Nonlinear Dyn (2012) 70:1273–1281
DOI 10.1007/s11071-012-0532-x

ORIGINAL PAPER

Stochastic solution to a time-fractional attenuated wave equation

Mark M. Meerschaert · Peter Straka · Yuzhen Zhou · Robert J. McGough

Received: 14 April 2012 / Accepted: 9 July 2012 / Published online: 19 July 2012

Abstract The power law wave equation uses two different fractional derivative terms to model wave propagation with power law attenuation. This equation averages complex nonlinear dynamics into a convenient, tractable form with an explicit analytical solution. This paper develops a random walk model to explain the appearance and meaning of the fractional derivative terms in that equation, and discusses an application to medical ultrasound. In the process, a new strictly causal solution to this fractional wave equation is developed.

Keywords Fractional derivative · Wave equation · Complex media · Stable law · Inverse subordinator

M.M. Meerschaert (✉) · P. Straka · Y. Zhou
Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA
e-mail: mcubed@stt.msu.edu

P. Straka
e-mail: straka@msu.edu

Y. Zhou
e-mail: zhouyuzh@msu.edu

R.J. McGough
Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI 48824, USA
e-mail: mcgough@egr.msu.edu

1 Introduction

Sound wave propagation in a heterogeneous medium involves attenuation, which can be frequency dependent. The attenuation coefficient $\alpha(\omega)$ for sound waves in human tissue follows a power law $\alpha(\omega) = \alpha_0|\omega|^y$ depending on the frequency ω, where typically $1 \leq y \leq 1.5$ [1]. Since wave propagation in heterogeneous media is nonlinear with complex interactions, it is useful to develop simplified analytical models that attempt to capture the overall behavior. Fractional derivatives are useful for developing simple analytic models of complex nonlinear dynamics [2–6]. Several competing models for attenuated wave conduction have been proposed, all of which exhibit a power law relation between attenuation and frequency, and all of which rely on fractional derivatives. The Szabo wave equation [7] modifies the traditional wave equation by adding a fractional time derivative term. The power law wave equation [8] modifies the Szabo wave equation by adding a second fractional time derivative term, leading to an exact analytical solution in terms of stable probability densities.

Existence of solutions for time-fractional differential equations is discussed in [9]. The stable probability density is connected to space- and time-fractional diffusion equations in a continuous time random walk framework [10]. Power law waiting times between particle jumps lead to a fractional time derivative, while power law jump lengths lead to a fractional derivative in space. This paper proposes a random walk model for the power law wave equation, to explain the appearance of a stable density in the analytical solution, and provide a simple physical explanation for frequency dependent attenuation of waves propagating through complex media.

2 The power law wave equation

The traditional wave equation $\partial_t^2 p = c_0^2 \Delta_{\boldsymbol{x}} p$ models sound wave propagation in an ideal conducting medium, where c_0 is the speed of sound in that medium, and $p = p(\boldsymbol{x}, t)$ is the pressure. Szabo [7] proposed a model for attenuated (lossy) wave conduction, which was converted to the compact form [11, 12]

$$\Delta_{\boldsymbol{x}} p = \frac{1}{c_0^2}\partial_t^2 p + \frac{2\alpha_0}{c_0 \cos(\pi y/2)}\partial_t^{y+1} p \tag{1}$$

using a Riemann–Liouville fractional derivative in the time variable. This derivative can be most simply described by noting that $\partial_t^y f(t)$ has the Fourier transform $(-i\omega)^y \hat{f}(\omega)$, using the convention $\hat{f}(\omega) = \int e^{i\omega t} f(t)\, dt$. When $y = 2$, the Szabo wave equation (1) reduces to the Blackstock equation [13] for sound wave propagation in a viscous fluid. When $y = 0$, it reduces to the telegrapher's equation.

The power law wave equation [8] adds an additional fractional derivative term:

$$\frac{1}{c_0^2}\partial_t^2 p + \frac{2\alpha_0}{c_0 b}\partial_t^{y+1} p + \frac{\alpha_0^2}{b^2}\partial_t^{2y} p = \Delta_{\boldsymbol{x}} p \tag{2}$$

where $b = \cos(\pi y/2)$ is a constant that changes sign at $y = 1$. For small α_0, the additional term is negligible, leading to an approximate solution to the Szabo wave equation (1).

Equation (2) admits an exact analytical solution in terms of stable densities: Assume a delta function initial condition

$$\frac{1}{c_0^2}\partial_t^2 p + \frac{2\alpha_0}{c_0 b}\partial_t^{y+1} p + \frac{\alpha_0^2}{b^2}\partial_t^{2y} p = \Delta_{\boldsymbol{x}} p + \delta(\boldsymbol{x})\delta(t)$$

and take Fourier transforms to get

$$\frac{1}{c_0^2}(-i\omega)^2 \hat{p} + \frac{2\alpha_0}{c_0 b}(-i\omega)^{y+1}\hat{p} + \frac{\alpha_0^2}{b^2}(-i\omega)^{2y}\hat{p} = \Delta_{\boldsymbol{x}}\hat{p} + \delta(\boldsymbol{x})$$

Rewrite in the form $-k(\omega)^2\hat{p} = \Delta_{\boldsymbol{x}}\hat{p} + \delta(\boldsymbol{x})$ where

$$k(\omega) = \frac{\omega}{c_0} - \frac{\alpha_0(-i)^{y+1}\omega^y}{b}$$

and apply the well-known Green's function solution in three dimensions to arrive at

$$\hat{p}(\boldsymbol{x}, \omega) = \frac{1}{4\pi r}e^{ik(\omega)r} = \frac{e^{i\omega r/c_0}}{4\pi r}\cdot e^{-\alpha_0(r/b)(-i\omega)^y} \tag{3}$$

where $r = \|\boldsymbol{x}\|$ is the radial distance from the source. Since the Fourier transform is a convolution with the forcing function $e^{-i\omega t}\delta(\boldsymbol{x})$, it can be interpreted as the system response to a sinusoidal pressure wave starting at $\boldsymbol{x} = 0$. Since $(-i\omega)^y = |\omega|^y \exp(-i\,\mathrm{sgn}(\omega)\pi y/2)$ has real part $b|\omega|^y$, it follows that this model displays power law attenuation: The first term in (3) models lossless wave motion, and the second term has real part $e^{-\alpha(\omega)r}$ with attenuation coefficient $\alpha(\omega) = \alpha_0|\omega|^y$ for any real frequency ω. The second term is also the Fourier transform of a positively skewed stable density $g_y(t, r)$ with index $0 < y \le 2$, see [14, Propositions 3.10, 3.12]. When $0 < y < 1$, this density is concentrated on the half-line $t > 0$. When $y = 2$, $g_y(t, r)$ is a normal density in t with mean zero and variance $2\alpha_0 r$.

Invert both terms in (3) to arrive at

$$p(\boldsymbol{x}, t) = p_0(\boldsymbol{x}, t) * g_y(t, r) \tag{4}$$

where $p_0(\boldsymbol{x}, t) = \delta(t - r/c_0)/(4\pi r)$ is the Green's function solution of the traditional wave equation in three dimensions, and $*$ denotes a convolution in the t variable. This exact analytical solution (4) to the power law wave equation (2) is causal for $0 < y < 1$, but for $1 \le y \le 2$ the solution is noncausal, since the stable density $g_y(t, r)$ is supported on the entire real line $-\infty < t < \infty$ for every $r > 0$. See [8] for complete details.

3 Continuous time random walks

The stable density is well known from continuous time random walk (CTRW) limit theory. In this stochastic model, each random particle jump J_n is preceded by a random waiting time W_n. The particle arrives at location $S_n = J_1 + \cdots + J_n$ at time $T_n = W_1 + \cdots + W_n$. Since $N(t) = \max\{n : T_n \le t\}$ is the number of jumps by time $t > 0$, the CTRW $S_{N(t)}$ gives the particle location at time $t > 0$. For particle jumps with mean

$\mathbb{E}[J_n] = 0$ and finite variance $\mathbb{E}[J_n^2] = \sigma^2$, the central limit theorem implies that $n^{-1/2} S_{[nt]}$ converges to a Brownian motion $B(t)$, whose probability densities $q(x,t)$ solve the diffusion equation $\partial_t q(x,t) = a\partial_x^2 q(x,t)$ with $\sigma^2 = 2a$. If

$$\mathrm{P}(W > t) \sim \frac{t^{-y}}{\Gamma(1-y)} \quad \text{as } t \to \infty$$

for some $0 < y < 1$, then the extended central limit theorem, Theorem 3.37 in [14], implies that the rescaled random walk in time $n^{-1/y} T_{[nt]}$ converges to the stable subordinator $D(t)$, a Lévy flight whose probability density $g = g_y(x,t)$ is stable with index y. This density solves a space-fractional diffusion equation

$$\partial_u g(u,t) = -\partial_t^y g(u,t);$$

see [14], p. 57.

The central limit theorem for the inverse process, Theorem 3.2 in [15], shows that the inverse process $N(t)$ has an inverse limit: $n^{-y} N(nt)$ converges to

$$E_t = \inf\{u : D(u) > t\}$$

the inverse (hitting time, first passage time) of the process $D(t)$. Since the process $D(t)$ has power-law jumps which are not exponentially distributed, the inverse process E_t rests for power-law distributed lengths of time. Since the residence times in these states are not exponential, the process E_t is not a Markov process.

The rescaled CTRW converges to a time-changed Brownian motion:

$$n^{y/2} S_{N(nt)} \approx (n^y)^{1/2} S_{n^y E_t} \approx B(E_t)$$

The time variable in the Brownian motion is replaced by the inverse or hitting time of a stable motion. The outer process $x = B(u)$ models the jumps, and the inner process $u = E_t$ models the waiting times. See Fig. 1 for an illustration. The particle resting times are evident in this picture.

The densities $h(u,t)$ of the inverse process E_t solve

$$\partial_u h(u,t) = -\partial_t^y h(u,t) + \delta(u)\frac{t^{-y}}{\Gamma(1-y)} \tag{5}$$

To see this, note that since $t = D(u)$ and $u = E_t$ are inverse processes, and $D(u)$ is strictly increasing, we have

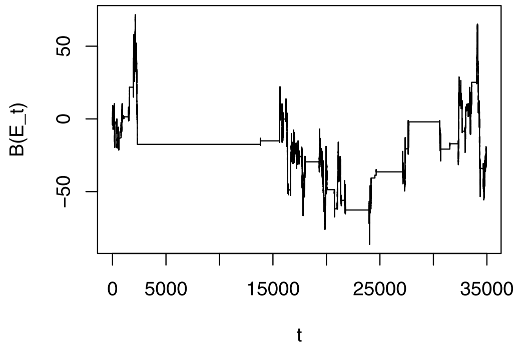

Fig. 1 Typical particle trace for the CTRW limit process $B(E_t)$ in the case $y = 0.7$

$$\mathrm{P}(E_t \le u) = \mathrm{P}\big(D(u) \ge t\big)$$

Then the probability density $h(u,t)$ of E_t is given by

$$\begin{aligned} h(u,t) &= \frac{d}{du}\mathrm{P}(E_t \le u) \\ &= \frac{d}{du}\mathrm{P}\big(D(u) \ge t\big) \\ &= \frac{d}{du}\big[1 - \mathrm{P}\big(D(u) \le t\big)\big] \end{aligned} \tag{6}$$

for $t > 0$ and $u > 0$. The density $g(t,u)$ of $D(u)$ has Laplace transform

$$\tilde{g}(s,u) = \int_0^\infty e^{-st} g(t,u)\,du = e^{-us^y}$$

and so the integral

$$\mathrm{P}\big(D(u) \le t\big) = \int_0^t g\big(t', u\big)\,dt'$$

has Laplace transform $s^{-1}\tilde{g}(s,u)$. Take Laplace transforms in (6) to see that

$$\tilde{h}(u,s) = \frac{d}{du}\big[s^{-1} e^{-us^y}\big] = s^{y-1} e^{-us^y} \tag{7}$$

and then take Laplace transforms in the other variable to see that

$$\bar{h}(\lambda,s) = \int_0^\infty e^{-\lambda u}\tilde{h}(u,s)\,du = \frac{s^{y-1}}{\lambda + s^y}$$

Rewrite in the form

$$s^y \bar{h}(\lambda,s) = -\lambda\bar{h}(\lambda,s) + s^{y-1}$$

and invert the (distributional) Laplace transforms, using the fact that $t^{-y}/\Gamma(1-y)$ has Laplace transform s^{y-1} by Example 2.9 in [14], to see that (5) holds. Note that Eq. (5) can also be viewed as the governing equation of a CTRW with deterministic jumps $J_n = 1$; see [20]. Then $S(n) = n$, the outer process $B(u) = u$, and the CTRW scaling limit $B(E_t) = E_t$. See [21] for additional discussion, and alternative forms of the governing equation (5).

It is also easy to check that the densities $p(x,t)$ of the CTRW limit process $B(E_t)$ solve the fractional Cauchy problem

$$\partial_t^y p(x,t) = a\partial_x^2 p(x,t) + \delta(x)\frac{t^{-y}}{\Gamma(1-y)} \tag{8}$$

The fractional equation (8) was originally applied by Zaslavsky [2] as a model for Hamiltonian chaos. To see that the CTRW densities $p(x,t)$ solve this equation, first note that a simple conditioning argument yields

$$p(x,t) = \int_0^\infty q(x,u)h(u,t)\,du \tag{9}$$

Next, note that the density $q(x,u)$ of the outer process $B(u)$ has the Fourier transform

$$\hat{q}(k,u) = \int e^{ikx} q(x,u)\,dx = e^{-uak^2}$$

by the well-known formula for the Fourier transform of a normal density. Take Laplace and Fourier transforms in (9) to see that the CTRW limit density $p(x,t)$ has Fourier–Laplace transform

$$\bar{p}(k,s) = \int_0^\infty e^{-uak^2} s^{y-1} e^{-us^y}\,du = \frac{s^{y-1}}{s^y + ak^2}$$

rewrite in the form

$$s^y \bar{p}(k,s) = a(ik)^2 \bar{p}(k,s) + s^{y-1}$$

and then invert the Fourier and (distributional) Laplace transforms to arrive at (8). The main insight to be gained here is that the time-fractional derivative of order β codes long waiting times, with a power law probability tail of the same order. See [14], [10], and [22] for more details and multivariate extensions.

4 Subordinator model

A continuous time random walk (CTRW) model for the power law wave equation (2) in three dimensions explains the appearance of the stable density $g_y(t,r)$ in the solution (4). Write out the convolution in the form

$$\begin{aligned} p(\boldsymbol{x},t) &= \int \frac{\delta(t-t'-r/c_0)}{4\pi r} g_y\big(t',r\big)\,dt' \\ &= \frac{\langle \delta(t - D_0(r) - r/c_0)\rangle}{4\pi r} \end{aligned} \tag{10}$$

where $D_0(r)$ is a stable random variable with density $g_y(t,r)$. This is the solution $p_0(\boldsymbol{x},t)$ to the traditional wave equation, with the deterministic time r/c_0 required for a wave to travel distance r at speed c_0 replaced by the random time $D(r) = D_0(r) + r/c_0$. The random variable $D(r)$ represents the time required for a randomly selected packet of wave energy to reach the spherical shell at distance r from the source, taking into account the variations in travel time due to the heterogeneous conducting medium.

Subdivide the interval $[0,r]$ into increments of size Δr. Let $\Delta r^{1/y} W_n + \Delta r/c_0$ represent the time required to traverse the nth spherical shell of thickness Δr. Take

$$\mathrm{P}(W_n > t) \sim \frac{\alpha_0 t^{-y}}{b\Gamma(1-y)} \quad \text{as } t \to \infty \tag{11}$$

and note that $b\Gamma(1-y) > 0$ for both $0 < y < 1$ and $1 < y < 2$, since $b = \cos(\pi y/2)$ also changes sign at $y = 1$. Assume $\langle W_n \rangle = 0$ if $y > 1$. For $y = 2$, suppose that $\langle W_n^2 \rangle = 2\alpha_0$. In this model, the time required for a randomly selected packet of wave energy to reach a distance $r > 0$ from the source is given by $\Delta r^{1/y} W_1 + \Delta r/c_0 + \cdots + \Delta r^{1/y} W_n + \Delta r/c_0 \approx D(r)$, where $n = r/\Delta r$, using the extended central limit theorem [14, Theorem 3.37] when $0 < y < 2$, and the traditional central limit theorem [14, Theorem 3.36] for $y = 2$.

For space-fractional diffusion models in hydrology, the order of the fractional derivative has been connected to the level of heterogeneity in the porous medium [16]. For the power law wave equation, it is hence reasonable to interpret the index y as a measure of heterogeneity. In the case $y = 2$, $D(r)$ is a normal random variable with mean $\langle D(r)\rangle = r/c_0$, and variance $\langle D(r)^2\rangle = 2\alpha_0 r$. The random variable W_n represents deviation from a typical travel time at a nominal length scale. Any distribution with an asymptotically power law probability tail for $0 < y < 2$, or any finite variance distribution for $y = 2$, leads to the same universal stochastic model for lossy wave propagation in a heterogeneous medium, since the (extended) central limit theorem is universal. In human tissue, $1 \le y \le 1.5$ is typical. A smaller y index indicates more extreme variations in travel time, reflecting a more highly heterogeneous conducting medium. The power law attenuation in this stochastic model is a result of random variations in travel time, whose probability distribution reflects medium heterogeneity.

Since the waiting time $\Delta r^{1/y} W_n + \Delta r/c_0$ to traverse the nth spherical shell of thickness Δr is always positive, the total travel time $\Delta r^{1/y} W_1 + \cdots + \Delta r^{1/y} W_n + r/c_0$ is also positive. However, if $1 < y \le 2$, the approximate travel time $D(r)$ can be either positive or negative. This phenomenon is well known in diffusion models: A random walk with positive jumps is approximated by a Brownian motion with positive drift. The random walker remains on the positive half-line, but the normal approximation, used to approximate the position of the random walker after many jumps, spreads over the entire line. Travel time in the microscopic statistical physics model is a random walk that is always positive, but the normal approximation (or stable if $1 < y < 2$) can be negative with some small probability, so the solution is not exactly causal.

If $0 < y < 1$, then the density $g_y(t, r)$ of the random variable $D_0(r)$ is concentrated on the positive half-line $t > 0$, and the solution (10) is causal, as noted in [8]. If $1 < y \le 2$, then $g_y(t, r)$ is positive for every real t, so the solution is noncausal, i.e., we have $p(\boldsymbol{x}, t) > 0$ even for $t < 0$. Next, we consider an alternative stochastic model, which leads to a strictly causal solution for all $0 < y \le 2$, $y \ne 1$.

5 Inverse subordinator model

The stochastic model (10) gives the probability distribution $p(\boldsymbol{x}, t)$ of the random time t at which a packet of wave energy reaches any fixed point $\boldsymbol{x}$ in space. Next, we develop an alternative model, which is simpler to interpret, since $p(\boldsymbol{x}, t)$ measures the random distribution of acoustic pressure in the space variable $\boldsymbol{x}$ at any given time point $t \ge 0$.

The random variable $D(r)$ from the previous section has probability density $p_1(t, r) = g_y(t - r/c_0, r)$, with Fourier transform $\hat{p}_1(\omega, r) = e^{-r\psi(-i\omega)}$ where $\psi(s) = s/c_0 + (\alpha_0/b)s^y = -ik(\omega)$ in our earlier calculations. Then $\partial_r \hat{p}_1 = -\psi(-i\omega)\hat{p}_1$. Inverting the Fourier transform yields $\partial_r p_1(t, r) = -\mathbb{D}_t p_1(t, r)$, where

$$\mathbb{D}_t = \psi(\partial_t) = \frac{1}{c_0}\partial_t + \frac{\alpha_0}{b}\partial_t^y \tag{12}$$

is a pseudo-differential operator [17].

The probability density $h(r, t)$ of the inverse process $E_t = \inf\{r > 0 : D(r) > t\}$ solves the inverse governing equation

$$\mathbb{D}_t h(r, t) = -\partial_r h(r, t) + \delta(r)\left[\frac{\delta(t)}{c_0} + \frac{\alpha_0}{b}\frac{t^{-y}}{\Gamma(1-y)}\right]$$

and hence on the open domain $r > 0, t > 0$ it solves the simpler equation

$$\mathbb{D}_t h(r, t) = -\partial_r h(r, t). \tag{13}$$

For example, if $0 < y < 1$, then since $\mathrm{P}(E_t \le r) = \mathrm{P}(D(r) \ge t)$, the density

$$h(r, t) = \frac{d}{dr}\left[1 - \mathrm{P}\big(D(r) \le t\big)\right]$$

has Laplace transform (set $s = -i\omega$)

$$\hat{h}(r, s) = -\frac{d}{dr}\left[s^{-1}e^{-r\psi(s)}\right] = s^{-1}\psi(s)e^{-r\psi(s)}$$

so that $\partial_r \hat{h} = -\psi(s)\hat{h}$. Invert to see that (13) holds. See [18, Eq. (25)] for extensions and explicit boundary conditions. The governing equation (13) remains valid for $1 < y \le 2$, but the situation is more complicated because $D(r)$ is not strictly increasing, see [19, Eq. (5)]. Since $t = D(r)$ represents the random time required for a packet of wave energy to travel a radial distance r from the source, the inverse process $r = E_t$ represents the random distance traveled by time t.

Now suppose that $q(\boldsymbol{x}, t)$ solves the traditional wave equation $\partial_t^2 q = \Delta_{\boldsymbol{x}} q$ with speed of sound $c_0 = 1$, and initial conditions $q(\boldsymbol{x}, t = 0) = 0$, $\partial_t q(\boldsymbol{x}, t = 0) = f(\boldsymbol{x})$. Then we will show that for $t > 0$

$$p(\boldsymbol{x}, t) = \int_0^\infty q(\boldsymbol{x}, u)h(u, t)\,du = \big\langle q(\boldsymbol{x}, E_t)\big\rangle \tag{14}$$

solves the power law wave equation with boundary term

$$\mathbb{D}_t^2 p(\boldsymbol{x}, t) = \Delta_{\boldsymbol{x}} p(\boldsymbol{x}, t) + f(\boldsymbol{x})\frac{\alpha_0 t^{-y}}{b\Gamma(1-y)} \tag{15}$$

for any $0 < y < 1$ or $1 < y < 2$, using the time-fractional operator notation (12). The solution (14) is causal, because the inverse stable density $h(r,t)$ is concentrated on $r > 0$ for every $t > 0$. The boundary term in (15) accounts for the memory of particle waiting times in this non-Markovian setting, and is typical for a time-fractional equation; compare [10, 18, 20]. The relation to (11) is discussed in [22, Sect. 4].

To see that (14) solves (15), first integrate by parts and use the initial condition $q(\boldsymbol{x},0) = 0$ to get

$$\begin{aligned}\mathbb{D}_t p(x,t) &= \int_0^\infty q(\boldsymbol{x},u)\mathbb{D}_t h(u,t)\,du \\ &= \int_0^\infty q(\boldsymbol{x},u)\big[-\partial_u h(u,t)\big]\,du \\ &= \int_0^\infty \partial_u q(\boldsymbol{x},u)h(u,t)\,du.\end{aligned}$$

Integrate by parts again, using the remaining initial condition $\partial_t q(\boldsymbol{x},0) = f(\boldsymbol{x})$, to get

$$\begin{aligned}\mathbb{D}_t^2 p(\boldsymbol{x},t) &= \int_0^\infty \partial_u q(\boldsymbol{x},u)\big[-\partial_u h(u,t)\big]\,du \\ &= \int_0^\infty \partial_u^2 q(\boldsymbol{x},u)h(u,t)\,du \\ &\quad + h(0+,t)\partial_t q(0+,\boldsymbol{x}) \\ &= \int_0^\infty \Delta_{\boldsymbol{x}} q(\boldsymbol{x},u)h(u,t)\,du \\ &\quad + h(0+,t)f(\boldsymbol{x}).\end{aligned}$$

This reduces to (15), since

$$h(0+,t) = \lim_{r\downarrow 0} h(r,t) = \frac{\alpha_0 t^{-y}}{b\Gamma(1-y)} \tag{16}$$

for any $0 < y < 1$ or $1 < y < 2$.

The CTRW model behind the causal solution (14) to the power law wave equation (15) tracks the radial distance r traveled by time $t > 0$: Take W_n as in (11), and let $T_n(r) = \Delta r^{1/y} W_1 + \cdots + \Delta r^{1/y} W_n + r/c_0$ denote the time required to travel a radial distance r in $n = r/\Delta r$ small increments. Then $N_n(t) = \max\{n \geq 0 : T_n(r) \leq t\}$ is the position of a randomly selected packet of wave energy at time $t > 0$, and an extended central limit theorem for the inverse process [26, Theorem 3.1] shows that $N_n(t) \approx E_t$ as $\Delta r \to 0$. Thus, the probability density $h(r,t)$ of E_t represents the radial pressure wave at time t. Even though $D(r) < 0$ is possible when $y > 1$, the hitting time E_t is always positive, so this model is always causal. The same causal solution (14) applies in any number of dimensions. Figure 2 graphs the positive half of a (symmetric) one-dimensional "planar wave" with initial conditions $p(x,0) = 2\delta(x)$ and $\partial_t p(x,0) = 0$. In this case, $q(x,t) = \delta(t-x) + \delta(t+x)$ and $p(x,t) = h(|x|,t)$, the probability density of E_t.

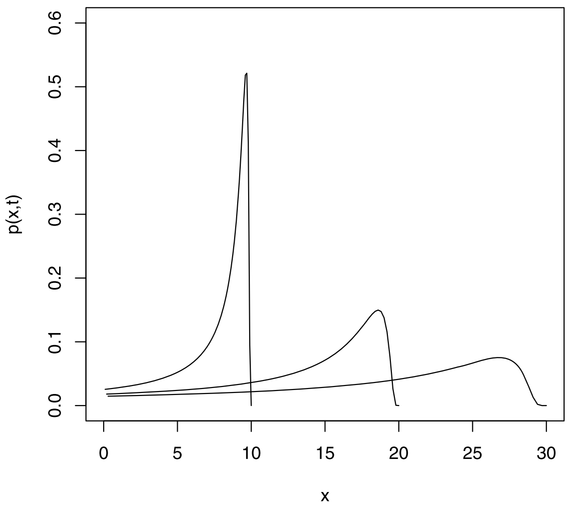

Fig. 2 Causal solution to the power law wave equation (15) in one dimension at time $t = 10, 20, 30$ with $c_0 = 1$, $\alpha_0 = 0.1$, and $y = 0.5$. As t increases, the pressure wave relaxes behind the sharp front at $x = c_0 t$

To check that (16) holds, let $g_y(t)$ be the probability density of a standard stable subordinator D with Fourier transform $\hat{g}_y(\omega) = e^{-(-i\omega)^y}$ in the case $0 < y < 1$. The random variable $D(r)$ has the same Fourier transform as $(r\alpha_0/b)^{1/y} D + r/c_0$, and hence the same probability density $(r\alpha_0/b)^{-1/y} g_y((r\alpha_0/b)^{-1/y}(t - r/c_0))$. Since $D(r)$ and E_t are inverse processes, $\mathrm{P}(E_t \leq r) = \mathrm{P}(D(r) \geq t)$. Then the probability density $h(r,t)$ of E_t is given by

$$\begin{aligned}h(r,t) &= \frac{d}{dr}\big[1 - \mathrm{P}\big(D(r) \leq t\big)\big] \\ &= g_y\big((r\alpha_0/b)^{-1/y}(t - r/c_0)\big)(r\alpha_0/b)^{-1-1/y} \\ &\quad \times \big[(r\alpha_0/bc_0) + (t - r/c_0)(\alpha_0/by)\big]\end{aligned}$$

A Tauberian theorem yields $g_y(t) \sim yt^{-1-y}/\Gamma(1-y)$ as $t \to \infty$ [23], and then (16) follows easily, as in [22, Eq. (3.21)]. For $1 < y < 2$, the calculation is similar, using [24, Eq. (6)]. For the general case $q(\boldsymbol{x},0) = g(\boldsymbol{x})$, there is an additional term $g(\boldsymbol{x})\mathbb{D}_t h(0+,t)$ on the right-hand side of (15).

When $y = 2$, the solution $h(r, t)$ is inverse Gaussian [25], and $h(0+, t) = 0$, so that (14) solves the original power law wave equation (2) without the fractional boundary term in (15) (a delta function term appears instead).

6 Discussion

A variety of fractional wave equations achieve power law attenuation. Caputo [27] and Wismer [28] proposed an alternative fractional wave equation

$$\Delta_{\boldsymbol{x}} p - \beta_0 \partial_t^{y-1} \Delta p = \frac{1}{c_0^2} \partial_t^2 p$$

Chen and Holm [29] recommend the equation

$$\Delta_{\boldsymbol{x}} p + \beta_0 \partial_t \Delta^{y/2} p = \frac{1}{c_0^2} \partial_t^2 p \tag{17}$$

using the fractional Laplacian. Treeby and Cox [30] suggest

$$\Delta_{\boldsymbol{x}} p + \beta_0 \partial_t \Delta^{y/2} p + \beta_1 \partial_t \Delta^{(y+1)/2} p = \frac{1}{c_0^2} \partial_t^2 p \tag{18}$$

Solutions to all of these equations exhibit power law attenuation $\alpha(\omega) = \alpha_0 |\omega|^y$ with frequency ω. It would be interesting to develop stochastic models for these equations.

In many practical applications, sound waves propagate though anisotropic media. For example, human tissue often presents in layers, and medical ultrasound in that setting is the main motivation for this research. All the wave equations discussed in this paper assume isotropic media, with sound moving in an expanding spherical wave. One way to develop anisotropic fractional wave equations would be to replace the isotropic fractional Laplacian in Eqs. (17) or (18) by its anisotropic analogue [14, Chap. 6]. The subtlety of the problem can be easily understood in terms of Fourier transforms. The fractional Laplacian $\Delta^{y/2} f(\boldsymbol{x})$ has Fourier transform $-k^2 \hat{f}(\boldsymbol{k})$, where $\hat{f}(\boldsymbol{k}) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d\boldsymbol{x}$, and $k = \|\boldsymbol{k}\|$ is the length of the wave vector. In 3 dimensions, it is true that $-\|\boldsymbol{k}\|^y = -|k_1|^y - |k_2|^y - |k_3|^y = (ik_1)^y + (ik_2)^y + (ik_3)^y$ only when $y = 2$. When $0 < y < 2$, the second and third expressions are the Fourier symbols for anisotropic analogues of the fractional Laplacian.

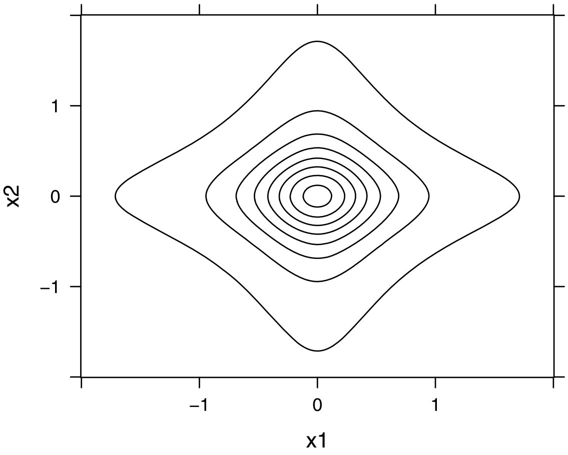

Fig. 3 Level sets for solutions of a fractional diffusion equation based on the one-dimensional Riesz fractional derivatives in each coordinate

Solutions of the space-fractional diffusion equation $p(\boldsymbol{x}, t) = a\Delta^{y/2} p(\boldsymbol{x}, t)$ have radially symmetric level sets. The anisotropic fractional derivative operator with Fourier symbol $-|k_1|^y - |k_2|^y - |k_3|^y$ corresponds to the sum of one-dimensional (Riesz) fractional Laplacians: $\partial^\alpha_{|x_1|} + \partial^\alpha_{|x_2|} + \partial^\alpha_{|x_3|}$, see [14, Chap. 6] for more details. Solutions to the corresponding fractional diffusion equation in the case $y = 1.1$ are illustrated in Fig. 3. These two-dimensional level sets are symmetric along each axis (i.e., $p(x_1, x_2, t) = p(-x_1, x_2, t) = p(x_1, -x_2, t)$), but exhibit anisotropy in the off-axis directions.

Another anisotropic fractional derivative operator with Fourier symbol $(ik_1)^y + (ik_2)^y + (ik_3)^y$ corresponds to the sum of one-dimensional (Riemann–Liouville) fractional derivatives: $\partial^\alpha_{x_1} + \partial^\alpha_{x_2} + \partial^\alpha_{x_3}$. Solutions to the corresponding fractional diffusion equation in the case $y = 1.2$ are illustrated in Fig. 4. These two-dimensional level sets exhibit anisotropy in every direction.

7 Conclusions

Sound wave propagation in complex anisotropic media has been observed to exhibit attenuation, at a rate proportional to a fractional power of the frequency. Fractional calculus models have been proposed to provide a simple analytical approximation to the complex nonlinear behavior. Previous research established that one of these models, the power law wave equation, has

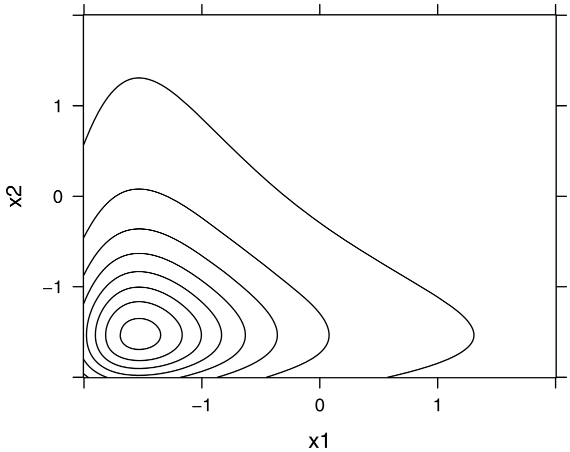

Fig. 4 Level sets for solutions of another anisotropic fractional diffusion equation, using a one-dimensional Riemann–Liouville fractional derivatives in each coordinate

an exact analytic solution that can be written in terms of a stable probability density.

This paper has proposed a random walk model for sound wave propagation, which explains the connection between the stable probability density and the power law wave equation. Because the speed of sound varies in a complex medium, travel times are essentially random. The continuous time random walk model proposed in this paper sums the random waiting times to traverse each incremental distance. In a complex medium, each incremental travel time can be expected to follow a power law. The power law index codes the degree of heterogeneity, and also determines the order of the fractional derivatives in the model. In past applications to flow and transport, this index has been related to the fractal dimension of the medium. The accumulated travel time approaches a stable random variable in the diffusion limit, which represents the travel time for a randomly selected packet of wave energy. The exact analytical solution for the power law wave equation comes from inserting this random time into the traditional wave equation solution, and taking expected values. The diffusion limit approximates a sum of positive waiting times by a Gaussian or stable limit, which can be either positive or negative. By focusing on distance traveled by a given time, a strictly causal solution was developed. Finally, some remaining challenges were discussed, including the development of anisotropic models to capture realistic variations in a complex conducting media.

Acknowledgements This research was partially supported by NIH grant R01-EB012079 and NSF grants DMS-1025486 and DMS-0803360.

References

1. Duck, F.A.: Physical Properties of Tissue: A Comprehensive Reference Book. Academic Press, San Diego (1990)
2. Zaslavsky, G.: Fractional kinetic equation for Hamiltonian chaos. Physica D **76**, 110–122 (1994)
3. Muslih, S.I., Baleanu, D.: Fractional Euler–Lagrange equations of motion in fractional space. J. Vib. Control **13**(9–10), 1209–1216 (2007)
4. Caputo, M., Carcione, J.M.: Wave simulation in dissipative media described by distributed-order fractional time derivatives. J. Vib. Control **17**(8), 1121–1130 (2011)
5. Herzallah, M., Baleanu, D.: Fractional Euler–Lagrange equations revisited. Nonlinear Dyn. **69**(3), 977–982 (2012)
6. Diethelm, K.: A fractional calculus based model for the simulation of an outbreak of dengue fever. Nonlinear Dyn. (2012, to appear). doi:10.1007/s11071-012-0475-2
7. Szabo, T.L.: Time domain wave equations for lossy media obeying a frequency power law. J. Acoust. Soc. Am. **96**, 491–500 (1994)
8. Kelly, J.F., McGough, R.J., Meerschaert, M.M.: Analytical time-domain Green's functions for power-law media. J. Acoust. Soc. Am. **124**(5), 2861–2872 (2008)
9. Agarwal, R.P., Benchohra, M., Slimani, B.A.: Existence results for differential equations with fractional order and impulses. Mem. Differ. Equ. Math. Phys. **44**, 1–21 (2008)
10. Meerschaert, M.M., Benson, D.A., Scheffler, H.-P., Baeumer, B.: Stochastic solution of space-time fractional diffusion equations. Phys. Rev. E **65**, 1103–1106 (2002)
11. Chen, W., Holm, S.: Modified Szabo's wave equation models for lossy media obeying frequency power law. J. Acoust. Soc. Am. **114**, 2570–2754 (2003)
12. Liebler, M., Ginter, S., Dreyer, T., Riedlinger, R.E.: Full wave modeling of therapeutic ultrasound: efficient time-domain implementation of the frequency power-law attenuation. J. Acoust. Soc. Am. **116**, 2742–2750 (2004)
13. Blackstock, D.T.: Transient solution for sound radiated into a viscous fluid. J. Acoust. Soc. Am. **41**, 1312–1319 (1967)
14. Meerschaert, M.M., Sikorskii, A.: Stochastic Models for Fractional Calculus. De Gruyter, Berlin (2012)
15. Meerschaert, M.M., Scheffler, H.-P.: Limit theorems for continuous time random walks with infinite mean waiting times. J. Appl. Probab. **41**(3), 623–638 (2004)
16. Clarke, D.D., Meerschaert, M.M., Wheatcraft, S.W.: Fractal travel time estimates for dispersive contaminants. Ground Water **43**(3), 401–407 (2005)
17. Jacob, N.: Pseudo Differential Operators & Markov Processes: Markov Processes And Applications. Imperial College Press, London (2005)
18. Kolokoltsov, V.N.: Generalized continuous-time random walks, subordination by hitting times, and fractional dynamics. Theory Probab. Appl. **53**(4), 594–609 (2009)
19. Baeumer, B., Benson, D.A., Meerschaert, M.M.: Advection and dispersion in time and space. Phys. A **350**(2–4), 245–262 (2005)

20. Saichev, A.I., Zaslavsky, G.M.: Fractional kinetic equations: solutions and applications. Chaos **7**(4), 753–764 (1997)
21. Meerschaert, M.M., Straka, P.: Inverse stable subordinators. Preprint, available at www.stt.msu.edu/users/mcubed/hittingTime.pdf
22. Meerschaert, M.M., Scheffler, H.-P.: Triangular array limits for continuous time random walks. Stoch. Process. Appl. **118**(9), 1606–1633 (2008)
23. Zolotarev, V.M.: One-dimensional Stable Distributions. Translations of Mathematical Monographs, vol. 65. AMS, Providence (1986)
24. Baeumer, B., Meerschaert, M.M.: Fractional diffusion with two time scales. Physica A **373**, 237–251 (2007)
25. Kumar, A., Meerschaert, M.M., Vellaisamy, P.: Fractional normal inverse Gaussian diffusion. Stat. Probab. Lett. **81**(1), 146–152 (2011)
26. Meerschaert, M.M., Scheffler, H.-P., Kern, P.: Limit theorem for continuous-time random walks with two time scales. J. Appl. Probab. **41**(2), 455–466 (2004)
27. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent-II. Geophys. J. R. Astron. Soc. **13**(5), 529–539 (1967)
28. Wismer, M.G.: Finite element analysis of broadband acoustic pulses through inhomogenous media with power law attenuation. J. Acoust. Soc. Am. **120**, 3493–3502 (2006)
29. Chen, W., Holm, S.: Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency. J. Acoust. Soc. Am. **115**, 1424–1430 (2004)
30. Treeby, B.E., Cox, B.T.: Modeling power law absorption and dispersion for acoustic propagation using the fractional Laplacian. J. Acoust. Soc. Am. **127**, 2741–2748 (2010)