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# Applications of inverse tempered stable subordinators

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ABSTRACT

The inverse tempered stable subordinator is a stochastic process that models power law waiting times between particle movements, with an exponential tempering that allows all moments to exist. This paper shows that the probability density function of an inverse tempered stable subordinator solves a tempered time-fractional diffusion equation, and its "folded" density solves a tempered time-fractional telegraph equation. Two explicit formulae for the density function are developed, and applied to compute explicit solutions to tempered fractional Cauchy problems, where a tempered fractional derivative replaces the first derivative in time. Several examples are given, including tempered fractional diffusion equations on bounded or unbounded domains, and the probability distribution of a tempered fractional Poisson process. It is shown that solutions to the tempered fractional diffusion equation have a cusp at the origin.



## 1. Introduction

Fractional calculus is a very old field, dating back to a letter from Leibniz to L'Hôpital in 1695. In recent decades, the subject has expanded rapidly, due to the discovery of interesting mathematical connections, and real world applications. See for example [1–8]. The close connection between fractional calculus and probability is outlined in [9–11]. A famous paper of Einstein [12] outlined the classical link between random walks, Brownian motion, and the diffusion equation. In the modern theory, the probability of a jump exceeding length $x$ falls off like a power law $x^{-\alpha}$ for some $0 < \alpha < 2$, the random walk limit is an $\alpha$-stable Lévy motion whose particle traces are fractals of dimension $\alpha$, and whose particle density solves a diffusion equation involving a fractional derivative of order $\alpha$ in the space variable. If particles wait a random time between jumps, with a probability that falls off like $t^{-\beta}$ for some $0 < \beta < 1$, the non-Markovian limit density solves a space–time fractional diffusion equation that involves a fractional derivative of order $\beta$ in the time variable. Particle traces follow a random process obtained by replacing the time variable in the $\alpha$-stable Lévy motion by an inverse $\beta$-stable subordinator. The fractal dimension $\alpha$ of the particle paths remains the same, since the inverse stable subordinator is continuous and nondecreasing [13].

The space–time fractional diffusion model implies that the mean waiting time, and the second moment of the particle jump distribution, are both infinite. The tempered fractional diffusion model was developed as an alternative with finite moments [14–19]. This model has proven useful in applications to geophysics [20–23] and finance [24,25]. Fractional Cauchy

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problems govern the long time limiting behavior of particle motions [26–28], assuming a power law distributed waiting time between movements, in a general framework that can accommodate complex boundary conditions and confining potentials. Tempered fractional Cauchy problems modify this general model, tempering the power law waiting times, so that the mean waiting time remains finite [29]. A tempered fractional derivative in time replaces the usual first derivative in the classical Cauchy problem. The Cauchy problem governs a Markov process, but the tempered fractional Cauchy problem governs a non-Markovian process, since the resting times are not exponentially distributed. Particle motions follow a time-changed Markov process, using an inverse tempered stable subordinator introduced in [30] as a subdiffusion model with finite moments. The inverse tempered subordinator was applied to a tempered fractional Fokker–Planck equation in [31] and applied to financial data [32,33].

The goal of this paper is to develop properties of the inverse tempered stable subordinator, to facilitate practical applications of tempered fractional Cauchy problems. Section 2 reviews some basic facts about tempered fractional calculus. In Section 3, we show that the probability density function of an inverse tempered stable subordinator solves a tempered time-fractional diffusion equation, and its ''folded'' density solves a tempered time-fractional telegraph equation. In Section 4, we develop two explicit formulae for the inverse tempered stable density. In Section 5, we prove scaling and asymptotic properties for the tempered stable subordinator and its inverse. Section 6 applies the inverse tempered stable density to solve several tempered fractional Cauchy problems, including tempered fractional diffusion equations on bounded and unbounded domains. There we also prove that solutions to the tempered fractional diffusion equation are non-differentiable at the center of mass. Section 7 discusses the tempered fractional Poisson process, and applies a formula from Section 4 to compute its probability distribution.

## 2. Tempered fractional calculus

The standard $\beta$-stable subordinator $D_x$ is a Lévy process (i.e., it has stationary and independent increments) whose probability density function (pdf) $g(t,x)$ has Laplace transform

$$\tilde{g}(s,x) = \int_0^\infty e^{-st} g(t,x)\,dt = e^{-xs^\beta} \tag{2.1}$$

for some $0 < \beta < 1$. It follows easily that

$$g^\lambda(t,x) := e^{-\lambda t} g(t,x) e^{x\lambda^\beta} \tag{2.2}$$

is also a pdf. In fact, it is infinitely divisible [9, p. 208], and there is another Lévy process $D_x^\lambda$ called a *tempered stable subordinator* with pdf $g^\lambda(t,x)$ for each $t > 0$. Using (2.1) it follows that

$$\tilde{g}^\lambda(s,x) = e^{-x\psi^\lambda(s)} \tag{2.3}$$

where the *Laplace symbol*

$$\psi^\lambda(s) = (s+\lambda)^\beta - \lambda^\beta. \tag{2.4}$$

Taking derivatives in (2.3) yields

$$\partial_x \tilde{g}^\lambda(s,x) = -\psi^\lambda(s)\tilde{g}^\lambda(s,x). \tag{2.5}$$

Define the *Riemann–Liouville tempered fractional derivative*

$$\mathbb{D}_t^{\beta,\lambda} g(t) = e^{-\lambda t}\, \mathbb{D}_t^\beta \left[e^{\lambda t} g(t)\right] - \lambda^\beta g(t), \tag{2.6}$$

where

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)\,ds}{(t-s)^{\beta+1-n}}$$

is the usual Riemann–Liouville fractional derivative of order $\beta > 0$, and $n = \lceil \beta \rceil$ is the ceiling function, so that $n-1 < \beta \le n$. A simple argument [9, p. 209] using the shift property of the Laplace transform shows that

$$\mathcal{L}[\mathbb{D}_t^{\beta,\lambda} g](s) = \int_0^\infty e^{-st}\, \mathbb{D}_t^{\beta,\lambda} g(t)\,dt = \psi^\lambda(s)\tilde{g}(s), \tag{2.7}$$

and then inverting the Laplace transform in (2.5) shows that the tempered stable subordinator pdf solves the tempered fractional diffusion equation

$$\partial_x g^\lambda(t,x) = -\mathbb{D}_t^{\beta,\lambda} g^\lambda(t,x).$$

Note that, while the argument in [9, p. 209] assumes $0 < \beta < 1$, exactly the same argument goes through for any $\beta > 0$. Hence the definition (2.6), and the Laplace transform formula (2.7), are valid for any $\beta > 0$.

Next we define the *Caputo tempered fractional derivative* of order $\beta > 0$ by

$$\partial_t^{\beta,\lambda} g(t) = \mathbb{D}_t^{\beta,\lambda}\left[g(t) - \sum_{j=0}^{n-1}\frac{t^j}{j!}g^{(j)}(0)\right] \tag{2.8}$$

where $g^{(j)}(t)$ is the derivative of order $j$, and again $n = \lceil\beta\rceil$. When $\lambda = 0$, (2.8) reduces to the usual Caputo fractional derivative [34, Eq. (2.4.1)]. One advantage of the Caputo form is that it allows initial conditions involving integer-ordered derivatives to be included in the formulation of a fractional differential equation.

**Proposition 2.1.** *The Laplace transform of the Caputo tempered fractional derivative of order $\beta > 0$ is given by*

$$\mathcal{L}[\partial_t^{\beta,\lambda} g](s) = \psi^\lambda(s)\tilde{g}(s) - \left[\sum_{j=0}^{n-1} s^{-j-1}\psi^\lambda(s)g^{(j)}(0)\right] \tag{2.9}$$

*for any $\lambda > 0$.*

**Proof.** Using the well-known formula $\mathcal{L}[t^j/j!](s) = s^{-j-1}$ (e.g., see [9, Example 2.7]), it follows that the

$$\mathcal{L}\left[g(t) - \sum_{j=0}^{n-1}\frac{t^j}{j!}g^{(j)}(0)\right] = \tilde{g}(s) - \sum_{j=0}^{n-1} s^{-j-1}g^{(j)}(0).$$

Then apply formula (2.7) to finish the proof. □

**Proposition 2.2.** *When $0 < \beta < 1$, the definition (2.8) of the Caputo tempered fractional derivative reduces to the definition in [29, Eq. (3.9)]:*

$$\partial_t^{\beta,\lambda} g(t) = \mathbb{D}_t^{\alpha,\lambda} g(t) - \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\,dr \quad \text{for } 0 < \beta < 1. \tag{2.10}$$

**Proof.** Apply (2.9) with $n = 1$ to see that

$$\mathcal{L}[\partial_t^{\beta,\lambda} g](s) = \psi^\lambda(s)\tilde{g}(s) - s^{-1}\psi^\lambda(s)g(0).$$

The Lévy measure of the infinitely divisible random variable $D_1^\lambda$ is $\phi^\lambda(dy)$ where

$$\phi^\lambda(y,\infty) = \frac{\beta}{\Gamma(1-\beta)}\int_y^\infty e^{-\lambda u}u^{-\beta-1}\,du, \tag{2.11}$$

see [9, Eq. (7.9)]. Apply [28, Eq. (3.12)] to see that the Laplace transform

$$\mathcal{L}[\phi^\lambda(t,\infty)] := \int_0^\infty e^{-st}\phi^\lambda(t,\infty)\,dt = s^{-1}\psi^\lambda(s). \tag{2.12}$$

Hence the right-hand side of (2.10) has the same Laplace transform as that of (2.8), and since both are continuous, it follows from the uniqueness of the Laplace transform that they are equal. □

**Remark 2.3.** Li et al. [35] investigate numerical solutions of tempered fractional differential equations, using a different definition

$$\mathbf{D}_t^{\beta,\lambda} g(t) = e^{-\lambda t}\,\mathbb{D}_t^\beta\left[e^{\lambda t}g(t)\right] \tag{2.13}$$

of the tempered Riemann–Liouville fractional derivative. Then $\mathbb{D}_t^{\beta,\lambda} g(t) = \mathbf{D}_t^{\beta,\lambda} g(t) - \lambda^\beta g(t)$, and

$$\mathcal{L}[\mathbf{D}_t^{\beta,\lambda} g](s) = (s+\lambda)^\beta \tilde{g}(s).$$

The fractional derivative $\mathbf{D}_t^{\beta,\lambda} g(t)$ was also applied in [36] to construct stochastic integrals with respect to tempered fractional Brownian motion [37]. The advantage is that $\mathbf{D}_t^{\beta,\lambda}\mathbf{I}_t^{\beta,\lambda} g(t) = g(t)$, where the tempered fractional integral

$$\mathbf{I}_t^{\beta,\lambda} g(t) = \frac{1}{\Gamma(\beta)}\int_0^t g(u)(t-u)^{\beta-1}e^{-\lambda(t-u)}du.$$

When $1 < \beta < 2$, another definition [9, Eq. (7.16)] has been proposed:

$$\mathcal{D}_t^{\beta,\lambda} g(t) = \mathbb{D}_t^{\beta,\lambda} g(t) - \beta\lambda^{\beta-1}g'(t).$$

The advantage is that the solution to the tempered fractional diffusion equation $\partial_t p(x,t) = \mathcal{D}_x^{\beta,\lambda} p(x,t)$ is a pdf with mean zero.

## 3. Governing equations

In this section, we define the inverse tempered stable subordinator, and discuss two fractional partial differential equations for its probability density. Suppose $0 < \beta < 1$ and define the *inverse tempered stable subordinator* [28,30,31]

$$E_t^\lambda = \inf\{x > 0 : D_x^\lambda > t\}. \tag{3.1}$$

Sample paths of the Lévy process $D_x^\lambda$ are continuous from the right, with left hand limits, strictly increasing, with $D_0^\lambda = 0$ and $D_x^\lambda \to \infty$ as $x \to \infty$. Hence the inverse process (3.1) is well-defined, with almost surely continuous sample paths. It follows from [28, Theorem 3.1] that the random variable $E_t^\lambda$ has a probability density function $h^\lambda(x,t)$ on $x > 0$ for all $t > 0$.

**Proposition 3.1.** *The probability density function $h^\lambda(x,t)$ of the inverse tempered stable subordinator solves the tempered time-fractional equation*

$$\mathbb{D}_t^{\beta,\lambda} h^\lambda(x,t) = -\partial_x h^\lambda(x,t) + \delta(x)\phi^\lambda(t,\infty), \tag{3.2}$$

*where $\phi^\lambda$ is the Lévy measure (2.11) and $\mathbb{D}_t^{\beta,\lambda}$ is the Riemann–Liouville tempered fractional derivative (2.6). Therefore it also solves*

$$\partial_t^{\beta,\lambda} h^\lambda(x,t) = -\partial_x h^\lambda(x,t), \tag{3.3}$$

*using the Caputo tempered fractional derivative (2.8).*

**Proof.** Recall that the Riemann–Liouville tempered fractional derivative (2.6) is the inverse Laplace transform of $\psi^\lambda(s)\tilde{g}(s)$, where the Laplace symbol $\psi^\lambda(s) = (s+\lambda)^\beta - \lambda^\beta$. Recall from [9, p. 208] that the pdf $g^\lambda(u,t)$ of the tempered stable subordinator $D_x^\lambda$ has Laplace transform $\tilde{g}^\lambda(s,x) = e^{-x\psi^\lambda(s)}$ for all $t > 0$. The Lévy measure of this infinitely divisible random variable is $x\phi^\lambda(dy)$ using (2.11), see [9, Eq. (7.9)]. Then [28, Theorem 4.1] shows that $h^\lambda(x,t)$ solves (3.2) using the Riemann–Liouville tempered fractional derivative (2.6). Eq. (3.2) can then be written in the more compact form (3.3) using the Caputo tempered fractional derivative (2.8), see [29, Proposition 3.2]. □

Define the "folded" pdf

$$v(x,t) = \frac{1}{2}h^\lambda(|x|,t) \quad x \in \mathbb{R} \tag{3.4}$$

so that $h^\lambda(x,t) = 2v(x,t)$ for $x > 0$. Next we develop a fractional differential equation that governs the folded pdf. When $\lambda = 0$, this result is due to Beghin and Orsingher [38, Eq. (2.12)].

**Theorem 3.2.** *The folded inverse tempered stable pdf $v(x,t)$ in (3.4) solves the tempered fractional telegraph equation*

$$\partial_t^{2\beta,\lambda} v(x,t) - 2\lambda^\beta \partial_t^{\beta,\lambda} v(x,t) = \partial_x^2 v(x,t) \tag{3.5}$$

*for any $0 < \beta < 1$, with initial condition $v(x,0) = \delta(x)$. If $\beta \geq 1/2$, we also require the initial condition $\partial_t v(x,0) = 0$.*

**Proof.** Apply the Laplace transform in the $t$ variable to the governing equation (3.2) using (2.7) and (2.12), and then apply the Fourier transform in the $x$ variable, to see that the Fourier–Laplace transform (FLT)

$$\bar{h}^\lambda(k,s) := \int_{-\infty}^{\infty}\int_0^\infty e^{-ikx}e^{-st}h^\lambda(x,t)\,dt\,dx$$

of the inverse tempered stable pdf satisfies

$$\psi^\lambda(s)\bar{h}(k,s) = -ik\,\bar{h}(k,s) + s^{-1}\psi^\lambda(s).$$

Rearrange to obtain the equivalent formula

$$\bar{h}^\lambda(k,s) = \frac{s^{-1}\psi^\lambda(s)}{ik + \psi^\lambda(s)}. \tag{3.6}$$

Since the inverse tempered stable pdf $h^\lambda(x,t)$ has FLT (3.6), it follows easily that the folded pdf (3.4) has FLT

$$\bar{v}(k,s) = \frac{1}{2}\frac{s^{-1}\psi^\lambda(s)}{\psi^\lambda(s)+ik} + \frac{1}{2}\frac{s^{-1}\psi^\lambda(s)}{\psi^\lambda(s)-ik} = \frac{s^{-1}\psi^\lambda(s)^2}{\psi^\lambda(s)^2 + k^2}. \tag{3.7}$$

Rearrange to obtain the equivalent expression

$$\psi^\lambda(s)^2\bar{v}(k,s) - s^{-1}\psi^\lambda(s)^2 = (ik)^2\bar{v}(k,s),$$

and recall that the Laplace symbol $\psi^\lambda(s) = (s+\lambda)^\beta - \lambda^\beta$. Then we have

$$\begin{aligned}\psi^\lambda(s)^2 &= [(s+\lambda)^\beta - \lambda^\beta]^2\\ &= (s+\lambda)^{2\beta} - 2\lambda^\beta(s+\lambda)^\beta + \lambda^{2\beta}\\ &= \left[(s+\lambda)^{2\beta} - \lambda^{2\beta}\right] - 2\lambda^\beta\left[(s+\lambda)^\beta - \lambda^\beta\right]\\ &= \psi^{2\beta,\lambda}(s) - 2\lambda^\beta\psi^{\beta,\lambda}(s)\end{aligned}$$

where we add the superscript $\beta$ to distinguish the two terms in the final line. Hence,

$$\psi^{2\beta,\lambda}(s)\bar{v}(k,s) - s^{-1}\psi^{2\beta,\lambda}(s) - 2\lambda^\beta[\psi^\lambda(s)\bar{v}(k,s) - s^{-1}\psi^\lambda(s)] = (ik)^2\bar{v}(k,s).$$

Now apply (2.9) to the Fourier transform $\hat{v}(k,t)$, recalling that $\hat{v}(k,0)=1$, and $\partial_t v(x,0) = 0$ if $2\beta \ge 1$, to see that

$$\partial_t^{2\beta,\lambda}\hat{v}(k,t) - 2\lambda^\beta\partial_t^{\beta,\lambda}\hat{v}(k,t) = (ik)^2\hat{v}(k,t).$$

Then invert the Fourier transform to arrive at (3.5). □

**Remark 3.3.** The tempered time-fractional telegraph equation (3.5) extends the time-fractional telegraph equation in [39, Eq. (1.1)], replacing the Caputo time derivative there with a tempered Caputo derivative.

## 4. Computing the density

In this section, we develop two explicit formulae for the inverse tempered stable density. Recall that the upper incomplete Gamma function is defined for $x > 0$ and $a$ real by

$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\,dt. \tag{4.1}$$

Let $g(t) = g(t,1)$ be the pdf of the standard stable subordinator $D_1$, so that

$$\mathcal{L}[g](s) := \tilde{g}(t) = \int_0^\infty e^{-st}g(t)\,dt = e^{-s^\beta} \tag{4.2}$$

for $0 < \beta < 1$. Both of these functions can be computed using widely available computer codes [9,40].

**Theorem 4.1.** *Let $h^\lambda(x,t)$ be the pdf of the inverse tempered stable subordinator (3.1) with $0 < \beta < 1$. Then we can write*

$$h^\lambda(x,t) = \int_0^t \phi^\lambda(u,\infty)g^\lambda(t-u,x)\,du \tag{4.3}$$

*where*

$$\phi^\lambda(t,\infty) = \frac{\beta\lambda^\beta\Gamma(-\beta,\lambda t)}{\Gamma(1-\beta)}, \tag{4.4}$$

*using the upper incomplete gamma function (4.1), and*

$$g^\lambda(t,x) = e^{x\lambda^\beta}e^{-\lambda t}\left[\frac{1}{x^{1/\beta}}g\left(\frac{t}{x^{1/\beta}}\right)\right] \tag{4.5}$$

*is the pdf of the tempered stable subordinator $D_x^\lambda$.*

**Proof.** It follows from [28, Theorem 3.1] that $h^\lambda(x,t)$ is given by the formula (4.3), with $g^\lambda(t,x)$ the probability density function of the random variable $D_x^\lambda$ in (3.1), and $\phi^\lambda(t,\infty)$ the Lévy measure of $D_x^\lambda$. A change of variable in (2.11) shows that (4.4) holds for $0 < \beta < 1$, using (4.1). The tempered stable pdf $g^\lambda(t,x) = e^{-\lambda t}g(t,x)e^{x\lambda^\beta}$ where $g(t,x)$ is the pdf of the stable subordinator $D_x$ with Laplace transform (2.1). A change of variable in (4.2) leads to (2.1), which implies that

$$g(t,x) = \frac{1}{x^{1/\beta}}g\left(\frac{t}{x^{1/\beta}}\right),$$

reflecting the self-similarity of the stable subordinator as a stochastic process: $D_x$ has the same pdf as $x^{1/\beta}D_1$. □

Although (4.3) can be used to numerically compute the density of the inverse tempered stable subordinator, the convolution integral is numerically poorly conditioned; therefore, we seek an alternative form.

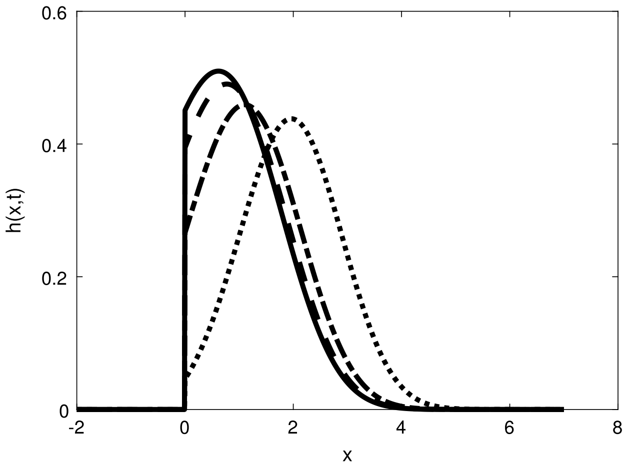


**Fig. 1.** Tempered inverse stable pdf $h^\lambda(x,t)$ versus $x$ for $\beta = 0.6$ and $t = 1$, for $\lambda = 0$ (solid), 0.01 (dashed), 0.1 (dot-dashed) and 1.0 (dotted). The pdf was calculated using (4.6).

**Proposition 4.2.** *Let $h^\lambda(x,t)$ be the pdf of the inverse tempered stable subordinator (3.1) with $0 < \beta < 1$. Then*

$$h^\lambda(x,t) = e^{x\lambda^\beta}\left[e^{-\lambda t}h(x,t) + \lambda\int_0^t e^{-\lambda\tau}h(x,\tau)\,d\tau - \lambda^\beta\int_0^t e^{-\lambda\tau}g(\tau,x)\,d\tau\right] \tag{4.6}$$

*where $g(t,x)$ is the pdf of the stable subordinator with Laplace transform (2.1), and*

$$h(x,t) = \frac{t}{x\beta}g(t,x) \tag{4.7}$$

*is the density of the inverse stable subordinator.*

**Proof.** For $x > 0$, Eq. (3.6) is equivalent to

$$\tilde{h}^\lambda(x,s) = -s^{-1}\frac{\partial}{\partial x}\left[e^{-x\psi^\lambda(s)}\right]. \tag{4.8}$$

Since the Laplace transform of the tempered stable subordinator is given by $\mathcal{L}\left(g^\lambda(t,x)\right) = e^{-x\psi^\lambda(s)}$ and since multiplication by $s^{-1}$ in the frequency domain is equivalent to integration in the time-domain, we have

$$\begin{aligned} h^\lambda(x,t) &= -\int_0^t \frac{\partial}{\partial x}\left[g^\lambda(\tau,x)\right]d\tau \\ &= -\int_0^t \frac{\partial}{\partial x}\left[e^{x\lambda^\beta}e^{-\lambda\tau}g(\tau,x)\right]d\tau \\ &= -e^{x\lambda^\beta}\left[\lambda^\beta\int_0^t e^{-\lambda\tau}g(\tau,x)\,d\tau + \int_0^t e^{-\lambda\tau}\frac{\partial g(\tau,x)}{\partial x}\,d\tau\right]. \end{aligned}$$

Letting $\lambda = 0$ in the first line of the computation yields

$$h(x,t) = -\int_0^t \frac{\partial}{\partial x}\left[g(\tau,x)\right]d\tau. \tag{4.9}$$

Integrate the second term of $h^\lambda(x,t)$ by parts using Eq. (4.9). Since $h(x,0+) = 0$ for all $x > 0$, this integration yields Eq. (4.6). □

Fig. 1 plots the inverse tempered stable subordinator density $h^\lambda(x,t)$ as a function of $x$ for $\beta = 0.6$ and $t = 1$, by numerically evaluating the formula (4.6) for different values of the tempering parameter $\lambda$. The graph for $\lambda = 0$ was plotted in [41, Fig. 2]. Proposition 4.4 gives additional information on the jump at $x = 0$.

**Remark 4.3.** An alternative to Eq. (4.6) was given in [30]:

$$h^\lambda(x,t) = e^{x\lambda^\beta}\int_0^t\left[W\left(-\beta,-\beta;x/u^\beta\right)u^{-\beta-1} - \lambda^2 W\left(-\beta,0;x/u^\beta\right)/u\right]e^{-\lambda u}\,du$$

where

$$W(a, b; z) = \frac{1}{2\pi i}\int_H e^{s+zs^{-a}} s^{-b}\, ds \tag{4.10}$$

is the Fox–Wright function [7, Eq. (1.157)] and $H$ is the Hankel contour. Another form was given in [42].

**Proposition 4.4.** *The density $h^\lambda(x, t)$ is discontinuous at $x = 0$ for all $\lambda \geq 0$. In fact, we have $h^\lambda(x, t) = 0$ for $x < 0$ and*

$$h^\lambda(0+, t) = \phi^\lambda(t, \infty) = \frac{\beta\lambda^\beta \Gamma(-\beta, \lambda t)}{\Gamma(1-\beta)} \tag{4.11}$$

*where $\phi^\lambda(t, \infty)$ is the Lévy measure (2.11) and $\Gamma(a, x)$ is the incomplete gamma function (4.1).*

**Proof.** Invert the Fourier transform in (3.6) to see that the inverse tempered stable pdf has Laplace transform

$$\tilde{h}^\lambda(x, s) = s^{-1}\psi^\lambda(s)H(x)e^{-x\psi^\lambda(s)} \tag{4.12}$$

where $H(x)$ is the Heaviside function. It follows that $h^\lambda(x, t) = 0$ for all $x < 0$. Letting $x \to 0+$ in (4.12), we have $\tilde{h}^\lambda(0+, s) = s^{-1}\psi^\lambda(s)$, and then it follows from (2.12) that $h^\lambda(0+, t) = \phi^\lambda(t, \infty) > 0$ for all $\lambda \geq 0$. $\square$

**Remark 4.5.** It follows easily from (2.11) that

$$h^\lambda(0^+, t) \to h^0(0^+, t) = \phi^0(t, \infty) = \frac{t^{-\beta}}{\Gamma(1-\beta)} \quad \text{as } \lambda \to 0, \tag{4.13}$$

see [41, Sec. 4] for the case $\lambda = 0$. To determine the behavior as $\lambda \to \infty$, note that $\Gamma(\alpha, z) \sim z^{\alpha-1}e^{-z}$ as $z \to \infty$ [43], implying

$$\phi^\lambda(t, \infty) \sim \frac{\beta t^{-\beta-1}e^{-\lambda t}}{\lambda\Gamma(1-\beta)} \quad \text{as } \lambda \to \infty.$$

Hence $\phi^\lambda(t, \infty) \to 0$ as $\lambda \to \infty$, implying that $h^\lambda(0^+, t) \to 0$ as $\lambda \to \infty$.

**Lemma 4.6.** *For any $t > 0$ the inverse tempered stable density function $h^\lambda(x, t)$ is a bounded continuous function of $x > 0$, with $h^\lambda(x, t) \to 0$ as $x \to \infty$, and in fact for some $C > 0$, $K > 0$, and $x_0 > 0$ we have*

$$h^\lambda(x, t) \leq Cx^{1/(2-2\beta)} \exp\left[-Kx^{1/(1-\beta)}\right] \tag{4.14}$$

*for all $x > x_0$.*

**Proof.** In order to show that $h^\lambda(x, t)$ is continuous at any $x > 0$, we use the representation (4.6). Note that $g(\tau, x) = \tau^{-1/\beta}g(\tau^{-1/\beta}x)$ where $g(x)$ is a smooth function of $x$ real, and $g(x) \leq Kx^{-\beta-1}$ for all $x > x_0$, for some $x_0$, $K$ depending on $\beta$ [44, p. 143]. Then the first term in (4.6) is continuous in view of (4.7). A straightforward dominated convergence argument shows that the second and third terms are also continuous in $x$. Hence $h^\lambda(x, t)$ is continuous at any $x > 0$. To show that $h^\lambda(x, t) \to 0$ as $x \to \infty$, we use the representation (4.3). Note that $\phi^\lambda(du) = e^{-\lambda u}\phi(du)$ where $\phi$ is the Lévy measure of the stable subordinator prior to tempering [14, Theorem 2]. Then $\phi^\lambda(u, \infty) \leq \phi(u, \infty) = u^{-\beta}/\Gamma(1-\beta)$ by [9, Proposition 3.10]. Also $g(x) \leq Kx^{(1-\beta/2)/(\beta-1)} \exp[-Cx^{\beta/(\beta-1)}]$ for all $0 < x < 1$ by [27, Eq. (2.4)], and after a little algebra we have that

$$g^\lambda(y, x) \leq e^{x\lambda^\beta} x^{-1/\beta}g(yx^{-1/\beta}) \leq Ky^{(1-\beta/2)/(\beta-1)}x^{1/(2-2\beta)} \exp[-C_2x^{1/(1-\beta)}]$$

for all $0 < y < t$ and all $x \geq x_0 > 1$. Then

$$\begin{aligned} h^\lambda(x, t) &\leq C_3x^{1/(2-2\beta)} \exp\left[-C_2x^{1/(1-\beta)}\right] \int_0^t (t-y)^{-\beta}y^{(1-\beta/2)/(\beta-1)}dy \\ &\leq C_4x^{1/(2-2\beta)} \exp\left[-C_2x^{1/(1-\beta)}\right] \end{aligned}$$

so that (4.14) holds. Hence $h^\lambda(x, t) \to 0$ as $x \to \infty$. Since $h(0+, t)$ is bounded for any $t > 0$ in view of Proposition 4.4, it follows that $h^\lambda(x, t)$ is a bounded continuous function of $x > 0$ for any $t > 0$. $\square$

## 5. Scaling and asymptotic properties

Recall that a stochastic process $X_t$ is called *self-similar* if $X_{ct}$ and $c^H X_t$ have the same finite dimensional distributions. Since the scaling property in Proposition 5.1 also involves the tempering parameter, it is weaker than self-similarity.

**Proposition 5.1.** *Let $D_x^\lambda$ denote the tempered stable subordinator given by (2.3), and let $E_t^\lambda$ denote the inverse tempered stable subordinator (3.1). Then for any $c \geq 0$, the following scaling properties hold:*

$$D_{cx}^\lambda \overset{f.d.}{=} c^{1/\beta} D_x^{c^{1/\beta}\lambda} \tag{5.1a}$$

$$E_{ct}^\lambda \overset{f.d.}{=} c^\beta E_t^{c\lambda} \tag{5.1b}$$

*where $\overset{f.d.}{=}$ denotes equality in the sense of finite dimensional distributions. Hence the densities of $D_x^\lambda$ and $E_t^\lambda$ scale as*

$$g^\lambda(t, cx) = c^{-1/\beta} g^{c^{1/\beta}\lambda}\left(c^{-1/\beta}t, x\right) \tag{5.2a}$$

$$h^\lambda(x, ct) = c^{-\beta} h^{c\lambda}\left(c^{-\beta}x, t\right). \tag{5.2b}$$

**Proof.** Recall from (2.3) and (2.4) that the tempered stable subordinator $D_x^\lambda$ has Laplace transform $\mathbb{E}[e^{-sD_x^\lambda}] = e^{-x\psi^\lambda(s)}$ for all $x > 0$ and $s > 0$, where $\psi^\lambda(s) = (s+\lambda)^\beta - \lambda^\beta$. For any $c > 0$ we have

$$c\psi^\lambda(s) = \left(c^{1/\beta}s + c^{1/\beta}\lambda\right)^\beta - \left(c^{1/\beta}\lambda\right)^\beta = \psi^{c^{1/\beta}\lambda}\left(c^{1/\beta}s\right),$$

and it follows that

$$\mathbb{E}\left[e^{-sD_{cx}^\lambda}\right] = e^{-cx\psi^\lambda(s)} = e^{-x\psi^{c^{1/\beta}\lambda}(c^{1/\beta}s)} = \mathbb{E}\left[e^{-sc^{1/\beta}D_x^{c^{1/\beta}\lambda}}\right].$$

Since the Laplace transform uniquely determines the distribution, it follows that $D_{cx}^\lambda$ and $c^{1/\beta}D_x^{c^{1/\beta}\lambda}$ are identically distributed. Now a standard argument yields equality in finite dimensional distribution: Given $x_0 = 0 < x_1 < \cdots < x_m$, use the fact that $D_x^\lambda$ has stationary and independent increments to conclude that $\{D_{cx_i}^\lambda - D_{cx_{i-1}}^\lambda : 1 \leq i \leq m\}$ and $\{c^{1/\beta}D_{x_i}^{c^{1/\beta}\lambda} - c^{1/\beta}D_{x_{i-1}}^{c^{1/\beta}\lambda} : 1 \leq i \leq m\}$ are identically distributed. Then (5.1a) follows using the continuous mapping $f(y_1, \ldots, y_m) = (y_1, y_1 + y_2, \ldots, y_1 + \cdots + y_m)$.

If $D_x^\lambda \geq t$ then $D_y^\lambda > t$ for all $y > x$ so that $E_t^\lambda \leq x$. On the other hand, if $D_x^\lambda < t$ then $D_y^\lambda < t$ for all $y > x$ sufficiently close to $x$, so that $E_t^\lambda > x$. Then it follows easily that for any $0 \leq t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$ we have

$$\{E_{t_i}^\lambda \leq x_i : 1 \leq i \leq m\} = \{D_{x_i}^\lambda \geq t_i : 1 \leq i \leq m\}. \tag{5.3}$$

From (5.1a) we have $c^{-1/\beta}D_{cx}^\lambda \overset{f.d.}{=} D_x^{c^{1/\beta}\lambda}$ and hence

$$\begin{aligned}
P\{c^{-\beta}E_{ct_i}^\lambda \leq x_i : 1 \leq i \leq m\} &= P\{E_{ct_i}^\lambda \leq c^\beta x_i : 1 \leq i \leq m\} \\
&= P\{D_{c^\beta x_i}^\lambda \geq ct_i : 1 \leq i \leq m\} \\
&= P\{(c^\beta)^{-1/\beta}D_{c^\beta x_i}^\lambda \geq t_i : 1 \leq i \leq m\} \\
&= P\{D_{x_i}^{c\lambda} \geq t_i : 1 \leq i \leq m\} \\
&= P\{E_{t_i}^{c\lambda} \leq x_i : 1 \leq i \leq m\},
\end{aligned}$$

which is equivalent to (5.1b). Recall that if a random variable $X$ has pdf $f(x)$, then $cX$ has pdf $c^{-1}f(c^{-1}x)$. Then (5.2a) and (5.2b) follow. □

We will write $X_t^n \overset{f.d.}{\Longrightarrow} X_t$ to mean that the stochastic processes $X_t^n$ converge to $X_t$ in the sense of finite dimensional distributions. Recall that $D_x = D_x^0$ is the standard stable subordinator defined by (2.1), and $E_t = E_t^0$ is its inverse defined by (3.1). The next result shows that, as the tempering parameter $\lambda \to 0$, we recover the untempered case. Also, the tempered (inverse) stable subordinator process approximates the untempered process at early time.

**Proposition 5.2.** *Let $D_x^\lambda$ denote the tempered stable subordinator given by (2.3), and let $E_t^\lambda$ denote the inverse tempered stable subordinator (3.1). Then*

$$D_x^\lambda \overset{f.d.}{\Longrightarrow} D_x \quad \text{and} \quad E_t^\lambda \overset{f.d.}{\Longrightarrow} E_t \quad \text{as } \lambda \to 0. \tag{5.4}$$

*Furthermore*

$$c^{-1/\beta} D^{\lambda}_{cx} \overset{f.d.}{\Longrightarrow} D_x \quad \textit{and} \quad c^{-\beta} E^{\lambda}_{ct} \overset{f.d.}{\Longrightarrow} E_t \quad \textit{as } c \to 0. \tag{5.5}$$

**Proof.** Write $X \simeq Y$ to mean that $X, Y$ are identically distributed, and $X_n \Rightarrow Y$ for convergence in distribution. Note that $\psi^{\lambda}(s) \to \psi^0(s)$ as $\lambda \to 0$. Then it follows from the continuity theorem for the Laplace transform that $D^{\lambda}_x \Rightarrow D^0_x = D_x$ as $\lambda \to 0$. Now a standard argument yields convergence of finite dimensional distributions: Given $x_0 = 0 < x_1 < \cdots < x_m$, use the fact that $D^{\lambda}_x$ has stationary and independent increments to conclude that

$$\{D^{\lambda}_{x_i} - D^{\lambda}_{x_{i-1}} : 1 \le i \le m\} \Rightarrow \{D_{x_i} - D_{x_{i-1}} : 1 \le i \le m\}$$

as $\lambda \to 0$. Then the first part of (5.4) follows using the Continuous Mapping Theorem [45, Theorem 5.2] with $f(y_1, \ldots, y_m) = (y_1, y_1 + y_2, \ldots, y_1 + \cdots + y_m)$. Now it follows using (5.1a) that

$$c^{-1/\beta} D^{\lambda}_{cx} \simeq D^{c^{1/\beta}\lambda}_x \overset{f.d.}{\Longrightarrow} D^0_x = D_x$$

as $c \to 0$.

Then for any $0 \le t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \ge 0$ we have

$$\begin{aligned} P\{E^{\lambda}_{t_i} \le x_i : 1 \le i \le m\} &= P\{D^{\lambda}_{x_i} \ge t_i : 1 \le i \le m\} \\ &\to P\{D^0_{x_i} \ge t_i : 1 \le i \le m\} \\ &= P\{E^0_{t_i} \le x_i : 1 \le i \le m\}, \end{aligned}$$

which proves the second part of (5.4). Then apply this fact along with (5.1b) to see that

$$c^{-\beta} E^{\lambda}_{ct} \simeq E^{c\lambda}_t \overset{f.d.}{\Longrightarrow} E^0_t = E_t$$

as $c \to 0$. □

**Remark 5.3.** Note that the sample paths of $E^{\lambda}_t$ are nondecreasing, and that the process $E_t$ has continuous sample paths, so that it is continuous in probability. Then Proposition 5.2 together with [46, Theorem 3] shows that we also get convergence $E^{\lambda}_t \Rightarrow E_t$ as $\lambda \to 0$ and $c^{-\beta} E^{\lambda}_{ct} \Rightarrow E_t$ as $c \to 0$ in the Skorokhod space $D([0, \infty), [0, \infty))$ with the $J_1$ topology.

**Remark 5.4.** The first part of (5.4) was originally proved in [19, Theorem 3.1], in the case of a more general tempering function. Here the proof is simpler, because the exponential tempering leads to a nice scaling property.

**Remark 5.5.** We now consider the asymptotic behavior of $g^{\lambda}(t, x)$ and $h^{\lambda}(x, t)$ in terms of $\lambda$. As $\lambda \to 0$, (2.2) reduces to the pdf of the stable subordinator $g(t, x)$. Likewise, the second and third terms in (4.6) vanish, yielding the density of the inverse stable subordinator $h(x, t)$. Hence, we recover the standard (untempered) densities as $\lambda \to 0$.

For the case $\lambda \to \infty$, note that for the Lévy measure (2.11) we have by the dominated convergence theorem that $\phi^{\lambda}(y, \infty) \to 0$ for all $y > 0$ as $\lambda \to \infty$. This vague convergence $\phi^{\lambda} \to 0$ implies that $D^{\lambda}_x \Rightarrow 0$ as $\lambda \to \infty$ [47, Theorem 3.1.16]. Hence we have $g^{\lambda}(t, x) \to \delta_0(x)$, the Dirac delta function at $x = 0$, as $\lambda \to \infty$. Since $E^{\lambda}_t$ is the inverse function of $D^{\lambda}_x$, this implies that $E^{\lambda}_t \to \infty$ in probability as $\lambda \to \infty$: To see this, write

$$P(E^{\lambda}_t \le x) = P(D^{\lambda}_x \ge t) \to \int_t^{\infty} \delta_0(x)\, dx = 0 \quad \text{as } \lambda \to \infty.$$

Hence we have $h^{\lambda}(x, t) \to 0$ as $\lambda \to \infty$ for all $t > 0$ and $x > 0$.

## 6. Tempered fractional Cauchy problems

Let $X$ be a Banach space (say $C_0(\mathbb{R})$) with norm $\|\cdot\|$. A family of linear operators $\{T_t : t > 0\}$ is called a *semigroup* if $T_0 f = f$ for all $f \in X$ and $T_{t+s} = T_t T_s$ for all $t, s > 0$. The semigroup is *bounded* if there exists a constant $M_t > 0$ such that $\|T_t f\| \le M_t \|f\|$ for all $f \in X$. The semigroup is *strongly continuous* if $\|T_t f - f\| \to 0$ as $t \to 0$. A bounded, strongly continuous semigroup is called a $C_0$ semigroup. The *generator* $L$ of the semigroup $T_t$ is a linear operator defined as

$$Lf(x) = \lim_{t \to 0} \frac{T_t f(x) - f(x)}{t} \tag{6.1}$$

for all $f \in D(L)$, the domain of the generator $L$, which is the set of all $f \in X$ for which the limit (in the Banach space norm) exists. Then the *tempered fractional Cauchy problem*

$$\partial_t^{\beta,\lambda} q(x, t) = Lq(x, t); \qquad q(x, 0) = f(x) \tag{6.2}$$

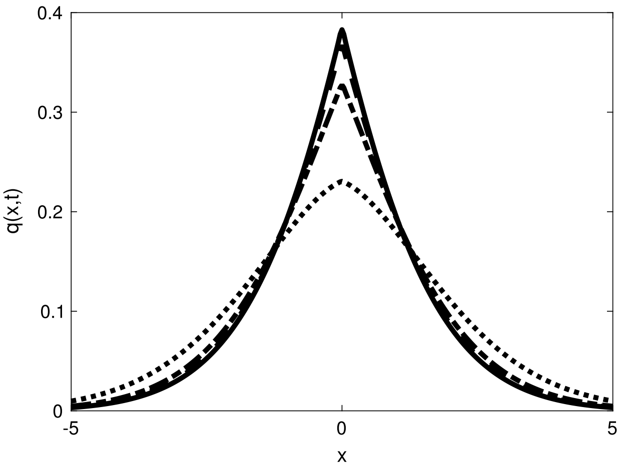


**Fig. 2.** Solutions (6.3) of the tempered fractional diffusion equation (6.4) with $D = 0.1$ and $\beta = 0.6$ at time $t = 1$, for $\lambda = 0$ (solid), 0.01 (dashed), 0.1 (dot-dashed) and 1.0 (dotted).

with $0 < \beta < 1$ and $\lambda > 0$ has a unique solution

$$q(x,t) = \int_0^\infty p(x,u) h^\lambda(u,t)\, du \tag{6.3}$$

where $p$ solves the traditional Cauchy problem $\partial_t p = Lp$ with initial condition $p(0) = f \in D(L)$, and $h^\lambda(u,t)$ is the pdf of the inverse tempered stable subordinator (3.1). This follows from [28, Theorem 4.1] for a Lévy generator, and from [29, Theorem 4.1] for a uniformly elliptic generator $L$ in divergence form on a bounded domain.

If $X(t)$ is a Markov process with forward generator $L$, so that the point source solution to $\partial_t p = Lp$ is the pdf of $X(t)$ (e.g., see [9, p. 62]), then the pdf $q(x,t)$ of the non-Markovian stochastic process $Y_t = X(E_t^\lambda)$ solves the tempered fractional Cauchy problem (6.2). Using the computational formula (4.6) for the inverse tempered stable subordinator pdf, we can provide explicit solutions for a wide variety of tempered fractional Cauchy problems (6.2).

Fig. 2 shows the solutions to the tempered fractional diffusion equation

$$\partial_t^{\beta,\lambda} q(x,t) = D\partial_x^2 q(x,t); \qquad q(x,0) = \delta(x) \tag{6.4}$$

for several values of the tempering parameter $\lambda$. The solutions were computed by substituting the exact point source solution

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \tag{6.5}$$

to the original Cauchy problem $\partial_t p(x,t) = D\partial_x^2 q(x,t)$ into the formula (6.3), using the explicit formula (4.6) for the inverse tempered stable pdf. All solutions exhibit a cusp, or a non-differentiable point, at $x = 0$. This cusp has been noted previously for the (untempered) time-fractional diffusion equation [5, Figure 6]. The next result provides more detail on the cusp in Fig. 2.

**Proposition 6.1.** *The solution to the tempered fractional diffusion equation (6.4) is the probability density function of a time-changed Brownian motion $B(E_t^\lambda)$ with mean zero and variance $2Dt$, where $E_t^\lambda$ is the inverse tempered stable subordinator (3.1). This density function $q(x,t)$ is not differentiable at $x = 0$ for any $t > 0$, and in fact we have*

$$\partial_x q(0+,t) = -2Dh^\lambda(0+,t) \quad \text{and} \quad \partial_x q(0-,t) = 2Dh^\lambda(0+,t).$$

**Proof.** Substitute (6.5) into (6.3) to see that

$$q(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi Du}} e^{-x^2/(4Du)} h^\lambda(u,t)\, du.$$

Now consider the difference quotient, and make a change of variable $s = \delta^2/(4Du)$ to get

$$\begin{aligned} \delta^{-1}[q(\delta,t) - q(0,t)] &= \frac{1}{\sqrt{4\pi D}} \int_0^\infty \delta^{-1}[e^{-\delta^2/(4Du)} - 1] u^{-1/2} h^\lambda(u,t)\, du \\ &= \frac{1}{4D\sqrt{\pi}} \int_0^\infty [e^{-s} - 1] s^{-3/2} h^\lambda(\delta^2/(4Ds), t)\, ds \\ &\to \frac{1}{4D\sqrt{\pi}} h^\lambda(0+,t) \int_0^\infty [e^{-s} - 1] s^{-3/2}\, ds \end{aligned}$$

as $\delta \to 0+$ by the dominated convergence theorem, using Lemma 4.6. Since

$$\int_0^\infty [e^{-s} - 1]s^{-3/2}\,ds = -2\sqrt{\pi}$$

by [9, Proposition 3.10], it follows that $\partial_x q(0+,t) = -2Dh^\lambda(0+,t)$. The proof for $\partial_x q(0-,t) = 2Dh^\lambda(0+,t)$ is quite similar. □

**Remark 6.2.** From Proposition 4.4 we know that $h^\lambda(0+,t) = \phi^\lambda(t,\infty) > 0$ where $\phi^\lambda(t,\infty)$ is given by (2.11) in terms of the incomplete gamma function. Since $\phi^\lambda(t,\infty) \to 0$ as $\lambda \to \infty$, the slope at the cusp decreases as the tempering parameter $\lambda$ increases.

Next consider the tempered fractional Dirichlet problem

$$\partial_t^{\beta,\lambda} q(x,t) = \partial_x^2 q(x,t)$$

on the bounded domain $(0, 1)$ with boundary conditions $q(0,t) = q(1,t) = 0$ and initial condition $q(x, 0) = f(x)$. The general solution to the Cauchy problem $\partial_t p(x,t) = \partial_x^2 u(x,t)$; $p(x, 0) = f(x)$ is given by

$$p(x,t) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\psi_n(x)$$

where $f_n = 2\int \psi_n(x)f(x)\,dx$. Here $\lambda_n = (n\pi)^2$ and $\psi_n(x) = \sin(n\pi x)$ are the eigenvalues and eigenfunctions of the generator $L = \partial_x^2$ with the zero boundary conditions, so that $\psi_n''(x) = \lambda_n\psi_n(x)$ for all $0 < x < 1$ and $\psi_n(0) = \psi_n(1) = 0$, see for example [48, Eq. (8) with $\alpha = 1$]. Then for example if the initial condition $f(x) = \sin(\pi x)$, the solution to the tempered fractional Dirichlet problem is given by (6.3) with

$$p(x,t) = e^{-\pi^2 t}\sin(\pi x). \tag{6.6}$$

Fig. 3 displays numerical solutions to this fractional Dirichlet problem for several different values of $\lambda$.

**Remark 6.3.** As noted in [30,49], the tempered fractional Cauchy problem Eq. (6.2) may be recast as an integral equation. Take the LT of (6.2) using (2.9), yielding

$$\psi^\lambda(s)\tilde{q}(x,s) - s^{-1}\psi^\lambda(s)f(x) = L\tilde{q}(x,s). \tag{6.7}$$

Divide (6.7) by $\psi^\lambda(s)$ and define $\tilde{M}^\lambda(s) = 1/\psi^\lambda(s)$, yielding

$$\tilde{q}(x,s) = s^{-1}f(x) + \tilde{M}^\lambda(s)L\tilde{q}(x,s). \tag{6.8}$$

Invert the LT in (6.8) using $\mathcal{L}^{-1}[s^{-1}] = H(t)$ and the convolution theorem to get

$$q(x,t) = f(x)H(t) + \int_0^t M^\lambda(t-\tau)Lq(x,\tau)\,d\tau \tag{6.9}$$

where the kernel function $M^\lambda(t)$ is given by

$$\begin{aligned} M^\lambda(t) &= \mathcal{L}^{-1}\left[\tilde{M}^\lambda(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+\lambda)^\beta - \lambda^\beta}\right] \\ &= e^{-\lambda t}\mathcal{L}^{-1}\left[\frac{1}{s^\beta - \lambda^\beta}\right] = e^{-\lambda t}t^{\beta-1}E_{\beta,\beta}\left(\lambda^\beta t^\beta\right) \end{aligned}$$

and $E_{\beta,\beta}(z)$ is the two-parameter Mittag-Leffler function [7, Eq. (1.56)]. Note that (6.9) corresponds to [30, Eq. (12)] for $t > 0$, an integral form equivalent to the tempered fractional Cauchy problem, with solution given by (6.3).

## 7. The tempered fractional Poisson process

The fractional Poisson process $N_\beta(t) = \max\{n \ge 0 : T_n \le t\}$ was studied in [38,50–57]. Here $T_n = J_1 + \cdots + J_n$ is the time of the $n$th arrival, with independent interarrival times such that

$$\mathbb{P}(J_n > t) = E_\beta(-\alpha t^\beta) = \sum_{k=0}^{\infty}\frac{(-\alpha t^\beta)^k}{\Gamma(1+\beta k)} \tag{7.1}$$

for $0 < \beta \le 1$, using the one-parameter Mittag-Leffler function $E_\beta(z) = E_{\beta,1}(z)$. The special case $\beta = 1$ is a traditional Poisson process. Theorem 2.2 in [55] shows that one can also construct the fractional Poisson process by replacing the time $t$ in the traditional Poisson process $N_1(t)$ with an independent inverse stable subordinator.

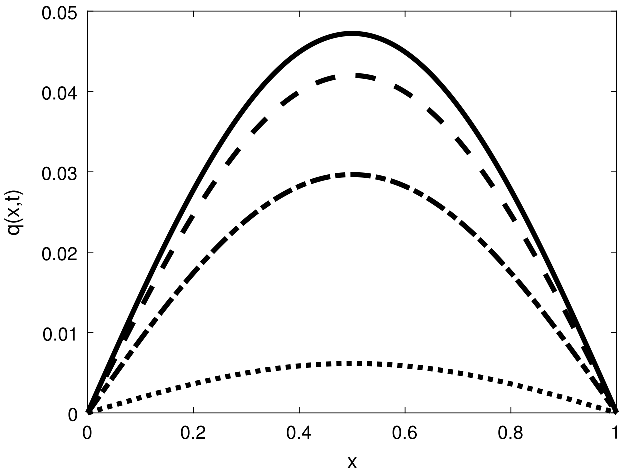


**Fig. 3.** Solutions to the tempered fractional Dirichlet problem at $t = 1$ and $\beta = 0.6$ using (6.3). The tempering parameter $\lambda =$ is 0 (solid), 0.01 (dashed), 0.1 (dot-dashed) and 1.0 (dotted).

In this section, we discuss the *tempered fractional Poisson process* $N^\lambda(t) = N_1(E_t^\lambda)$, where $E_t^\lambda$ is the inverse tempered stable subordinator (3.1). Theorem 4.1 in [55] shows that one can also write

$$N^\lambda(t) = \max\{n \geq 0 : T_n \leq t\} \tag{7.2}$$

where $T_n = J_1 + \cdots + J_n$ is the sum of independent and identically distributed waiting times with

$$\mathbb{P}(J_n > t) = \mathbb{E}[e^{-\alpha E_t^\lambda}]. \tag{7.3}$$

Example 5.7 in [55] shows that the pdf $w^\lambda(t)$ of the waiting times $J_n$ has Laplace transform

$$\int_0^\infty e^{-st} w^\lambda(t)\, dt = \frac{\alpha}{\alpha + (s+\lambda)^\beta - \lambda^\beta}.$$

Inverting this Laplace transform as in [55, Example 5.7] shows that

$$w^\lambda(t) = w(t) e^{-\lambda t}(\eta + \lambda^\beta)/\eta$$

when $\eta = \alpha - \lambda^\beta > 0$, where $w(t)$ is the Mittag-Leffler pdf of the fractional Poisson process waiting times (7.1).

It follows from [55, Theorem 5.2] along with (2.10) and (2.11) that the probability distribution $q(m, t) = \mathbb{P}[N^\lambda(t) = m]$ of the tempered fractional Poisson process solves the tempered fractional Cauchy problem

$$\partial_t^{\beta,\lambda} q(m, t) = \alpha\big[q(m-1, t) - q(m, t)\big] \tag{7.4}$$

for all integers $m$, with initial condition $q(0, 0) = 1$ and $q(m, 0) = 0$ for $m \neq 0$. The solution to this tempered fractional Cauchy problem yields the probability distribution of the tempered fractional Poisson process $N^\lambda(t) = N_1(E_t^\lambda)$: First recall that the traditional Poisson process with rate $\alpha$ has pmf

$$p(m, t) = \mathbb{P}[N_1(t) = m] = e^{-\alpha t}\frac{(\alpha t)^m}{m!}$$

for all $m \geq 0$ and all $t \geq 0$, with the convention that $0^0 = 1$. This is the solution to the Cauchy problem $\partial_t p(m, t) = \alpha\big[p(m-1, t) - p(m, t)\big]$ with the same initial condition $q(0, 0) = 1$ and $q(m, 0) = 0$ for $m \neq 0$. Then apply (6.3) to see that

$$q(m, t) = \int_0^\infty e^{-\alpha u}\frac{(\alpha u)^m}{m!}\, h^\lambda(u, t)\, du \tag{7.5}$$

for all $m \geq 0$ and all $t \geq 0$, where the inverse tempered stable pdf $h^\lambda(u, t)$ can be computed using the explicit formula (4.6).

Eq. (7.5) is numerically evaluated using (4.6) in Fig. 4 using $\beta = 0.6$ for $\alpha = 1$, $t = 3$ and $\lambda = 0$ (thick bars) and $\lambda = 0.1$ (thin bars).

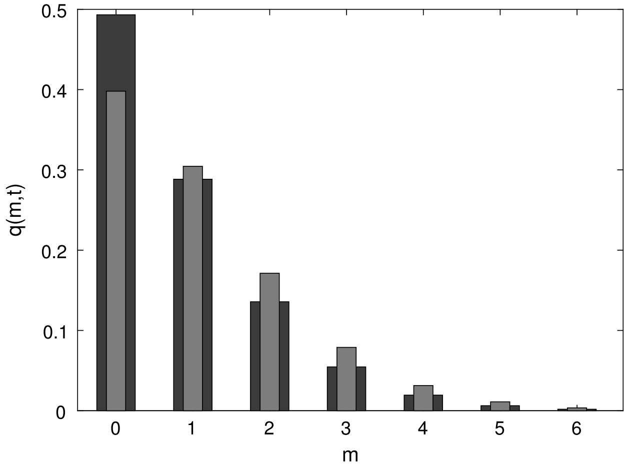


**Fig. 4.** Probability distribution (7.5) of the tempered fractional Poisson process with $\beta = 0.6$ for $\alpha = 1$, $t = 3$, for $\lambda = 0$ (thick bars) and $\lambda = 0.1$ (thin bars).

## 8. Summary

The inverse or hitting time of a tempered fractional subordinator is a useful stochastic process. It can be used as a time change to construct stochastic solutions to tempered fractional Cauchy problems, and its probability density appears in an integral formula (6.3) for explicit solutions. This paper develops two methods to compute the probability density: (1) a convolution integral (4.3); and (2) a computational formula (4.6). Scaling and asymptotic properties of the density are described, and the "folded" density is shown to solve a tempered fractional telegraph equation. Explicit solutions to a few tempered fractional Cauchy problems are presented as an illustration. The existence of a cusp at $x = 0$ for solutions to the tempered fractional diffusion equation is proven. A tempered fractional Poisson process is defined using the inverse tempered stable subordinator, and its probability distribution is computed.

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