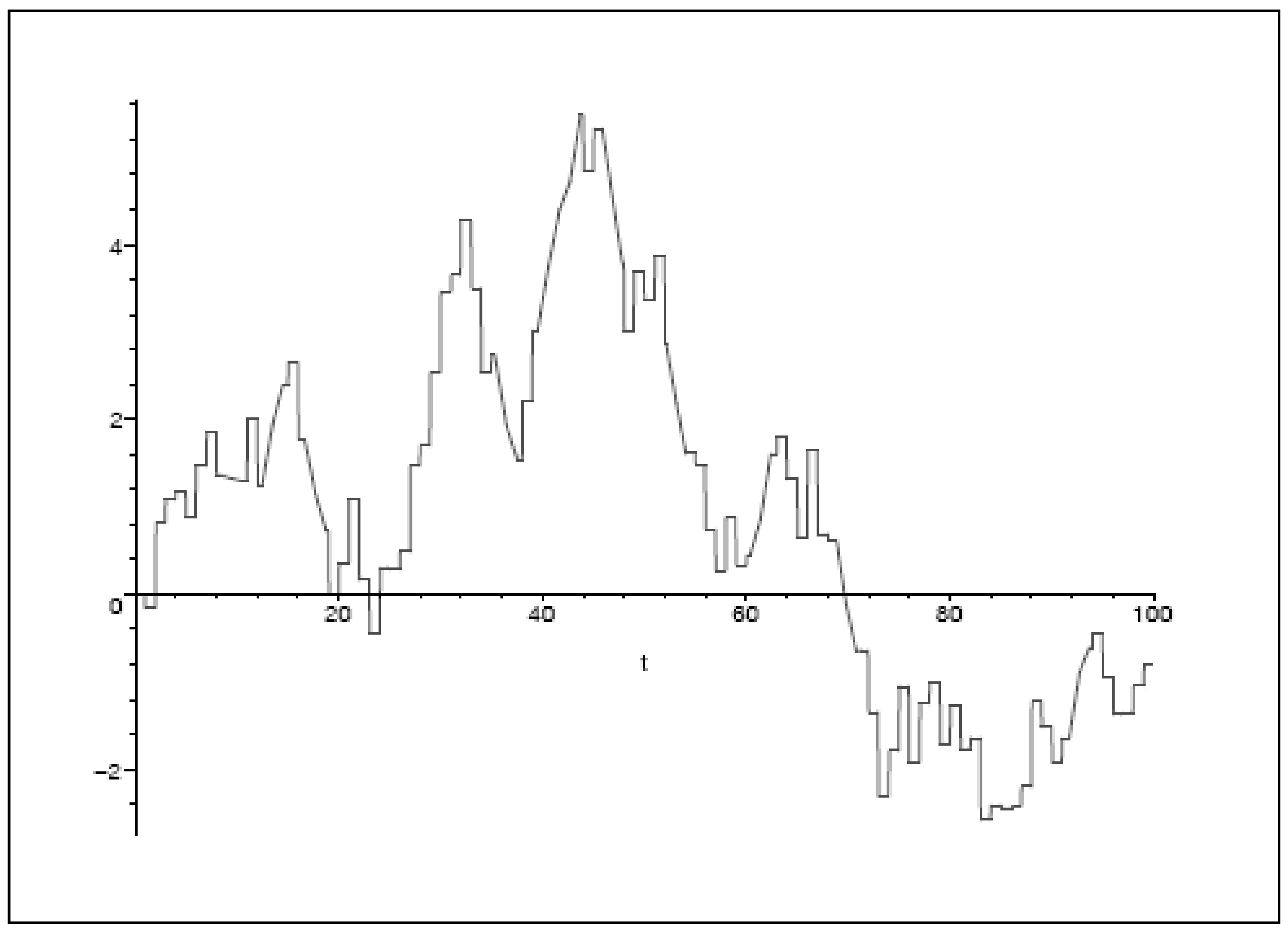
invert

$$\boxed{C(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{(x-vt)^2}{4Dt}\right)}$$

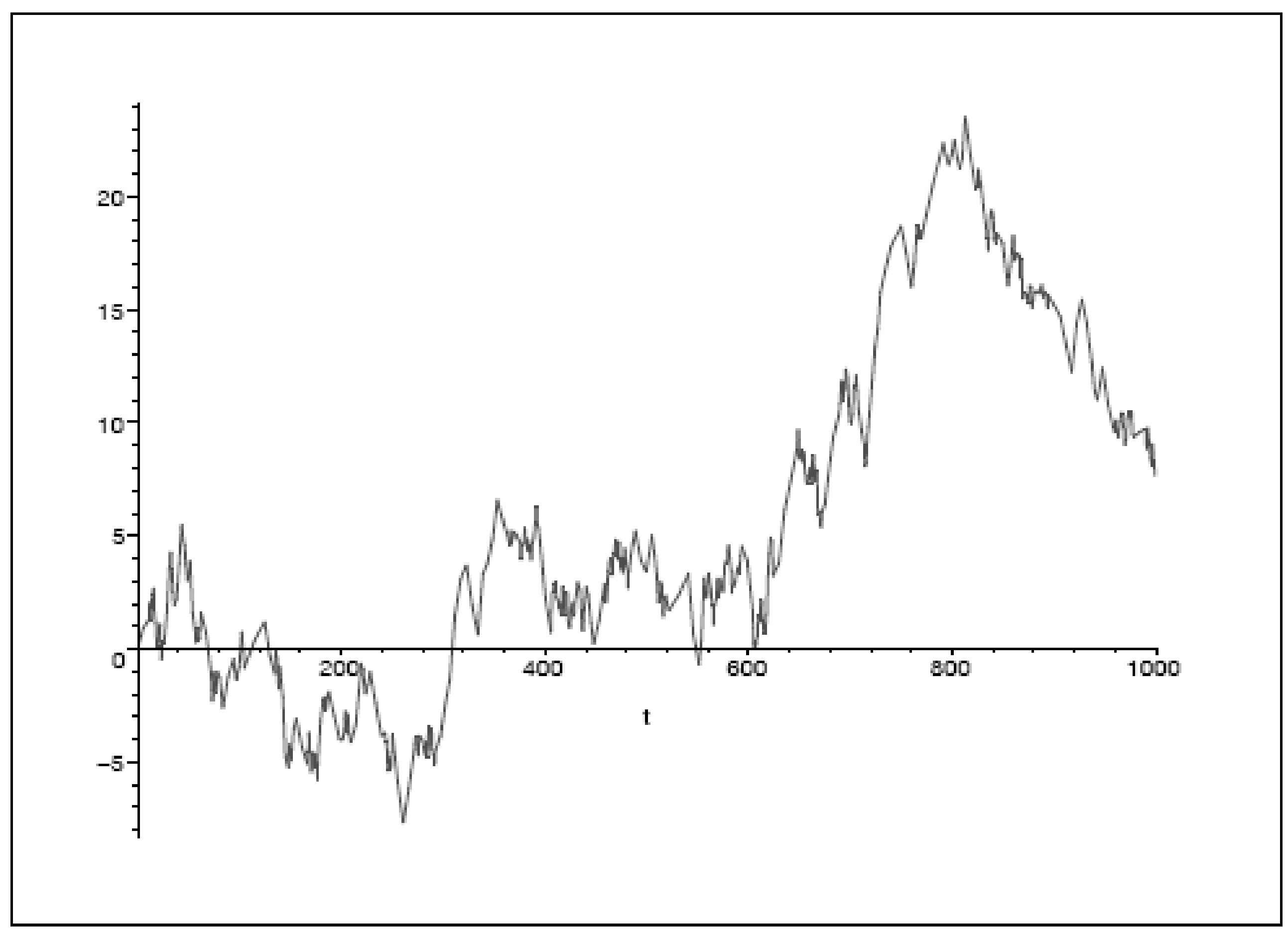
Random walk simulation



Longer time scale



Scaling limit: Brownian motion



Random graph of fractal dimension 1.5 and no jumps.

Heavy tailed particle jumps

For a random walk with heavy tailed particle jumps

$$P(|Y_n| > r) \approx C r^{-\alpha} \qquad (0 < \alpha < 2)$$

the scaling limit is an α-stable Levy motion

$$c^{-1/\alpha} S(ct) \Rightarrow W(t)$$

leading to a Levy motion with drift

$$L(t) = vt + W(t)$$

Fractional advection-dispersion equation (fADE)

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^{\alpha} C}{\partial x^{\alpha}}$$

Fourier transform

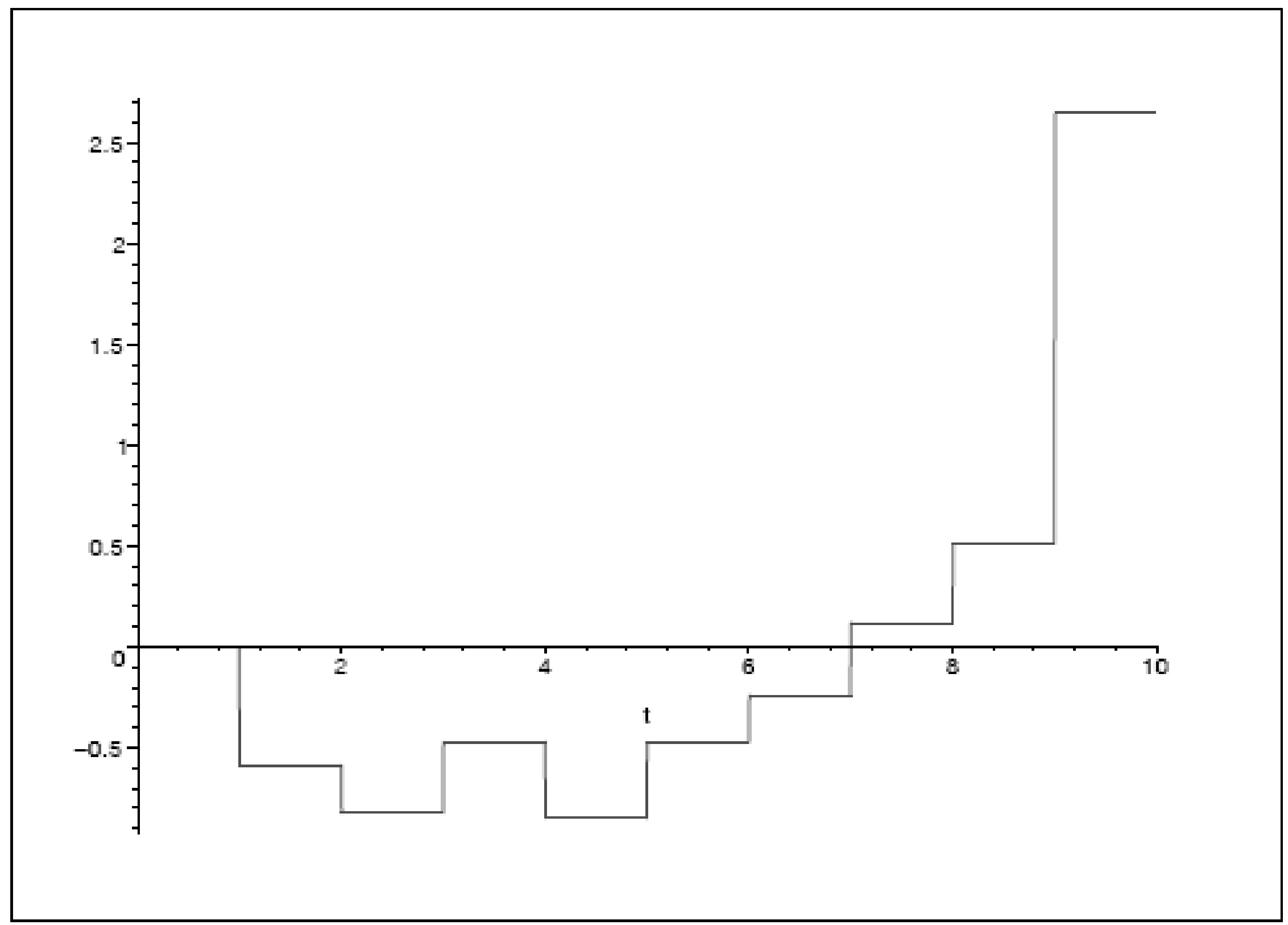
$$\boxed{\frac{d\hat{C}}{dt} = -v(ik)\hat{C} + D(ik)^{\alpha}\hat{C}}$$

$$\hat{C} = \exp\left(-v(ik)t + D(ik)^{\alpha} t\right)$$

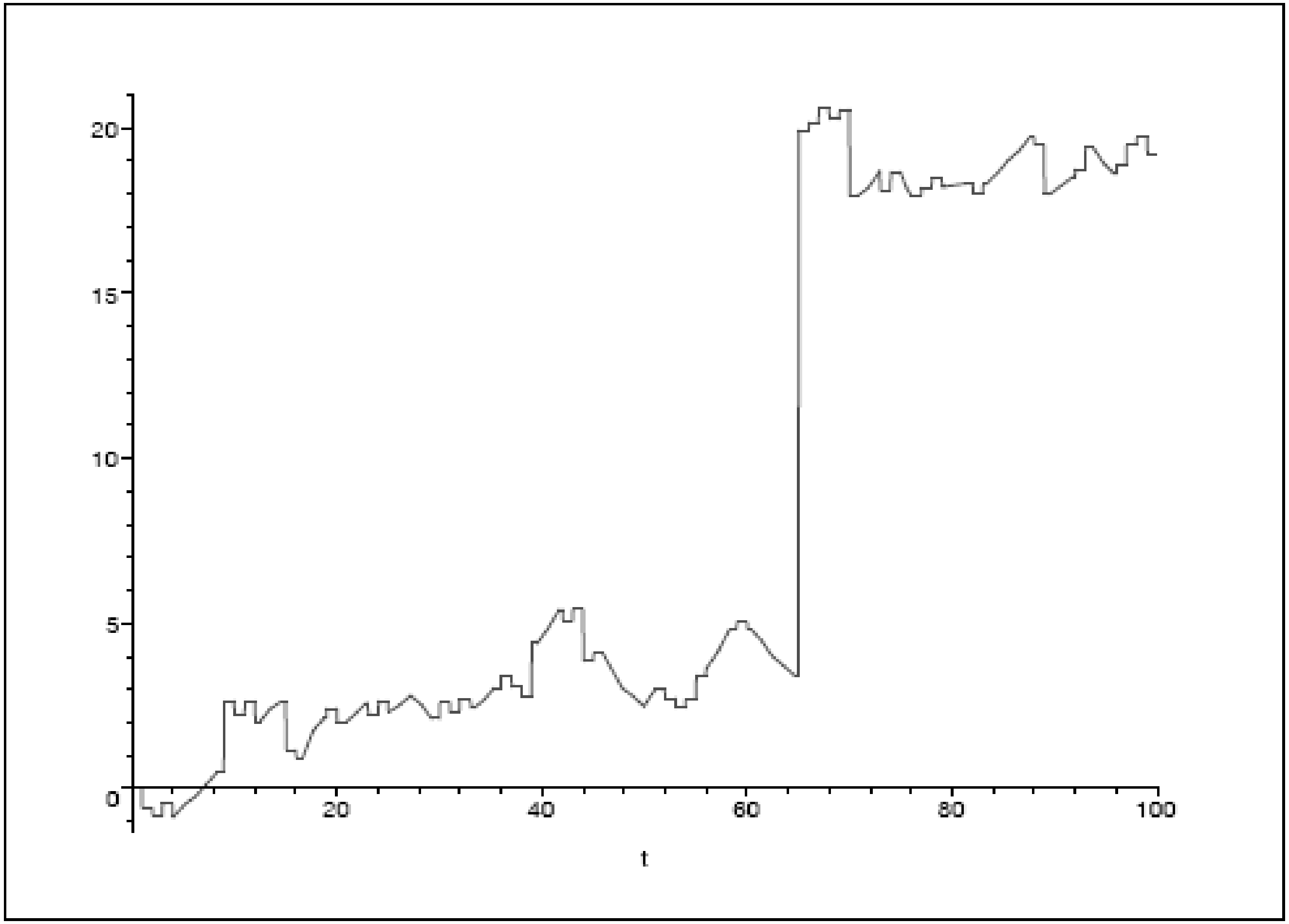
invert

$C(x,t)$ is an α-stable stable density with mean vt

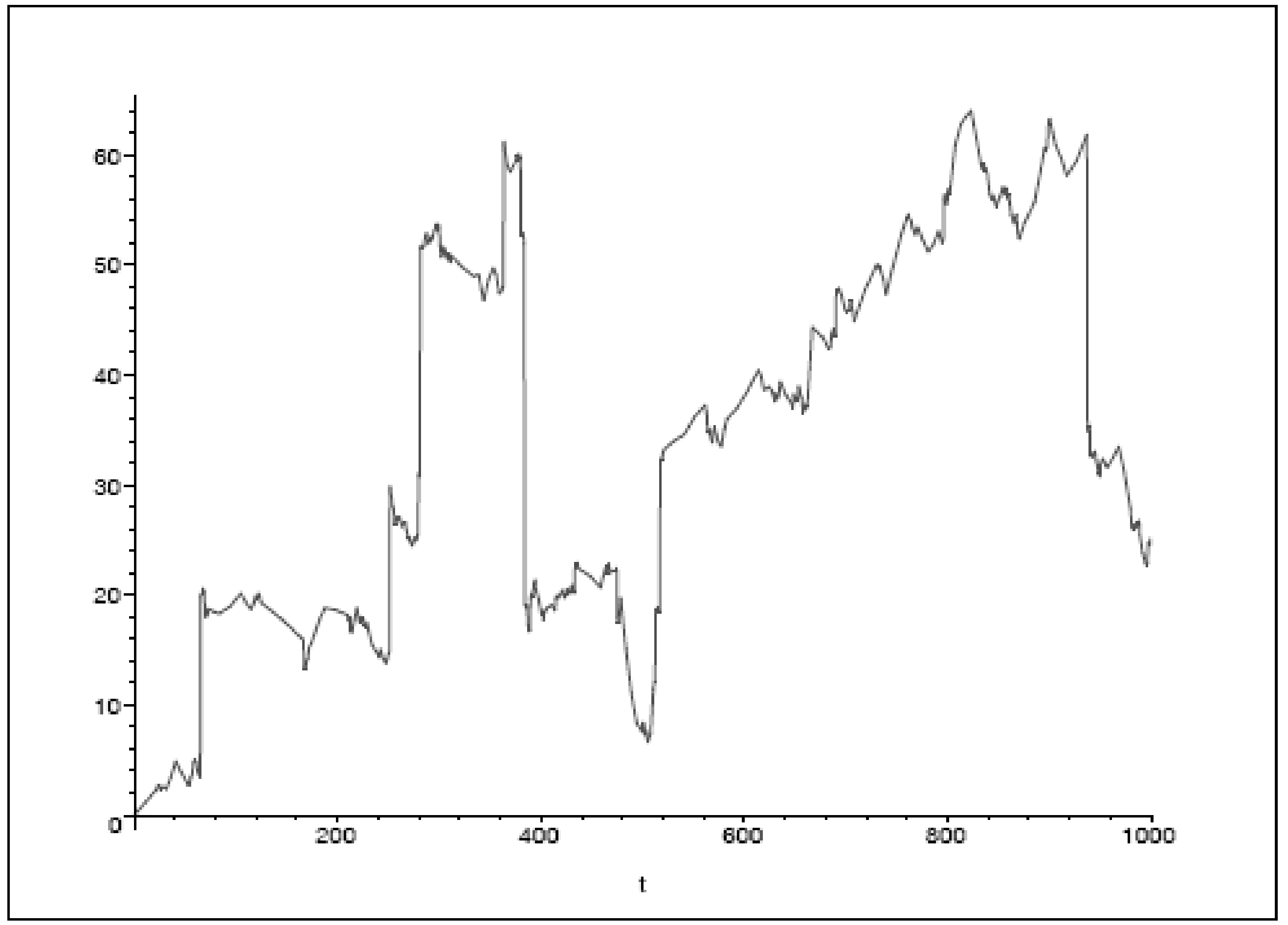
Heavy tailed random walk simulation



Longer time scale

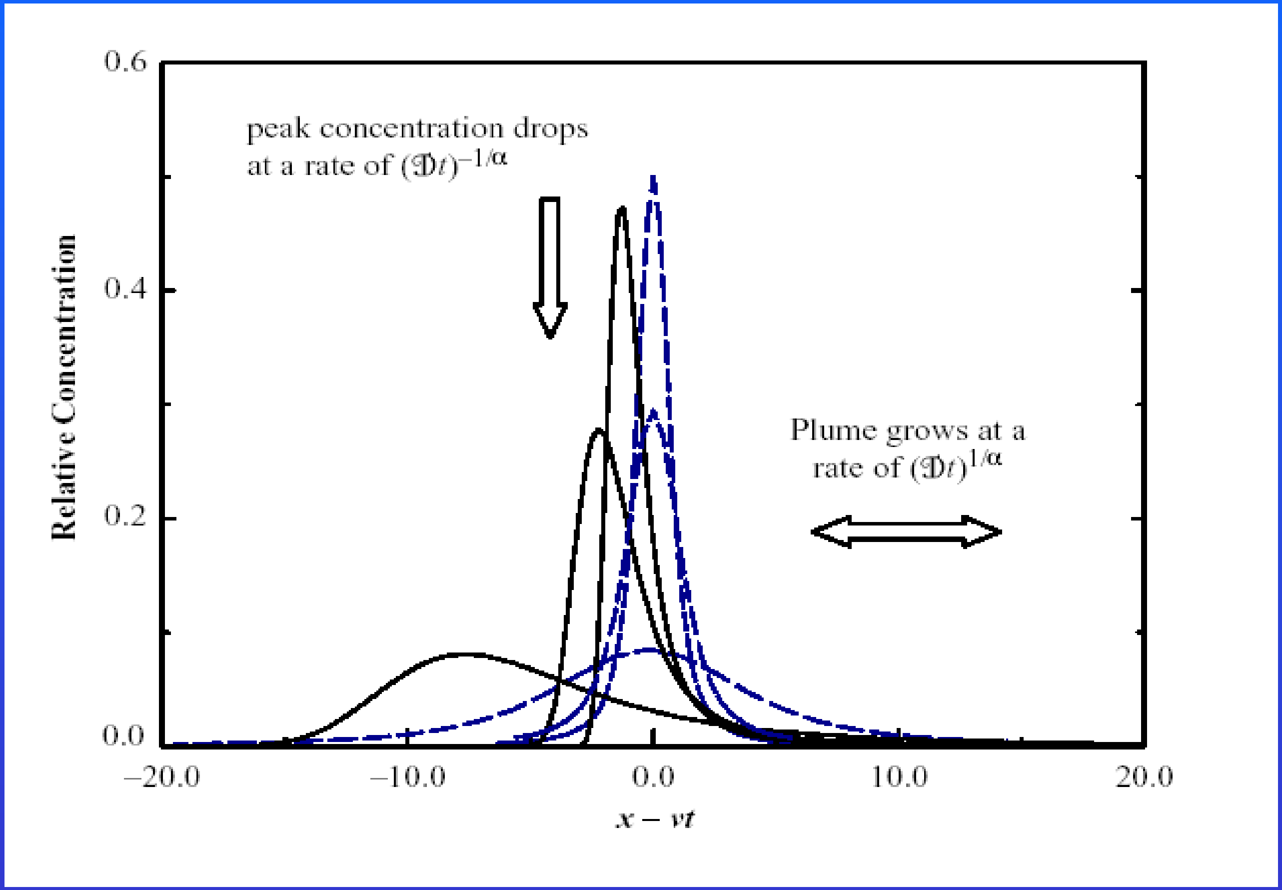


Scaling limit: Stable Levy motion



Random graph of fractal dimension 2-1/α retains jumps.

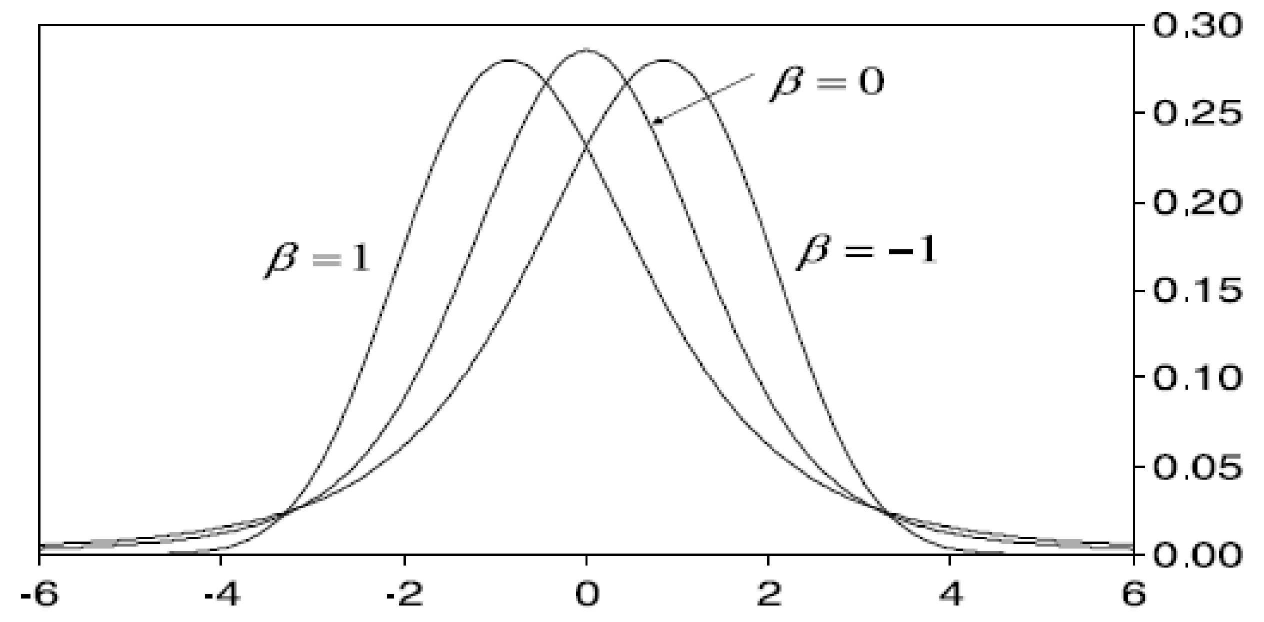
Comparison of normal and stable models



The α-stable stable density has skewness and power-law tails

Positive and negative fractional derivatives

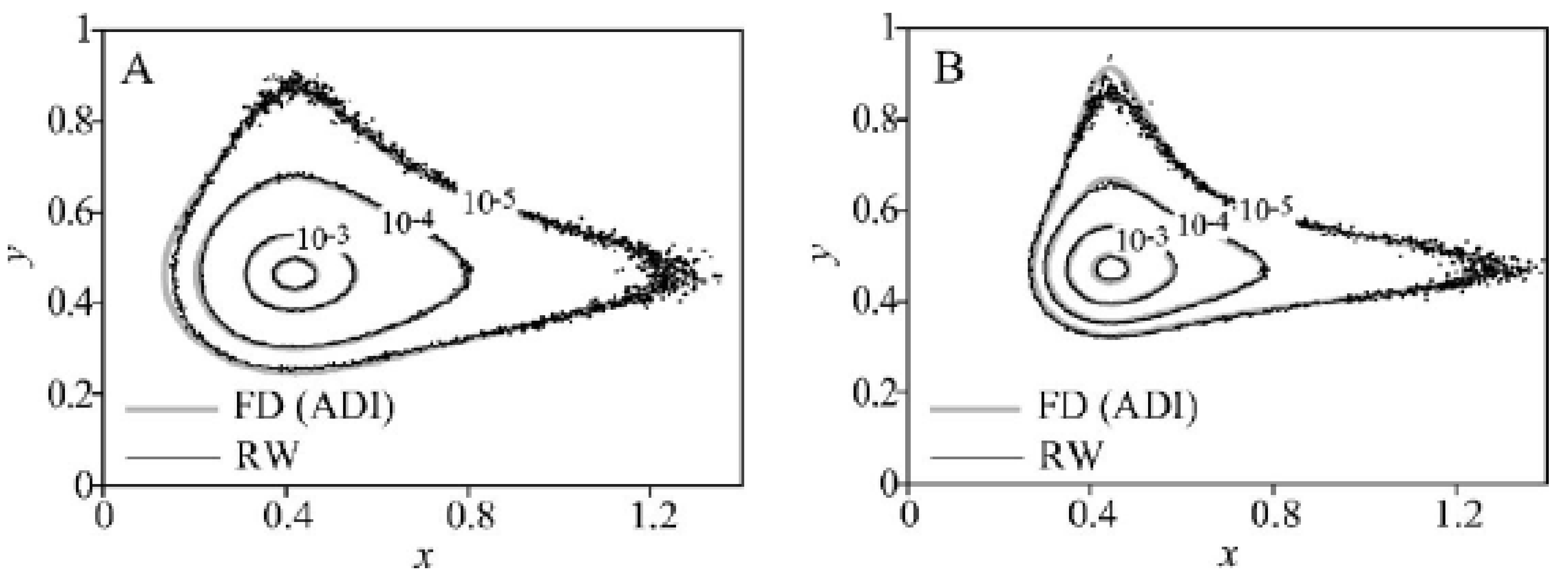
$$\frac{\partial C}{\partial t} = D\left(\frac{1-\beta}{2}\right)\frac{\partial^{\alpha} C}{\partial(-x)^{\alpha}} + D\left(\frac{1+\beta}{2}\right)\frac{\partial^{\alpha} C}{\partial x^{\alpha}}$$



$$\frac{\partial^{\alpha} C}{\partial(-x)^{\alpha}} \to (-ik)^{\alpha}\,\hat{C}$$

Numerical methods (Euler + Lagrange)

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^{1.6} C}{\partial x^{1.6}} + D_y \frac{\partial^{1.8} C}{\partial y^{1.8}}$$



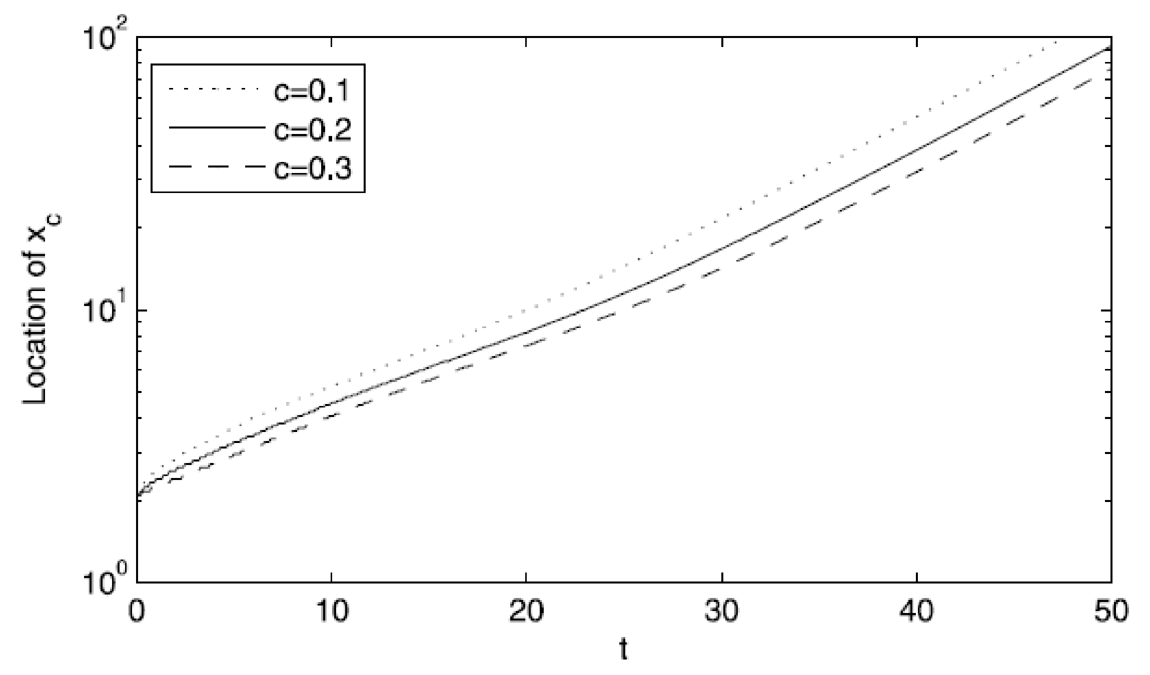
$$D_x = .02 \qquad D_x = .02x$$

$$D_y = .005 \qquad D_x = .005y$$

Exponential moving front

Fractional
Fisher
equation

$$\frac{\partial C}{\partial t} = D \frac{\partial^{\alpha} C}{\partial x^{\alpha}} + rC\left(1 - \frac{C}{K}\right)$$

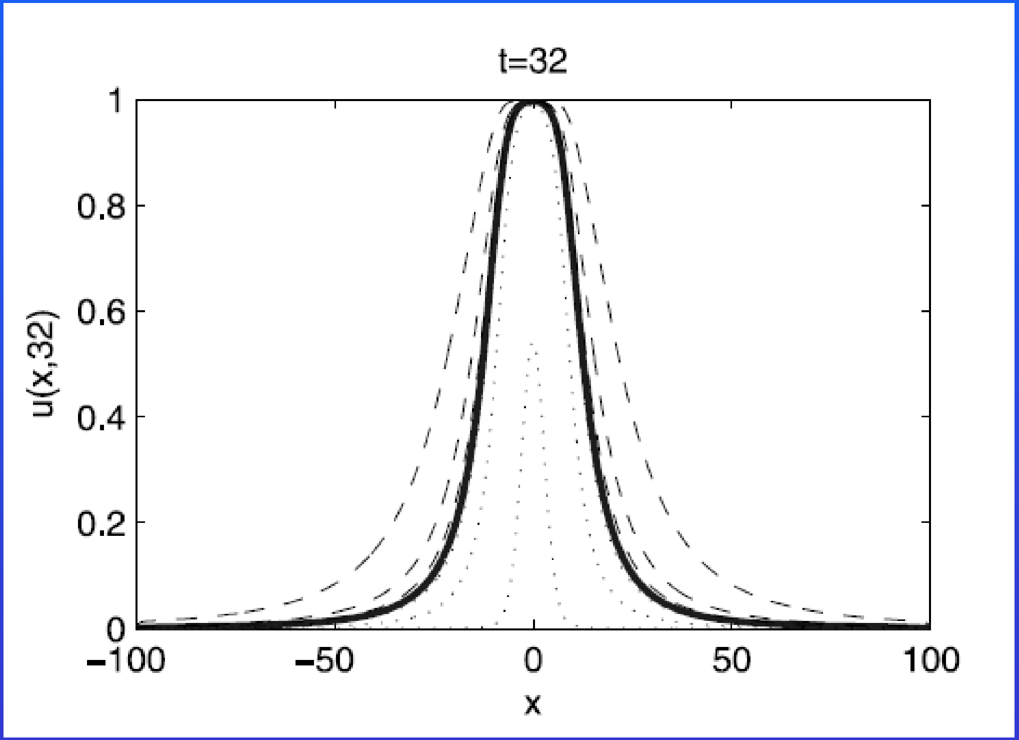


Implicit Euler solution via operator splitting.

Integro-difference equation approximates RDE

$$P(x,t+\Delta) = \int_{-\infty}^{\infty} K(x,y)G(P(y,t))\,dy$$

Symmetric
stable
dispersal
kernel

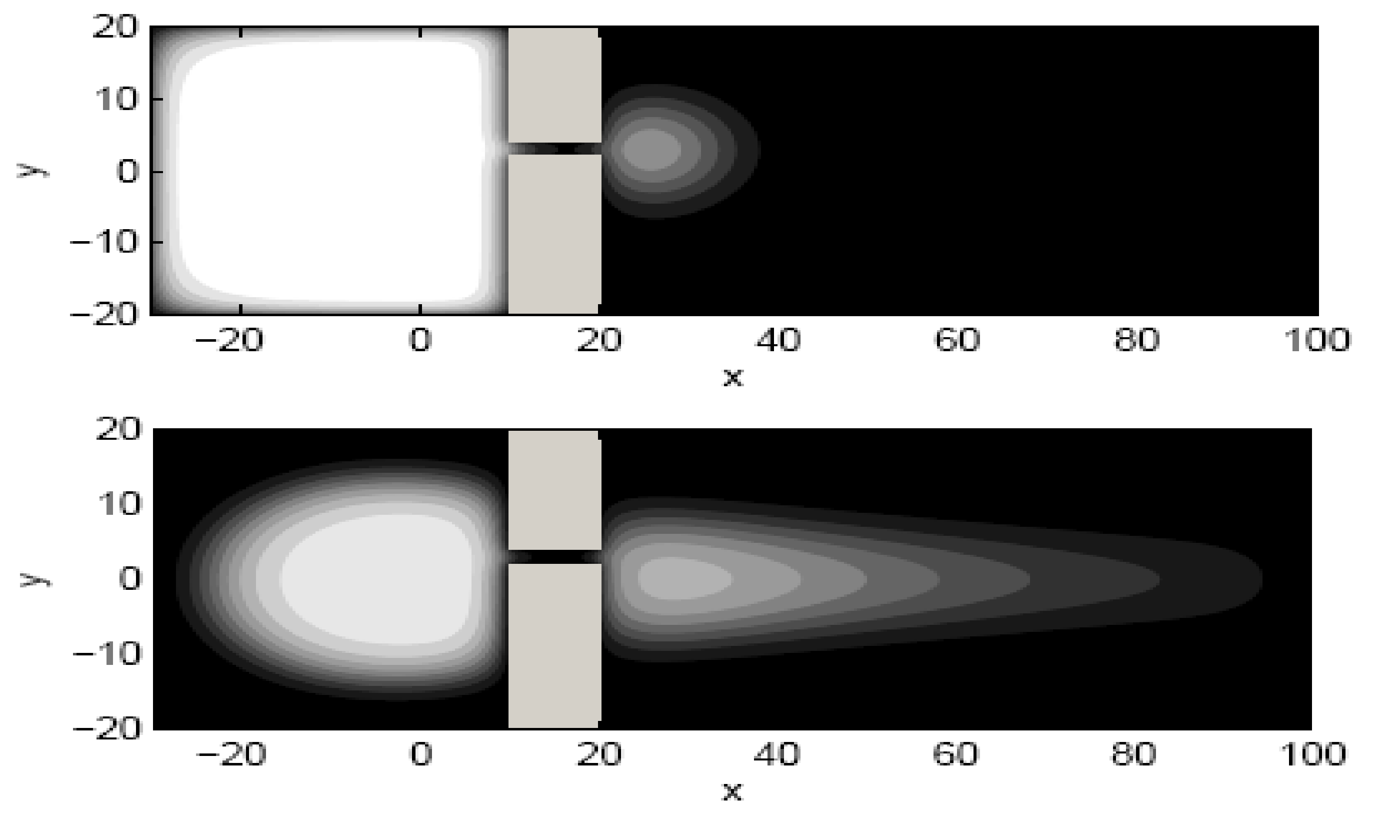


Fisher
growth
equation

Upper bounds (spread, then grow) and lower (grow, then spread).

Reaction-diffusion across a slit barrier

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^\alpha C}{\partial x^\alpha} + D_y \frac{\partial^2 C}{\partial y^2} + rC\left(1 - \frac{C}{K}\right)$$



$\alpha = 2$

$\alpha = 1.7$

Heavy tailed waiting times

Random wait J_n between jumps, nth jump time given by a random walk

$$T(n) = J_1 + \cdots + J_n$$

Number of jumps by time t is inverse $N(t) \geq n \longleftrightarrow T(n) \leq t$

For heavy tail waiting times $P(J_n > t) \approx Ct^{-\beta} \quad (0 < \beta < 1)$

$$c^{-1/\beta} T(ct) \Rightarrow P(t) \quad \Longleftrightarrow \quad c^{-\beta} N(ct) \Rightarrow Q(t)$$

Inverse processes have inverse scaling

$$P(ct) \approx c^{1/\beta} P(t) \quad \Longleftrightarrow \quad Q(ct) \approx c^{\beta} Q(t)$$

Continuous time random walks

Particle location at time *t* follows a CTRW

$$S(N(t)) = Y_1 + \cdots + Y_{N(t)}$$

$$\boxed{N(ct) \approx c^{\beta} Q(t)} \qquad \boxed{c^{-1/\alpha} S(ct) \Rightarrow W(t)}$$

$$\boxed{c^{-\beta/\alpha} S(N(ct)) \approx (c^{\beta})^{-\alpha} S(c^{\beta} Q(t)) \Rightarrow W(Q(t))}$$

Limit is self-similar $\quad W(Q(ct)) \approx c^{\beta/\alpha} W(Q(t))$

Time-fractional model for anomalous sub-diffusion

Caputo derivative

$$\boxed{\frac{\partial^{\beta} C}{\partial t^{\beta}} = -v\frac{\partial C}{\partial x} + D\frac{\partial^{2} C}{\partial x^{2}}}$$

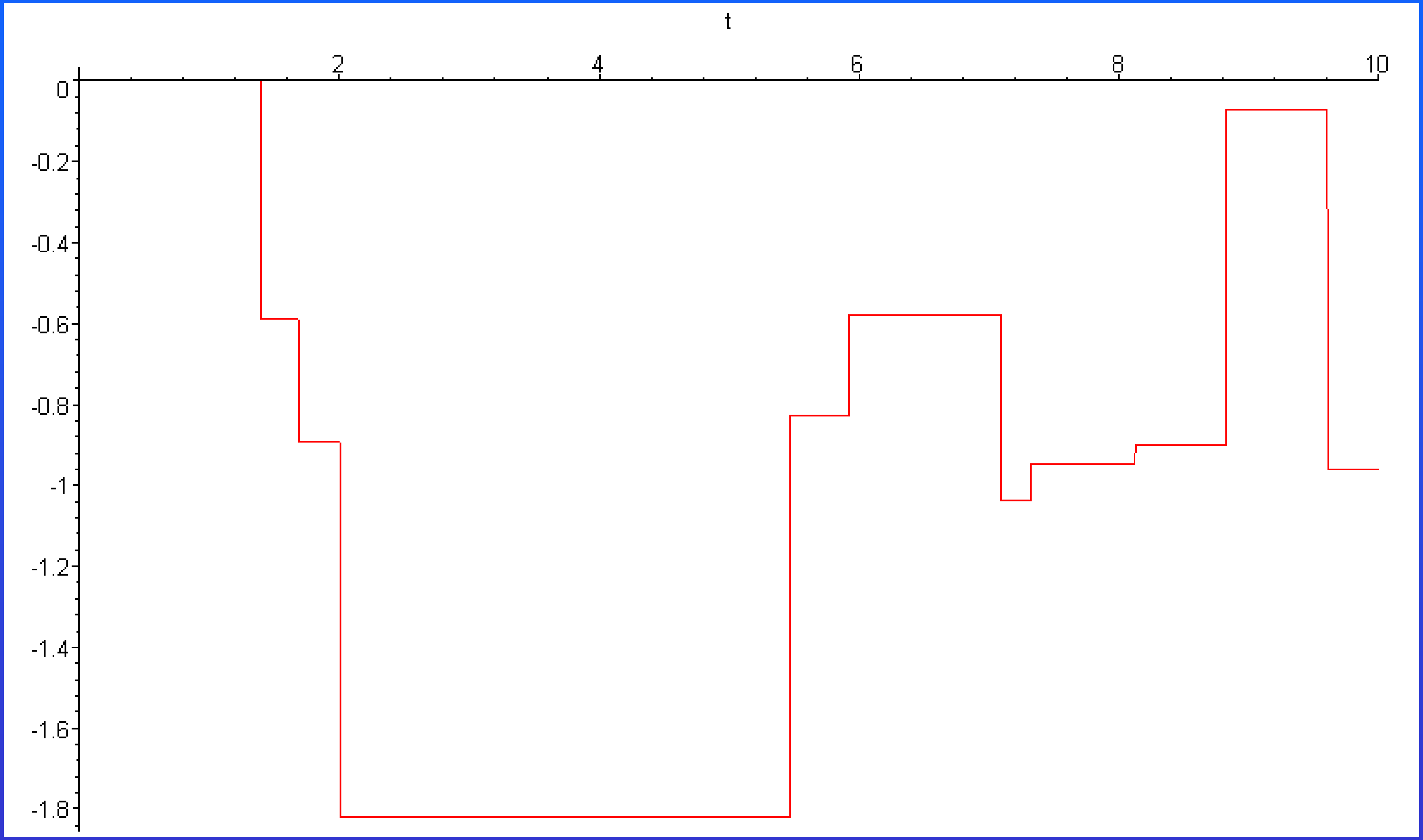
Semi-analytical solution $C(x,t) = \int_{0}^{\infty} p(x,u)h(u,t)\,du$

$p(x,u)$ stable density of $x = W(u)$

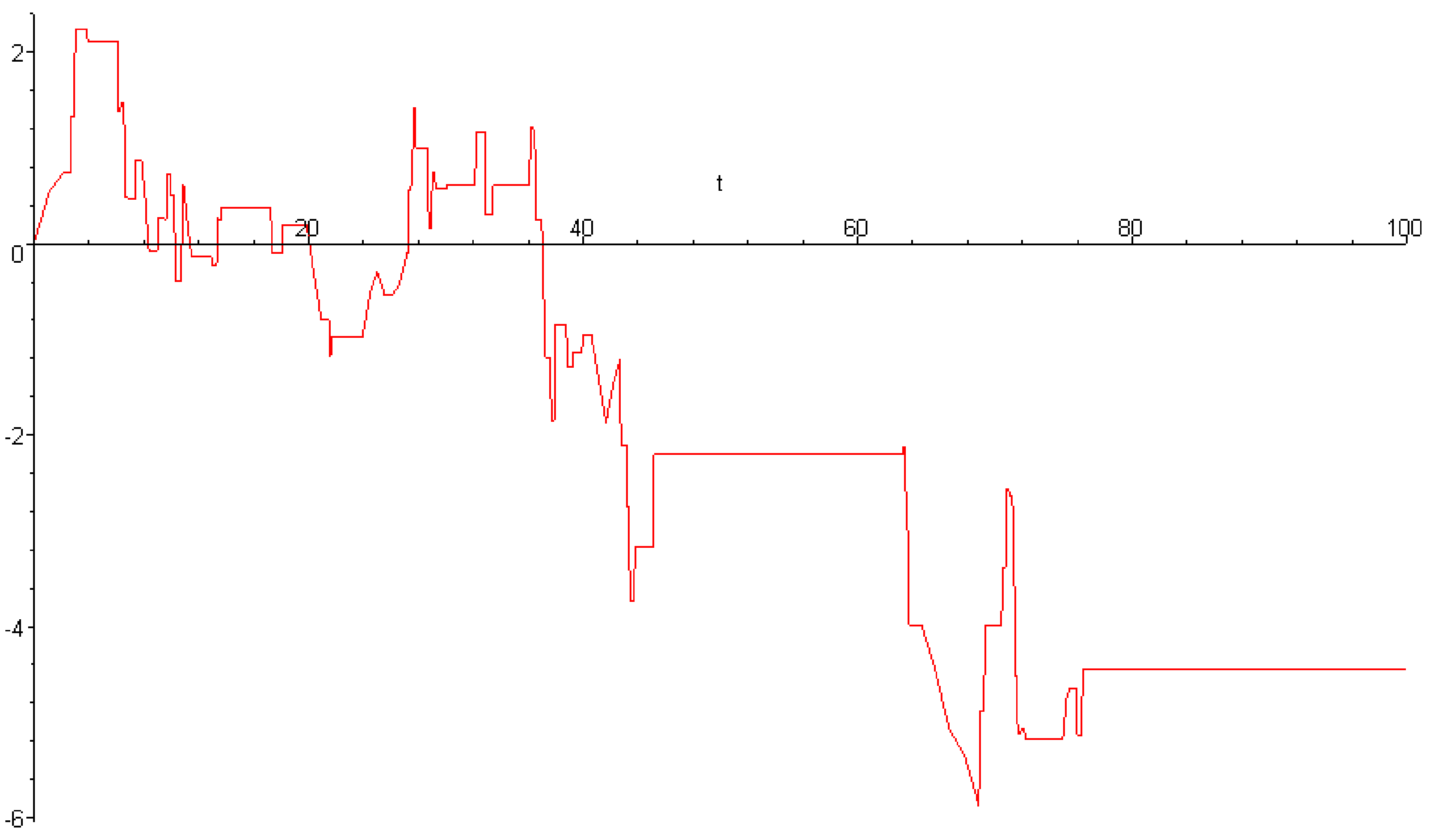
$h(u,t)$ inverse stable density of $u = Q(t)$

Markovian particle tracking $x=W(u)$ and $t=P(u)$.

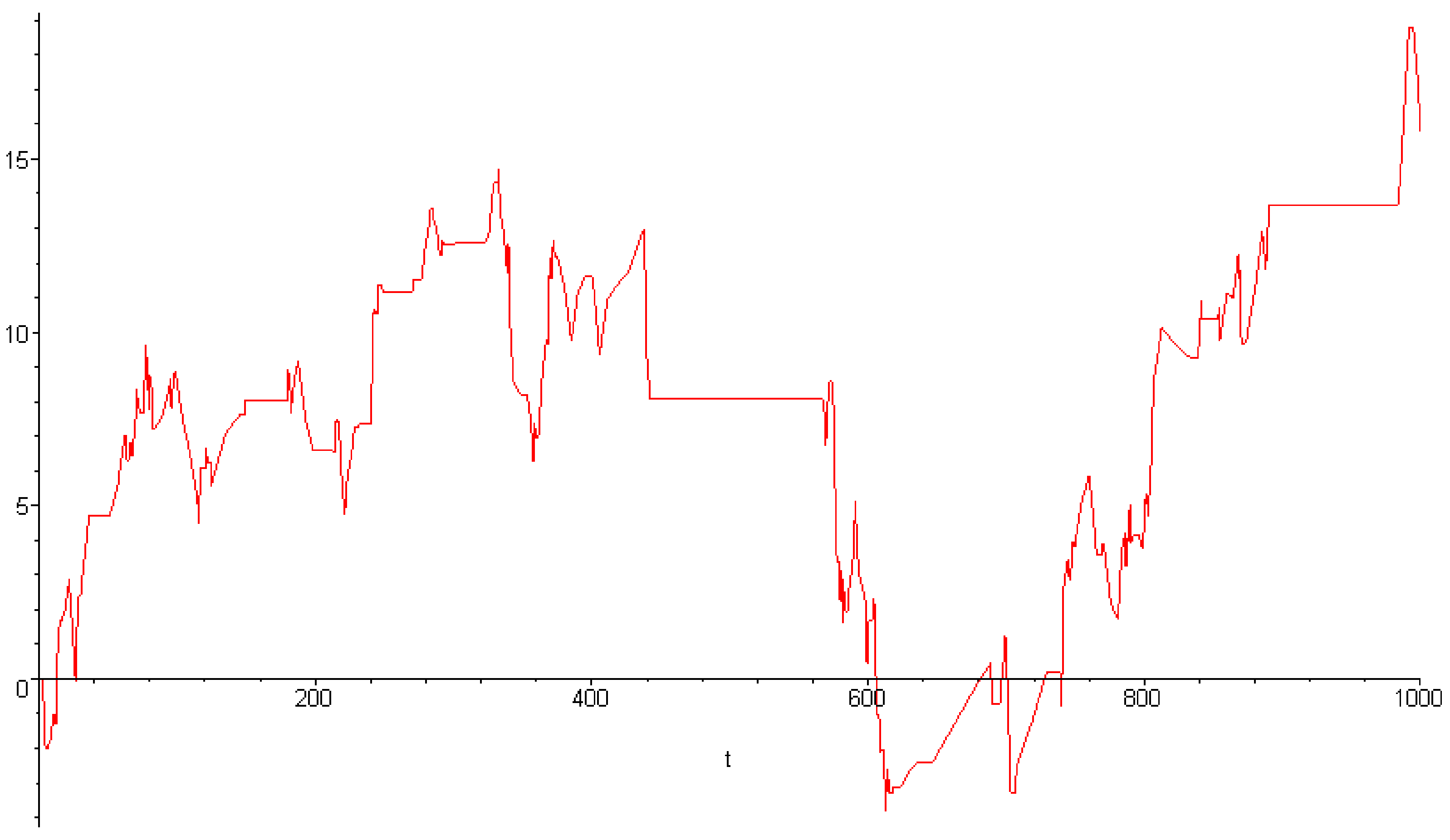
CTRW simulation with heavy tail waiting times



Longer time scale



Scaling limit: Subordinated motion



Limit retains long waiting times.

Coupled space-time derivatives

If waiting times and jumps are dependent random variables, *W(t)* and *Q(t)* are coupled, and so are the space and time fractional derivatives.

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)^{\beta} C(x,t)=\frac{t^{-\beta}\delta(x)}{\Gamma(1-\beta)}$$

This example (Shlesinger, Klafter, and Wong *JSP* 1982) assumes

$$P(J>t)=Ct^{-\beta} \qquad (Y\mid J=t)\approx N(0,2t).$$

Recent applications include tick-by-tick stock prices.

Conclusions

- Power law jumps ⟶ fractional in space
- Particle tracking and finite difference solutions
- Fractional RDE has exponential moving front
- IDE approximates/bounds RDE
- Anomalous dispersion ⟶ jump barriers
- Power law waiting times ⟶ fractional in time
- Open problem: Time fractional RDE

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