PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 2, February 2013, Pages 699–710
S 0002-9939(2012)11362-0
Article electronically published on June 26, 2012

TRANSIENT ANOMALOUS SUB-DIFFUSION ON BOUNDED DOMAINS

MARK M. MEERSCHAERT, ERKAN NANE, AND P. VELLAISAMY

(Communicated by Edward C. Waymire)

Abstract. This paper develops strong solutions and stochastic solutions for the tempered fractional diffusion equation on bounded domains. First the eigenvalue problem for tempered fractional derivatives is solved. Then a separation of variables and eigenfunction expansions in time and space are used to write strong solutions. Finally, stochastic solutions are written in terms of an inverse subordinator.

1. Introduction

Transient anomalous sub-diffusion equations replace the first time derivative by a tempered fractional derivative of order $0 < \beta < 1$ to model delays between movements [1, 7]. These governing equations have proven useful in finance [2, 6] and geophysics [18] to model sub-diffusive phenomena that eventually transition to diffusive behavior. The idea of tempering was introduced by Mantegna and Stanley [13, 14] and developed further by Rosiński [20]. A stochastic model for transient anomalous sub-diffusion replaces the time variable in a diffusion by an independent inverse tempered stable subordinator. This inverse subordinator grows like t^β at early time, and like t^1 at late time [23]. Since $0 < \beta < 1$, the time-changed process transitions from sub-diffusive to diffusive behavior; i.e., the diffusion is slowed by the time change at early time, and then later it proceeds as if there were no significant time change. The stochastic model is useful for particle tracking, a superior numerical method in the presence of irregular boundaries [24, 25].

Section 2 provides some background on diffusion and fractional calculus, to establish notation, and to make the paper relatively self-contained. Section 3 uses Laplace transforms and complex analysis to prove strong solutions to the eigenvalue problem for the tempered fractional derivative operator. Then Section 4 solves the tempered fractional diffusion equation on bounded domains. Separation of variables and eigenvalue expansions in space and time lead to explicit strong solutions in series form. Stochastic solutions are then developed, using an inverse tempered stable time change in the underlying diffusion process.

Received by the editors May 2, 2011 and, in revised form, July 8, 2011.

2010 *Mathematics Subject Classification.* Primary 60G52, 35Q86; Secondary 35P10.

Key words and phrases. Fractional diffusion, Cauchy problem, tempered stable, boundary value problem.

The first author was partially supported by NSF grants DMS-1025486, DMS-0803360, EAR-0823965 and NIH grant R01-EB012079-01.

This paper was completed while the third author was visiting Michigan State University.

2. Traditional and fractional diffusion

Suppose that D is a bounded domain in $\mathbb{R}^d$. A uniformly elliptic operator in divergence form is defined for $u \in C^2(D)$ by

$$L_D u = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) \tag{2.1}$$

with $a_{ij}(x) = a_{ji}(x)$ such that for some $\lambda > 0$ we have

$$\lambda \sum_{i=1}^{n} y_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \le \lambda^{-1} \sum_{i=1}^{n} y_i^2, \quad \text{for all } y \in \mathbb{R}^d. \tag{2.2}$$

Assume also that there exists a $\Lambda > 0$ satisfying

$$\sum_{i,j=1}^{n} |a_{ij}(x)| \le \Lambda, \quad \text{for all } x \in D. \tag{2.3}$$

Take $a = \sigma\sigma^T$, and $B(t)$ a Brownian motion. Let $X(t)$ solve the stochastic differential equation $dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$, and define the first exit time $\tau_D(X) = \inf\{t \ge 0 : X(t) \notin D\}$. An application of the Itô formula shows that the semigroup

$$T_D(t)f(x) = \mathbb{E}_x[f(X(t))I(\tau_D(X) > t)] \tag{2.4}$$

has generator (2.1); see Bass [4, Chapters 1 and 5]. Since $T_D(t)$ is intrinsically ultracontractive (see [8, Corollary 3.2.8, Theorems 2.1.4, 2.3.6, 4.2.4 and Note 4.6.10] and [10, Theorems 8.37 and 8.38]), there exist eigenvalues $0 < \eta_1 < \eta_2 \le \eta_3 \le \cdots$, with $\eta_n \to \infty$ and a complete orthonormal basis of eigenfunctions ψ_n in $L^2(D)$ satisfying

$$L_D \psi_n(x) = -\eta_n \psi_n(x), \ x \in D : \psi_n|_{\partial D} = 0. \tag{2.5}$$

Then $p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\eta_n t}\psi_n(x)\psi_n(y)$ is the heat kernel of the killed semigroup T_D. This series converges absolutely and uniformly on $[t_0,\infty)\times D \times D$ for all $t_0 > 0$.

Denote the Laplace transform (LT) $t \to s$ of $u(t,x)$ by

$$\tilde{u}(s,x) = \mathcal{L}_t[u(t,x)] = \int_0^\infty e^{-st}u(t,x)dt.$$

The ψ_n-transform is defined by $\bar{u}(t,n) = \int_D \psi_n(x)u(t,x)dx$ and the ψ_n-Laplace transform is defined by

$$\hat{u}(s,n) = \int_D \psi_n(x)\tilde{u}(s,x)dx. \tag{2.6}$$

Since $\{\psi_n\}$ is a complete orthonormal basis for $L^2(D)$, we can invert the ψ_n-transform to obtain $u(t,x) = \sum_n \bar{u}(t,n)\psi_n(x)$ for any $t > 0$, where the series converges in the L^2 sense [21, Proposition 10.8.27].

Suppose that D satisfies a uniform exterior cone condition, so that then each $x \in \partial D$ is regular for $D^{\complement}$ [3, Proposition 1, p. 89]. If f is continuous on $\bar{D}$, then

$$\begin{aligned} u(t,x) &= T_D(t)f(x) = \mathbb{E}_x[f(X(t))I(\tau_D(X) > t)] \\ &= \int_D p_D(t,x,y)f(y)dy = \sum_{n=1}^{\infty} e^{-\eta_n t}\psi_n(x)\bar{f}(n) \end{aligned} \tag{2.7}$$

solves the Dirichlet initial-boundary value problem [8, Theorem 2.1.4]:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= L_D u(t,x), \quad x \in D,\ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D, \\ u(0,x) &= f(x), \quad x \in D. \end{aligned} \tag{2.8}$$

This shows that the diffusion $X(t)$ killed at the boundary ∂D is the stochastic solution to the diffusion equation (2.8) on the bounded domain D.

Let $0 < \beta < 1$. The Riemann-Liouville fractional derivative [19, 22] is defined by

$$\frac{\partial^\beta}{\partial t^\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{g(s)ds}{(t-s)^\beta}. \tag{2.9}$$

Also, the Caputo fractional derivative [5] is defined by

$$\left(\frac{\partial}{\partial t}\right)^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{g'(s)\,ds}{(t-s)^\beta}. \tag{2.10}$$

It is easy to check using $\mathcal{L}[t^{-\beta}] = s^{\beta-1}\Gamma(1-\beta)$ that

$$\mathcal{L}_t \left[\frac{d^\beta}{dt^\beta} g(t)\right] = s^\beta \tilde{g}(s), \tag{2.11}$$

while the Caputo fractional derivative (2.10) has LT $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$. It follows that

$$\frac{\partial^\beta}{\partial t^\beta} g(t) = \left(\frac{\partial}{\partial t}\right)^\beta g(t) + \frac{g(0)t^{-\beta}}{\Gamma(1-\beta)}. \tag{2.12}$$

Substituting a Caputo fractional derivative of order $0 < \beta < 1$ for the first-order time derivative in (2.8) yields a fractional Cauchy problem. This fractional diffusion equation was solved in [16, Theorem 3.1] in the special case where L_D is the killed Laplacian, and extended to uniformly elliptic operators in [16, Theorem 3.6]. Those solutions exhibit anomalous sub-diffusion at all times, with a plume spreading rate that is significantly slower than (2.7). Many practical problems exhibit transient sub-diffusion, resembling the fractional problem at early time, and transitioning to (2.7) at late time [2, 6, 18]. Hence, the goal of this paper is to extend the results of [16] to transient sub-diffusions.

3. Eigenvalues for tempered fractional derivatives

Let $D(x)$ be a stable subordinator with Lévy measure $\phi(y,\infty) = y^{-\beta}/\Gamma(1-\beta)$ for $y > 0$ and $0 < \beta < 1$. Then $\mathbb{E}[e^{-sD(x)}] = e^{-x\psi(s)}$, where the Laplace symbol is given by $\psi(s) = s^\beta = \int_0^\infty (1 - e^{-sy})\,\phi(dy)$. If $f_x(t)$ is the density of $D(x)$, then $q_\lambda(t,x) = f_x(t)e^{-\lambda t}/e^{-x\lambda^\beta}$ is a density on $x > 0$ with LT

$$\tilde{q}_\lambda(s,x) = \int_0^\infty e^{-st} q_\lambda(t,x)\,dt = e^{x\lambda^\beta} \int_0^\infty e^{-(s+\lambda)t} f_x(t)\,dt = e^{-x\psi_\lambda(s)}, \tag{3.1}$$

where $\psi_\lambda(s) = (s+\lambda)^\beta - \lambda^\beta$. Rosiński [20] notes that the tempered stable subordinator $D_\lambda(x)$ with this Laplace symbol has Lévy measure $\phi_\lambda(dy) = e^{-\lambda y}\phi(dy)$.

Define the Riemann-Liouville tempered fractional derivative of order $0 < \beta < 1$ by

$$\frac{\partial^{\beta,\lambda}}{\partial t^{\beta,\lambda}} g(t) = e^{-\lambda t} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{e^{\lambda s} g(s)\,ds}{(t-s)^\beta} - \lambda^\beta g(t) \tag{3.2}$$

as in [1]. We say that a function is a mild solution to a pseudo-differential equation if its LT with respect to time solves the corresponding equation in transform space.

Proposition 3.1. *The density* $q_\lambda(t,x)$ *of the tempered stable subordinator* $D_\lambda(x)$ *is a mild solution to*

$$\frac{\partial}{\partial x} q_\lambda(t,x) = -\frac{\partial^{\beta,\lambda}}{\partial t^{\beta,\lambda}} q_\lambda(t,x). \tag{3.3}$$

Proof. Clearly $\tilde{q}_\lambda(s,x) = e^{-x\psi_\lambda(s)}$ solves

$$\frac{\partial}{\partial x} \tilde{q}_\lambda(s,x) = -\psi_\lambda(s)\,\tilde{q}_\lambda(s,x) \tag{3.4}$$

with initial condition $\tilde{q}_\lambda(s,0) = 1$. The right-hand side of (3.4) involves a pseudo-differential operator $\psi_\lambda(\partial_t)$ with Laplace symbol $\psi_\lambda(s)$; see Jacob [11]. To complete the proof, it suffices to show that $\psi_\lambda(s)\tilde{g}(s)$ is the LT of (3.2). Since $\mathcal{L}[e^{\lambda t}g(t)] = \tilde{g}(s-\lambda)$, we get

$$\mathcal{L}\left[\frac{d^\beta}{dt^\beta}\left(e^{\lambda t}g(t)\right)\right] = s^\beta \tilde{g}(s-\lambda), \tag{3.5}$$

which leads to

$$\mathcal{L}\left[e^{-\lambda t}\frac{d^\beta}{dt^\beta}\left(e^{\lambda t}g(t)\right)\right] = (s+\lambda)^\beta \tilde{g}(s). \tag{3.6}$$

Then (3.3) follows easily. This also shows that $\psi_\lambda(\partial_t)$ is the negative generator of the C_0 semigroup associated with the tempered stable process. $\square$

Define the inverse tempered stable subordinator

$$E_\lambda(t) = \inf\{x > 0 : D_\lambda(x) > t\}. \tag{3.7}$$

A general result on hitting times [17, Theorem 3.1] shows that, for all $t > 0$, the random variable $E_\lambda(t)$ has Lebesgue density

$$g_\lambda(t,x) = \int_0^t \phi_\lambda(t-y,\infty) q_\lambda(y,x)\,dy \tag{3.8}$$

and $(t,x) \mapsto g_\lambda(t,x)$ is measurable. Following [17, Remark 4.8], we define the Caputo tempered fractional derivative of order $0 < \beta < 1$ by

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} g(t) = \frac{\partial^{\beta,\lambda}}{\partial t^{\beta,\lambda}} g(t) - \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\,dr. \tag{3.9}$$

Proposition 3.2. *The density* (3.8) *of the inverse tempered stable subordinator* (3.7) *is a mild solution to*

$$\frac{\partial}{\partial x} g_\lambda(t,x) = -\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} g_\lambda(t,x). \tag{3.10}$$

Proof. Theorem 4.1 in [17] shows that (3.8) is a mild solution to the pseudo-differential equation

$$\frac{\partial}{\partial x} g_\lambda(t,x) = -\psi_\lambda(\partial_t) g_\lambda(t,x) + \delta(x)\phi_\lambda(t,\infty), \tag{3.11}$$

where $\psi_\lambda(\partial_t)$ is the pseudo-differential operator with Laplace symbol $\psi_\lambda(s)$, i.e., the Riemann-Liouville tempered fractional derivative (3.2). Use

$$\phi_\lambda(t,\infty) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty e^{-\lambda r} \beta r^{-\beta-1}\, dr \tag{3.12}$$

to rewrite (3.11) in the form

$$\frac{\partial}{\partial x} g_\lambda(t,x) = -\frac{\partial^{\beta,\lambda}}{\partial t^{\beta,\lambda}} g_\lambda(t,x) + \frac{\delta(x)}{\Gamma(1-\beta)} \int_t^\infty e^{-\lambda r} \beta r^{-\beta-1}\, dr, \tag{3.13}$$

and then apply (3.9) with $g_\lambda(0,x) = \delta(x)$ to get (3.10). □

The next two results establish eigenvalues for Caputo tempered fractional derivatives, which will then be used in Section 4 to solve tempered fractional diffusion equations by an eigenvalue expansion.

Lemma 3.3. *For any $\mu > 0$, the Laplace transform*

$$\breve{g}_\lambda(t,\mu) = \mathcal{L}_x[g_\lambda(t,x)] = \int_0^\infty e^{-\mu x} g_\lambda(t,x)\, dx = \mathbb{E}[e^{-\mu E_\lambda(t)}] \tag{3.14}$$

is a mild solution to the eigenvalue problem

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} \breve{g}_\lambda(t,\mu) = -\mu \breve{g}_\lambda(t,\mu) \tag{3.15}$$

with $\breve{g}_\lambda(0,\mu) = 1$, for the Caputo tempered fractional derivative (3.9).

Proof. Equation (3.12) in [17] shows that

$$\mathcal{L}_t[\phi_\lambda(t,\infty)] = s^{-1}\psi_\lambda(s). \tag{3.16}$$

Then (3.8) together with the LT convolution property shows that

$$\tilde{g}_\lambda(s,x) = \mathcal{L}_t[g_\lambda(t,x)] = \mathcal{L}_t[\phi_\lambda(t,\infty)]\mathcal{L}_t[q_\lambda(t,x)] = \frac{1}{s}\psi_\lambda(s) e^{-x\psi_\lambda(s)} \tag{3.17}$$

for any $x > 0$, and then a Fubini argument shows that the double LT

$$G_\lambda(s,\mu) = \mathcal{L}_t\mathcal{L}_x[g_\lambda(t,x)] = \frac{\psi_\lambda(s)}{s} \int_0^\infty e^{-(\mu+\psi_\lambda(s))x} dx = \frac{\psi_\lambda(s)}{s(\mu+\psi_\lambda(s))}. \tag{3.18}$$

Rearrange (3.18) to get

$$-\mu G_\lambda(s,\mu) = \psi_\lambda(s) G_\lambda(s,\mu) - s^{-1}\psi_\lambda(s). \tag{3.19}$$

Use (3.16) along with (3.9) and (3.12) to see that

$$\mathcal{L}_t\left[\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} \breve{g}_\lambda(s,\mu)\right] = \psi_\lambda(s) G_\lambda(s,\mu) - s^{-1}\psi_\lambda(s) \tag{3.20}$$

and then substitute into (3.19) to get

$$\mathcal{L}_t\left[\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} \breve{g}_\lambda(t,\mu)\right] = \mathcal{L}_t\left[-\mu \breve{g}_\lambda(t,\mu)\right].$$

This proves that (3.14) is the mild solution to (3.15). □

The next theorem is the main technical result of this paper. It shows that the LTs (3.14) of inverse tempered stable densities are the eigenvalues of the Caputo tempered fractional derivative, in the strong sense.

Theorem 3.4. *For $0 < \beta < 1$, let*

$$k(t) = \frac{e^{-t\lambda}}{\pi \sin(\beta\pi)} t^{\beta-1}\Gamma(1-\beta). \tag{3.21}$$

For any $\mu, \lambda > 0$, $\mu \neq \lambda^\beta$, the function $\check{g_\lambda}(t,\mu)$ in (3.14) *can be written in the form*

$$\check{g_\lambda}(t,\mu) = \frac{\mu}{\pi}\int_0^\infty (r+\lambda)^{-1} e^{-t(r+\lambda)}\Phi(r,1)dr, \tag{3.22}$$

where

$$\Phi(r,1) = \frac{r^\beta \sin(\beta\pi)}{r^{2\beta}\sin^2(\beta\pi) + (\mu - \lambda^\beta + r^\beta\cos(\beta\pi))^2}$$

and satisfies

$$|\partial_t \check{g_\lambda}(t,\mu)| \leq \mu k(t). \tag{3.23}$$

Then $\check{g_\lambda}(t,\mu)$ is a strong (classical) solution of the eigenvalue problem (3.15).

Proof. The proof extends Theorem 2.3 in Kochubei [12] using some probabilistic arguments. Since $E_\lambda(t)$ has continuous sample paths, a dominated convergence argument shows that $\check{g_\lambda}(t,\mu) = \mathbb{E}[e^{-\mu E_\lambda(t)}]$ is a continuous function of $t > 0$. Use (3.18) to write

$$G_\lambda(s,\mu) = \mathcal{L}_t[\check{g_\lambda}(t,\mu)] = \frac{\psi_\lambda(s)}{s(\mu + \psi_\lambda(s))} = \frac{[(s+\lambda)^\beta - \lambda^\beta]}{s[(s+\lambda)^\beta - \lambda^\beta + \mu]} \tag{3.24}$$

and note that $G_\lambda(s,\mu)$ is analytic off the branch cut $\arg(s) = \pi, |s| \geq 0$. The Laplace inversion formula [9, p. 25] shows that for $\gamma > 0$ and for almost all $t > 0$,

$$\check{g_\lambda}(t,\mu) = \frac{d}{dt}\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s}\frac{[(s+\lambda)^\beta - \lambda^\beta]/s}{[(s+\lambda)^\beta - \lambda^\beta + \mu]}ds. \tag{3.25}$$

Let $\frac{1}{2} < \omega < 1$ and consider the closed curve $C_{\gamma,\omega}$ in $\mathbb{C}$, formed by a circle of radius R_n with a counterclockwise orientation, cut off on the right side by the line $\mathrm{Re}\,(z) = \gamma$, and by the curve $S_{\gamma,\omega}$ on the left side, consisting of the arc

$$T_{\gamma,\omega} = \{\gamma e^{i\theta} : -\omega\pi \leq \theta \leq \omega\pi\}$$

and the two rays $\Gamma^+_{\gamma,\omega} = \{re^{i\omega\pi} : r \geq \gamma\}$ and $\Gamma^-_{\gamma,\omega} = \{re^{-i\omega\pi} : r \geq \gamma\}$.

By Cauchy's Theorem, the integral

$$\int_{C_{\gamma,\omega}} \frac{e^{st}}{s}\frac{[(s+\lambda)^\beta - \lambda^\beta]/s}{[(s+\lambda)^\beta - \lambda^\beta + \mu]}ds = 0$$

and then Jordan's Lemma [9, p. 27] implies that we can let $R_n \to \infty$ to get

$$\check{g_\lambda}(t,\mu) = -\frac{d}{dt}\frac{1}{2\pi i}\int_{S_{\gamma,\omega}} \frac{e^{st}}{s}\frac{[(s+\lambda)^\beta - \lambda^\beta]/s}{[(s+\lambda)^\beta - \lambda^\beta + \mu]}ds. \tag{3.26}$$

Now pass the derivative inside the integral to get

$$\check{g_\lambda}(t,\mu) = \frac{-1}{2\pi i}\int_{S_{\gamma,\omega}} e^{st}\frac{[(s+\lambda)^\beta - \lambda^\beta]/s}{[(s+\lambda)^\beta - \lambda^\beta + \mu]}ds, \tag{3.27}$$

which also implies the smoothness of the function $t \to \check{g_\lambda}(t,\mu)$.

It is not hard to show that

$$\lim_{\gamma\to 0}\left|\int_{T_{\gamma,\omega}} e^{st}\frac{[(s+\lambda)^\beta - \lambda^\beta]/s}{(s+\lambda)^\beta - \lambda^\beta + \mu}ds\right| = 0.$$

The remaining path integral is $\check{g_\lambda}(t,\mu) = \check{g_\lambda}^+(t,\mu) + \check{g_\lambda}^-(t,\mu)$, where

$$\check{g_\lambda}^{\pm}(t,\mu) = \frac{-1}{2\pi i}\int_{\Gamma^{\pm}_{\gamma,\omega}} e^{st}\frac{[(s+\lambda)^\beta - \lambda^\beta]/s}{[(s+\lambda)^\beta - \lambda^\beta + \mu]}ds.$$

Since the point $s = re^{i\omega\theta}$ in $\Gamma^+_{\gamma,\omega}$ is the complex conjugate of the point $s = re^{-i\omega\theta}$ in $\Gamma^-_{\gamma,\omega}$, the real parts of the integrals $\check{g_\lambda}^{\pm}(t,\mu)$ cancel out. Using the fact that $z(z+\mu)^{-1} = 1 - \mu(z+\mu)^{-1}$, we then get

(3.28)
$$\begin{aligned}\check{g_\lambda}^+(t,\mu) + \check{g_\lambda}^-(t,\mu) &= \frac{1}{\pi}\mathrm{Im}\left\{\int_\gamma^\infty e^{tre^{i\omega\pi}}\frac{[(re^{i\omega\pi}+\lambda)^\beta - \lambda^\beta]/(re^{i\omega\pi})}{(re^{i\omega\pi}+\lambda)^\beta - \lambda^\beta + \mu}e^{i\omega\pi}dr\right\}\\ &= \frac{1}{\pi}\mathrm{Im}\int_\gamma^\infty r^{-1}e^{tre^{i\omega\pi}}dr - \frac{\mu}{\pi}\mathrm{Im}\int_\gamma^\infty \frac{e^{tre^{i\omega\pi}}/r}{(re^{i\omega\pi}+\lambda)^\beta - \lambda^\beta + \mu}dr.\end{aligned}$$

The first term

$$\frac{1}{\pi}\mathrm{Im}\int_\gamma^\infty r^{-1}e^{tre^{i\omega\pi}}dr \to -\frac{1}{\pi}\int_0^\infty l^{-1}e^{-l}\sin(l\tan\omega\pi)dl := I(\omega)$$

as $\gamma \to 0$, and a dominated convergence argument shows that $I(\omega) \to 0$ as $\omega \to 1$. The second term in (3.28) is $I_1(\omega,\gamma) + I_2(\omega,\gamma)$, where

$$\begin{aligned}I_1(\omega,\gamma) &= -\frac{\mu}{\pi}\int_\gamma^\infty \mathrm{Im}\left(\frac{e^{tre^{i\omega\pi}}}{r}\right)\mathrm{Re}\left(\frac{1}{(re^{i\omega\pi}+\lambda)^\beta - \lambda^\beta + \mu}\right)dr,\\ I_2(\omega,\gamma) &= -\frac{\mu}{\pi}\int_\gamma^\infty \mathrm{Re}\left(\frac{e^{tre^{i\omega\pi}}}{r}\right)\mathrm{Im}\left(\frac{1}{(re^{i\omega\pi}+\lambda)^\beta - \lambda^\beta + \mu}\right)dr.\end{aligned} \tag{3.29}$$

Some elementary estimates lead to $\lim_{\omega\to1}\lim_{\gamma\to0} I_1(\omega,\gamma) = 0$ and

$$I_2(\omega,0) := \lim_{\gamma\to0} I_2(\omega,\gamma) = \frac{\mu}{\pi}\int_0^\infty r^{-1}e^{tr\cos\omega\pi}\cos(tr\sin\omega\pi)U(r,\omega)dr,$$

where

$$\begin{aligned}U(r,\omega) &= \mathrm{Im}\left(\frac{-1}{(re^{i\omega\pi}+\lambda)^\beta - \lambda^\beta + \mu}\right)\\ &= \frac{|re^{i\omega\pi}+\lambda|^\beta\sin(\beta\theta)}{\left[|re^{i\omega\pi}+\lambda|^\beta\cos(\beta\theta) - \lambda^\beta + \mu\right]^2 + |re^{i\omega\pi}+\lambda|^{2\beta}\sin^2(\beta\theta)}\end{aligned} \tag{3.30}$$

and $\theta = \arg(re^{i\omega\pi}+\lambda)$. Since $U(r,1) = 0$ when $0 \le r \le \lambda$, we have

$$\begin{aligned}\check{g_\lambda}(t,\mu) = \lim_{\omega\to1} I_2(\omega,0) &= \frac{\mu}{\pi}\int_\lambda^\infty r^{-1}e^{-tr}U(r,1)dr\\ &= \frac{\mu}{\pi}\int_0^\infty (r+\lambda)^{-1}e^{-t(r+\lambda)}U(r+\lambda,1)dr,\end{aligned}$$

which reduces to (3.22).

Differentiate (3.22) with respect to t and use $r^\beta \sin(\beta\pi)\Phi(r,1) \le 1$ to obtain

$$
\begin{aligned}
|\partial_t \check{g_\lambda}(t,\mu)| &= \left| \frac{\mu}{\pi} \int_0^\infty (r+\lambda)^{-1} [\partial_t e^{-t(r+\lambda)}] \Phi(r,1) dr \right| \\
&\le \frac{\mu}{\pi \sin(\beta\pi)} \int_0^\infty e^{-t(r+\lambda)} r^{-\beta} dr \qquad (3.31) \\
&= \frac{\mu e^{-t\lambda}}{\pi \sin(\beta\pi)} t^{\beta-1}\Gamma(1-\beta) = \mu k(t),
\end{aligned}
$$

so that (3.23) holds. Note that $|\check{g_\lambda}(t,\mu)| \le 1$, and write

$$
\begin{aligned}
\left| \left(\frac{\partial}{\partial t}\right)^\beta \left(e^{\lambda t}\check{g_\lambda}(t,\mu)\right) \right| &= \left| \frac{1}{\Gamma(1-\beta)} \int_0^t \left(\lambda e^{\lambda s} \check{g_\lambda}(s,\mu) + e^{\lambda s} \frac{\partial[\check{g_\lambda}(s,\mu)]}{\partial s} \right) \frac{ds}{(t-s)^\beta} \right| \\
&\le \frac{1}{\Gamma(1-\beta)} \int_0^t \left(\lambda e^{\lambda s} |\check{g_\lambda}(s,\mu)| + e^{\lambda s} \left| \frac{\partial[\check{g_\lambda}(s,\mu)]}{\partial s} \right| \right) \frac{ds}{(t-s)^\beta} \\
&= \frac{1}{\Gamma(1-\beta)} \int_0^t \left(\lambda e^{\lambda s} + \frac{\mu}{\pi\sin(\beta\pi)} s^{\beta-1}\Gamma(1-\beta) \right) \frac{ds}{(t-s)^\beta}.
\end{aligned}
$$

Then a simple dominated convergence argument shows that the Riemann-Liouville fractional derivative of $e^{\lambda t}\check{g_\lambda}(t,\mu)$ is a continuous function of $t > 0$. Now it follows from (3.2) and (3.9) that the Caputo tempered fractional derivative of $\check{g_\lambda}(t,\mu)$ is continuous in $t > 0$. Since both sides of (3.15) are continuous in $t > 0$, it follows from Lemma 3.3 and the uniqueness theorem for the Laplace transform that (3.15) holds pointwise in $t > 0$ for all $\mu > 0$. □

4. Tempered fractional diffusion

Replacing the time variable in a diffusion by an independent inverse tempered stable subordinator $E_\lambda(t)$ of index $0 < \beta < 1$ yields a useful stochastic model for transient anomalous sub-diffusion. For example, if $X(t)$ is a standard Brownian motion on $\mathbb{R}^1$, independent of $E_\lambda(t)$, Stanislavsky et al. [23] show that $X(E_\lambda(t))$ satisfies

$$
\begin{aligned}
\mathbb{E}[X(E_\lambda(t))^2] &\sim t^\beta/\Gamma(1+\beta), \quad \text{as } t \to 0; \\
\mathbb{E}[X(E_\lambda(t))^2] &\sim t/\beta, \quad \text{as } t \to \infty.
\end{aligned}
$$

Hence, the process $X(E_\lambda(t))$ occupies an intermediate place between sub-diffusion, in which the second moment grows like t^β, and traditional diffusion, where the second moment is proportional to t. Let $D_\infty = (0,\infty) \times D$, and define $\mathcal{H}_{L_D}(D_\infty) = \{u : D_\infty \to \mathbb{R} : \quad L_D u(t,x) \in C(D_\infty)\}$, and let $\mathcal{H}^b_{L_D}(D_\infty) = \mathcal{H}_{L_D}(D_\infty) \cap \{u : |\partial_t u(t,x)| \le k(t)g(x),$ for some $g \in L^\infty(D)$, and for all $t > 0\}$, where $k(t)$ is defined in (3.21).

Theorem 4.1. *Let D be a bounded domain with $\partial D \in C^{1,\alpha}$ for some $0 < \alpha < 1$. Let $X(t)$ be a continuous Markov process with generator given in* (2.1)*, where the a_{ij} are elements of $C^\alpha(\bar{D})$ and satisfy* (2.2) *and* (2.3)*. Then, for any $f \in D(L_D) \cap C^1(\bar{D}) \cap C^2(D)$ such that the eigenfunction expansion of $L_D f$ with respect to the complete orthonormal basis $\{\psi_n\}$ from* (2.5) *converges uniformly and absolutely,* *the (classical) solution of*

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} u(t,x) &= L_D u(t,x), \quad x \in D,\ t \geq 0; \\ u(t,x) &= 0,\ x \in \partial D,\ t \geq 0; \\ u(0,x) &= f(x),\ x \in D,\end{aligned} \tag{4.1}$$

for $u \in \mathcal{H}^b_{L_D}(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$, *is given by*

$$\begin{aligned}u(t,x) &= \mathbb{E}_x[f(X(E_\lambda(t)))I(\tau_D(X) > E_\lambda(t))] = \mathbb{E}_x[f(X(E_\lambda(t)))I(\tau_D(X(E_\lambda)) > t)] \\ \text{(4.2)} \quad &= \int_0^\infty T_D(l)f(x)g_\lambda(t,l)dl = \sum_{n=1}^\infty \bar{f}(n)\psi_n(x)\check{g_\lambda}(t,\eta_n),\end{aligned}$$

where $E_\lambda(t)$ *is defined by* (3.7), *independent of* $X(t)$, $\check{g_\lambda}(t,\eta) = \mathbb{E}(e^{-\eta E_\lambda(t)})$ *is its Laplace transform, and* $T_D(t)$ *is the killed semigroup* (2.4).

Proof. The proof is based on the method of separation of variables. Let $u(t,x) = G(t)F(x)$ be a solution of (4.1). Then substituting into (4.1), we get

$$F(x)\left(\frac{d}{dt}\right)^{\beta,\lambda} G(t) = G(t)L_D F(x).$$

Divide both sides by $G(t)F(x)$ to obtain

$$\frac{\left(\frac{d}{dt}\right)^{\beta,\lambda} G(t)}{G(t)} = \frac{L_D F(x)}{F(x)} = -\eta.$$

Then we have

$$\left(\frac{d}{dt}\right)^{\beta,\lambda} G(t) = -\eta G(t),\ t > 0 \tag{4.3}$$

and

$$L_D F(x) = -\eta F(x),\ x \in D,\ F|_{\partial D} = 0. \tag{4.4}$$

The eigenvalue problem (4.4) is solved by an infinite sequence of pairs $\{(\eta_n, \psi_n)\}$, where $0 < \eta_1 < \eta_2 \leq \eta_3 \leq \cdots$, $\eta_n \to \infty$, as $n \to \infty$, and ψ_n forms a complete orthonormal set in $L^2(D)$. In particular, the initial function f regarded as an element of $L^2(D)$ can be represented as

$$f(x) = \sum_{n=1}^\infty \bar{f}(n)\psi_n(x). \tag{4.5}$$

Use Lemma 3.3 to see that $G_n(t) = \bar{f}(n)\check{g_\lambda}(t,\eta_n)$ solves (4.3). Sum these solutions $\psi_n(x)G_n(t)$ to (4.1) to get

$$u(t,x) = \sum_{n=1}^\infty \bar{f}(n)\check{g_\lambda}(t,\eta_n)\psi_n(x). \tag{4.6}$$

It remains to show that (4.6) solves (4.1) and satisfies the conditions of Theorem 4.1.

The remainder of the proof is similar to [16, Theorem 3.1], so we only sketch the argument.

First note that (4.6) converges uniformly in $t \in [0,\infty)$ in the L^2 sense.

Next argue that $||u(t,\cdot) - f||_{2,D} \to 0$ as $t \to 0$ using the fact that, since $\check{g_\lambda}(t,\lambda)$ is the Laplace transform of the density of $E_\lambda(t)$, it is completely monotone and nonincreasing in $\lambda \geq 0$.

Use the Parseval identity, the fact that μ_n is increasing in n, and the fact that $\check{g_\lambda}(t,\mu_n)$ is nonincreasing in $n \geq 1$, to get $||u(t,\cdot)||_{2,D} \leq \check{g_\lambda}(t,\mu_1)||f||_{2,D}$.

A Fubini argument, which can be rigorously justified using the bound $|\partial_t u(t,x)| \leq k(t)g(x)$ from Theorem 3.4, shows that the ψ_n transform commutes with the Caputo tempered fractional derivative. For this, it suffices to show that the ψ_n-transform commutes with the Caputo fractional derivative of $e^{\lambda t}u(t,x)$. To check this, write

$$\begin{aligned}
&\int_D \psi_n(x)\left(\frac{\partial}{\partial t}\right)^\beta \left(e^{\lambda t}u(t,x)\right)dx \\
&= \int_D \psi_n(x)\frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial\left(e^{\lambda s}u(s,x)\right)}{\partial s}\frac{ds}{(t-s)^\beta}dx \\
&= \frac{1}{\Gamma(1-\beta)}\int_0^t \left(\int_D \psi_n(x)\frac{\partial}{\partial s}\left(e^{\lambda s}u(s,x)\right)dx\right)\frac{ds}{(t-s)^\beta} \\
&= \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial}{\partial s}\left(e^{\lambda s}\int_D \psi_n(x)u(s,x)dx\right)\frac{ds}{(t-s)^\beta} \\
&= \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial}{\partial s}(e^{\lambda s}\bar{u}(s,n))\frac{ds}{(t-s)^\beta} = \left(\frac{\partial}{\partial t}\right)^\beta \left(e^{\lambda t}\bar{u}(t,n)\right).
\end{aligned}$$

Then the Caputo tempered fractional time derivative and the generator L_D can be applied term by term in (4.6).

Next show that the series (4.6) is the classical solution to (4.1) by checking uniform and absolute convergence. Argue that $u \in C^1(\bar{D})$ using [10, Theorem 8.33], and the absolute and uniform convergence of the series defining f. Using Theorem 3.4 and the assumption about the convergence of the eigenfunction expansion of f, it follows easily that $u \in \mathcal{H}^b_{L_D}$.

Finally, obtain the stochastic solution by inverting the ψ_n-Laplace transform. Since $\{\psi_n\}$ forms a complete orthonormal basis for $L^2(D)$, the ψ_n-transform of the killed semigroup $T_D(t)f(x) = \sum_{m=1}^\infty e^{-\mu_m t}\psi_m(x)\bar{f}(m)$ from (2.7) is given by

$$\overline{[T_D(t)f]}(n) = e^{-t\mu_n}\bar{f}(n). \tag{4.7}$$

Use Fubini together with (4.6) and (4.7) to get

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^\infty \bar{f}(n)\psi_n(x)\check{g_\lambda}(t,\mu_n) = \sum_{n=1}^\infty \psi_n(x)\int_0^\infty \bar{f}(n)e^{-\lambda y}g_\lambda(t,y)dy \\
&= \sum_{n=1}^\infty \psi_n(x)\int_0^\infty \overline{[T_D(y)f]}(n)g_\lambda(t,y)dy \\
&= \int_0^\infty \left[\sum_{n=1}^\infty \psi_n(x)\bar{f}(n)e^{-y\mu_n}\right]g_\lambda(t,y)dy \\
&= \int_0^\infty T_D(y)f(x)g_\lambda(t,y)dy \\
&= \mathbb{E}_x[f(X(E_\lambda(t)))I(\tau_D(X) > E_\lambda(t))].
\end{aligned}$$

The argument that

$$\mathbb{E}_x[f(X(E_\lambda(t)))I(\tau_D(X) > E_\lambda(t))] = \mathbb{E}_x[f(X(E_\lambda(t)))I(\tau_D(X(E_\lambda)) > t)]$$

is similar to [16, Corollary 3.2].

Uniqueness follows by considering two solutions u_1, u_2 with the same initial data and showing that $u_1 - u_2 \equiv 0$. $\square$

Remark 4.2. In the special case where $L_D = \Delta$, the Laplacian operator, sufficient conditions for existence of strong solutions to (4.1) can be obtained from [16, Corollary 3.4]. Let $f \in C_c^{2k}(D)$ be a $2k$-times continuously differentiable function of compact support in D. If $k > 1+3d/4$, then (4.1) has a classical (strong) solution. In particular, if $f \in C_c^\infty(D)$, then the solution of (4.1) is in $C^\infty(D)$.

Remark 4.3. In the special case where $L_D = \Delta$ on an interval $(0, M) \subset \mathbb{R}$, eigenfunctions and eigenvalues are explicitly known, and solutions to the tempered fractional Cauchy problem can be made explicit. Eigenvalues of the Laplacian on $(0, M)$ are $(n\pi/M)^2$ for $n = 1, 2, \ldots$, and the corresponding eigenfunctions are $\frac{2}{M}\sin(n\pi x/M)$. Using this eigenfunction expansion, the solution reads

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n)\psi_n(x)\check{g_\lambda}(t,\mu_n) = \sum_{n=1}^{\infty} \bar{f}(n)\sin(n\pi x/M)\check{g_\lambda}(t,(n\pi/M)^2).$$

Acknowledgement

The authors thank the referee for some helpful comments and for pointing out an error in the original proof of Theorem 3.4.

References

[1] B. Baeumer and M. M. Meerschaert, Tempered stable Lévy motion and transient super-diffusion, *J. Comput. Appl. Math.* **233** (2010), 2438–2448. MR2577834 (2010m:60285)

[2] O. E. Barndorff-Nielsen and N. Shephard, Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics, *J. Roy. Statist. Soc. Ser. B* **63** (2001), 167–241. MR1841412 (2002c:62127)

[3] R. F. Bass, *Probabilistic Techniques in Analysis*, Springer-Verlag, New York, 1995. MR1329542 (96e:60001)

[4] R. F. Bass, *Diffusions and Elliptic Operators*, Springer-Verlag, New York, 1998. MR1483890 (99h:60136)

[5] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, *Geophys. J. R. Astr. Soc.* **13** (1967), 529–539.

[6] P. Carr, H. Geman, D. B. Madan and M. Yor, Stochastic volatility for Lévy processes, *Math. Finance* **13** (2003), 345–382. MR1995283 (2005a:91054)

[7] Á. Cartea and D. del Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, *Phys. Rev. E* **76** (2007), 041105.

[8] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics **92**, Cambridge Univ. Press, Cambridge, 1989. MR990239 (90e:35123)

[9] V. A. Ditkin and A. P. Prudnikov, *Integral Transforms and Operational Calculus*, Pergamon Press, Oxford, 1965. MR0196422 (33:4609)

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 ed., Springer, New York, 2001. MR1814364 (2001k:35004)

[11] N. Jacob, *Pseudo-differential operators and Markov processes*, Mathematical Research **94**, Akademie Verlag, Berlin, 1996. MR1409607 (97m:60109)

[12] A. N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, *J. Math. Anal. Appl.* **340** (2008), 252–281. MR2376152 (2009i:35177)

[13] R. N. Mantegna and H. E. Stanley, Stochastic process with ultraslow convergence to a Gaussian: The truncated Lévy flight, *Phys. Rev. Lett.* **73** (1994), 2946–2949. MR1303317 (95g:82062)

[14] R. N. Mantegna and H. E. Stanley, Scaling behavior in the dynamics of an economic index, *Nature* **376** (1995), 46–49.

[15] M. M. Meerschaert and H.-P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, *J. Appl. Probab.* **41** (2004), 623–638. MR2074812 (2005f:60105)

[16] M. M. Meerschaert, E. Nane and P. Vellaisamy, Fractional Cauchy problems on bounded domains, *Ann. Probab.* **37** (2009), 979–1007. MR2537547 (2010h:60121)
[17] M. M. Meerschaert and H.-P. Scheffler, Triangular array limits for continuous time random walks, *Stochastic Process Appl.* **118** (2008), 1606–1633. MR2442372 (2010b:60135)
[18] M. M. Meerschaert, Y. Zhang and B. Baeumer, Tempered anomalous diffusion in heterogeneous systems, *Geophys. Res. Lett.* **35** (2008), L17403.
[19] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993. MR1219954 (94e:26013)
[20] J. Rosiński, Tempering stable processes, *Stochastic Process Appl.* **117** (2007), 677–707. MR2327834 (2008g:60146)
[21] H. L. Royden, *Real Analysis*, 2nd Ed., MacMillan, New York, 1968. MR0151555 (27:1540)
[22] S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London, 1993. MR1347689 (96d:26012)
[23] A. Stanislavsky, K. Weron and A. Weron, Diffusion and relaxation controlled by tempered α-stable processes, *Phys. Rev. E* **3** 78 (2008), no. 5, 051106. MR2551366 (2010i:82145)
[24] Y. Zhang, D. A. Benson, M. M. Meerschaert and H.-P. Scheffler, On using random walks to solve the space-fractional advection-dispersion equations, *J. Stat. Phys.* **123** (2006), 89–110. MR2225237 (2006k:82142)
[25] Y. Zhang, M. M. Meerschaert and B. Baeumer, Particle tracking for time-fractional diffusion, *Phys. Rev. E* **78** (2008), 036705.

Department of Statistics and Probability, Michigan State University, East Lansing, Michigan 48823
E-mail address: mcubed@stt.msu.edu
URL: http://www.stt.msu.edu/∼mcubed/

Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, Auburn, Alabama 36849
E-mail address: nane@auburn.edu
URL: http://www.auburn.edu/∼ezn0001/

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India
E-mail address: pv@math.iit.ac.in