Continuous time random walks, fractional calculus, and applications

Department of Mathematics
Tufts University
02 October 2009

Mark M. Meerschaert
Department of Statistics and Probability
Michigan State University

mcubed@stt.msu.edu
http://www.stt.msu.edu/∼mcubed

Partially supported by USA NSF grants DMS-0803360 and EAR-0823965.

Abstract

Continuous time random walks model the motion of diffusing particles, with a random waiting time preceding each random particle jump. In the long time limit, the CTRW converges to an infinitely divisible process governing the particle jumps, subordinated to an inverse or hitting time process that accounts for the waiting times. Probability densities of the limit process solve space-time pseudo-differential equations. Power law jumps lead to fractional space derivatives, while power law waiting times yield a fractional derivative in time. Sample paths are random fractals whose dimension equals the order of the fractional derivative. Ongoing research is extending these models in several directions. Exponentially tempered power laws lead to tempered fractional derivatives. Mixtures of power laws with different order are coded by distributed order fractional derivatives. Distribution dependence between waiting times and particle jumps is modeled by coupled space-time fractional derivatives. Correlated particle jumps can lead to long-range dependence. Several applications to water pollution and finance will be discussed, to illustrate the modeling issues and opportunities.

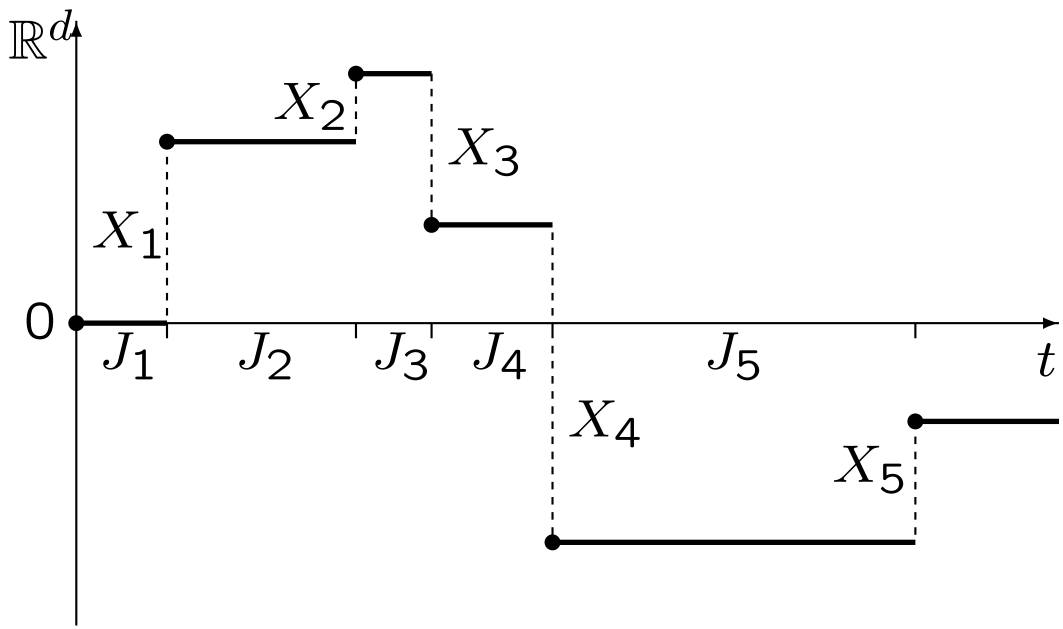
Collaborators

Chichi Aban, Biostatistics, U. Alabama Birmingham.
Paul Anderson, Math and Computer Science, Albion College.
Boris Baeumer, Maths & Stats, University of Otago, New Zealand.
Peter Becker-Kern, Math, Universität Dortmund, Germany.
David Benson, Colorado School of Mines.
P. Chakraborty, Math, California State U. Bakersfield.
James Kelly, Naval Postgraduate School.
Chae Young Lim, Statistics and Probability, Michigan State.
M. Kovács, Maths & Stats, University of Otago, New Zealand.
Robert McGough, Electrical Engineering, Michigan State.

Collaborators

Erkan Nane, Math and Stat, Auburn University.
Anna Panorska, Mathematics and Statistics, University of Nevada.
Parthanil Roy, Statistics and Probability, Michigan State
Enrico Scalas, Universita' del Piemonte Orientale, Italy.
Hans-Peter Scheffler, Math, Universität Siegen, Germany.
Rina Schumer, Desert Research Institute, Reno, Nevada.
Qin Shao, Mathematics, University of Toledo, Ohio
P. Vellaisamy, Mathematics, Indian Inst. Tech. Bombay, India
Stephen W. Wheatcraft, Geology, University of Nevada, Reno.
Yimin Xiao, Statistics and Probability, Michigan State University.
Yong Zhang, Desert Research Institute, Las Vegas NV.

Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

CTRW triangular arrays

Consider a sequence of CTRW at each scale $c > 0$

$S^{(c)}(n) = X_1^{(c)} + \cdots + X_n^{(c)}$ particle location after n jumps

$T^{(c)}(n) = J_1^{(c)} + \cdots + J_n^{(c)}$ time of the nth jump

$N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$ number of jumps by time $t > 0$

$S^{(c)}(N_t^{(c)})$ particle location at time $t > 0$ (CTRW)

Note $\{T^{(c)}(n) \leq t\} = \{N_t^{(c)} \geq n\}$ inverse processes

CTRW scaling limits

Assume $(S^{(c)}(cu), T^{(c)}(cu)) \Rightarrow (A(u), D(u))$ infinitely divisible

Write $\mathbb{E}(e^{-ik\cdot A(u)-sD(u)}) = e^{-u\psi(k,s)}$

$$\psi(k,s) = ia\cdot k + k\cdot Qk + \int \left(1 - e^{-ik\cdot x}e^{-st} - \frac{ik\cdot x}{1+\|x\|^2}\right)\phi(dx,dt)$$

$\mathbb{E}(e^{ik\cdot A(u)}) = e^{-u\psi_A(k)}$ with $\psi_A(k) = \psi(k,0)$
$\mathbb{E}(e^{-sD(u)}) = e^{-u\psi_D(s)}$ with $\psi_D(s) = \psi(0,s)$

Inverse mapping yields $N_{ct}^{(i)} \Rightarrow E(t)$
$E(t) = \inf\{u > 0 : D(u) > t\}$ inverse process.

CTRW scaling limit $S^{(c)}(N_t^{(c)}) \Rightarrow A(E(t))$

Semigroups and generators

The CTRW scaling limit defines a semigroup

$$T(u)f(x,t) = \int_0^t \int_{\mathbb{R}^d} f(x-y, t-r)\, P_{(A(u),D(u))}(dy, dr)$$

with generator

$$\psi(-i\mathbb{D}_x, \partial_t)f(x,t) = a \cdot \nabla f(x,t) - \nabla \cdot Q \nabla f(x,t)$$
$$- \int \left(f(x-y, t-u) - f(x,t) + \frac{\nabla f(x,t) \cdot y}{1 + \|y\|^2} \right) \phi(dy, du)$$

The pseudodifferential operator $\psi(-i\mathbb{D}_x, \partial_t)$ has symbol $\psi(k,s)$

Inverse subordinators

Let $g(t,u)$ be Lebesgue density of $t = D(u)$

Assume $\phi_D(0,\infty) = \infty$ and $\int_0^1 y|\ln y|\,\phi_D(dy) < \infty$ (technical).

Theorem $E(t) = \inf\{u > 0 : D(u) > t\}$ has Lebesgue density

$$f(u,t) = \int_0^t \phi_D(t-y,\infty) g(y,u)\,dy.$$

Moreover, the mapping $(u,t) \mapsto f(u,t)$ is measurable.

Idea: $f(u,t) = \frac{d}{du} P(E(t) \le u) = \frac{d}{du} P(D(u) \ge t)$
Compute with Laplace tranforms

Space-time decomposition

Suppose A, D are independent $\psi(k,s) = \psi_A(k) + \psi_D(s)$

Suppose $x = A(u)$ has Lebesgue density $p(x,u)$

CTRW scaling limit $A(E(t))$ has density

$$m(x,t) = \int_0^\infty p(x,u) f(u,t)\, du$$

Governing equations:

$$\partial_u p(x,u) = -\psi_A(-i\mathbb{D}_x) p(x,u); \quad p(x,0) = \delta(x)$$
$$\partial_u f(u,t) = -\psi_D(\partial_t) f(u,t) + \delta(u)\phi_D(t,\infty)$$
$$\psi_D(\partial_t) m(x,t) = -\psi_A(-i\mathbb{D}_x) m(x,t) + \delta(x)\phi_D(t,\infty)$$

Fractional derivatives

The Fourier transform $\hat{f}(k) = \int e^{-ikx} f(x)\, dx$ for $x \in \mathbb{R}$

$\mathbb{D}_x^\alpha f(x)$ has Fourier transform $(ik)^\alpha \hat{f}(k)$

In the simplest case $0 < \alpha < 1$

$$\int_0^\infty \left(e^{-iky} - 1\right) \alpha y^{-\alpha-1} dy = -\Gamma(1-\alpha)(ik)^\alpha$$

Since $f(x-y)$ has FT $e^{-iky}\hat{f}(k)$ we see that

$$\mathbb{D}_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y)\Big) y^{-\alpha-1} dy$$

The pseudodifferential operator $\mathbb{D}_x^\alpha$ has Fourier symbol $(ik)^\alpha$

Space-time fractional diffusion

Suppose $\psi_A(k) = (ik)^\alpha$ stable jump limit $0 < \alpha < 1$

$\psi_D(s) = s^\beta$ stable waiting time limit $0 < \beta < 1$

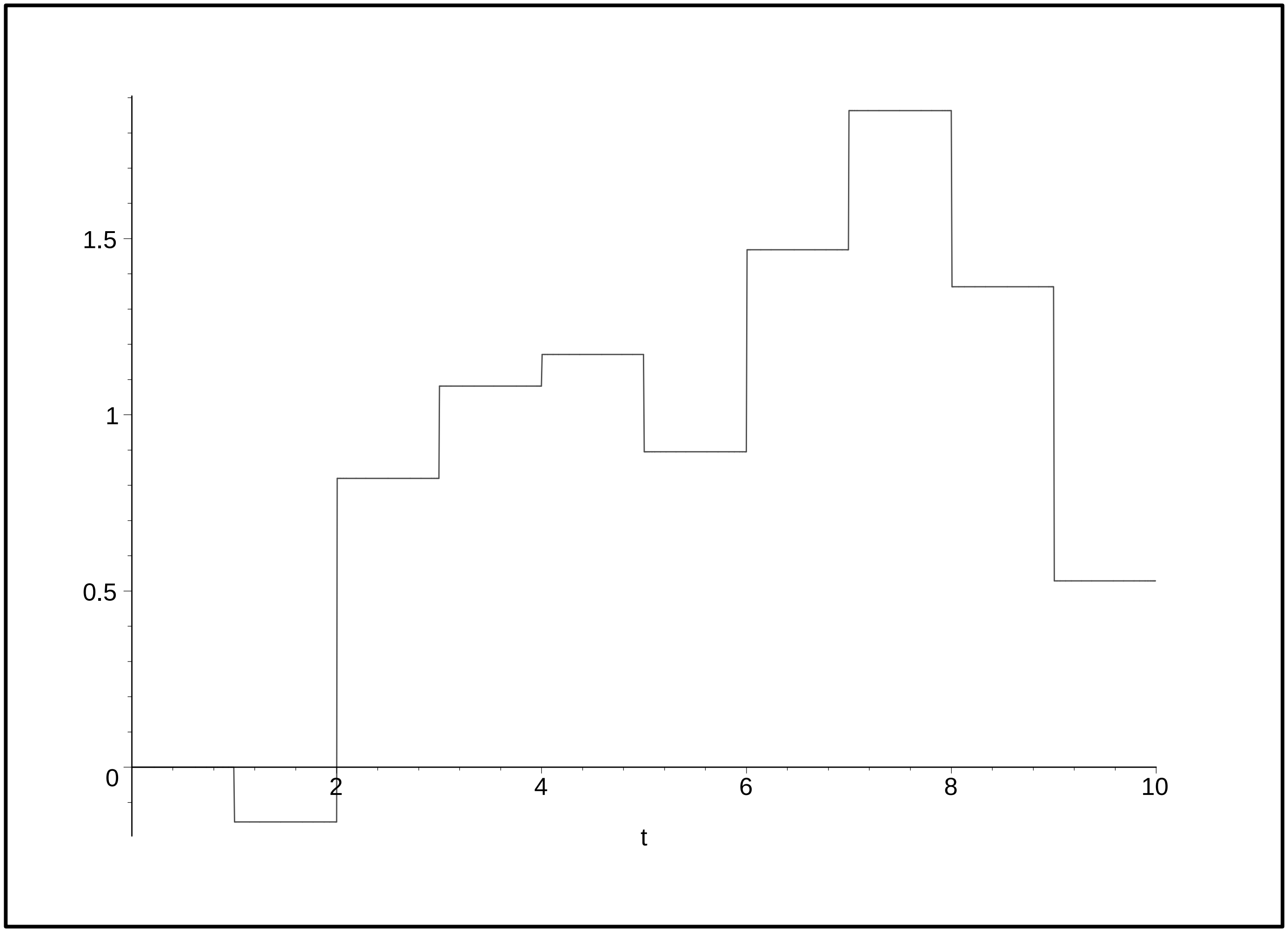
Then $\psi_A(-i\mathbb{D}_x) = \mathbb{D}_x^\alpha$ and $\psi_D(\partial_t) = \partial_t^\beta$ so

$$\partial_u p(x,u) = -\mathbb{D}_x^\alpha p(x,u); \quad p(x,0) = \delta(x)$$

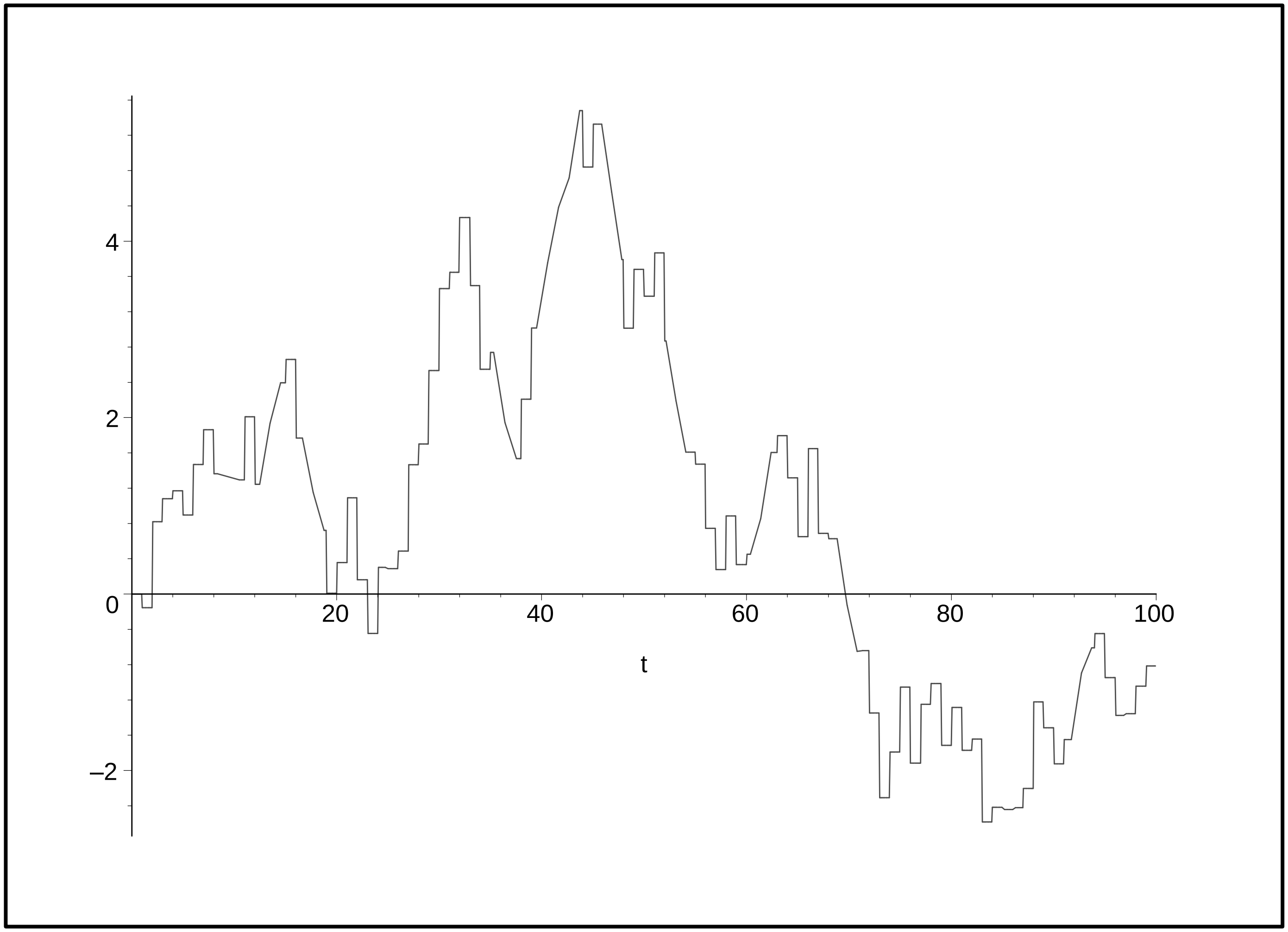
$$\partial_u f(u,t) = -\partial_t^\beta f(u,t) + \delta(u)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

$$\partial_t^\beta m(x,t) = -\mathbb{D}_x^\alpha m(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

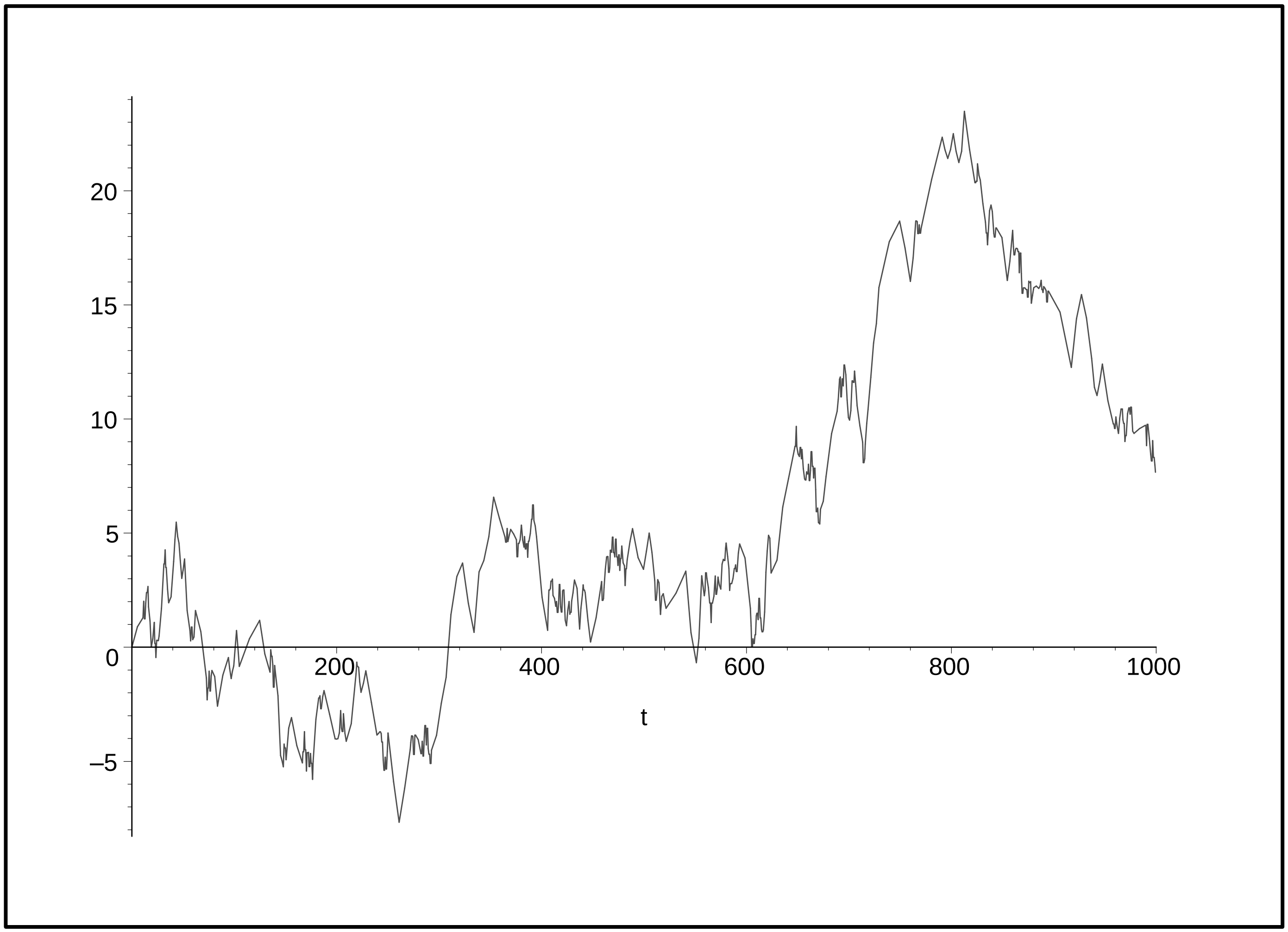
Simple random walk simulation



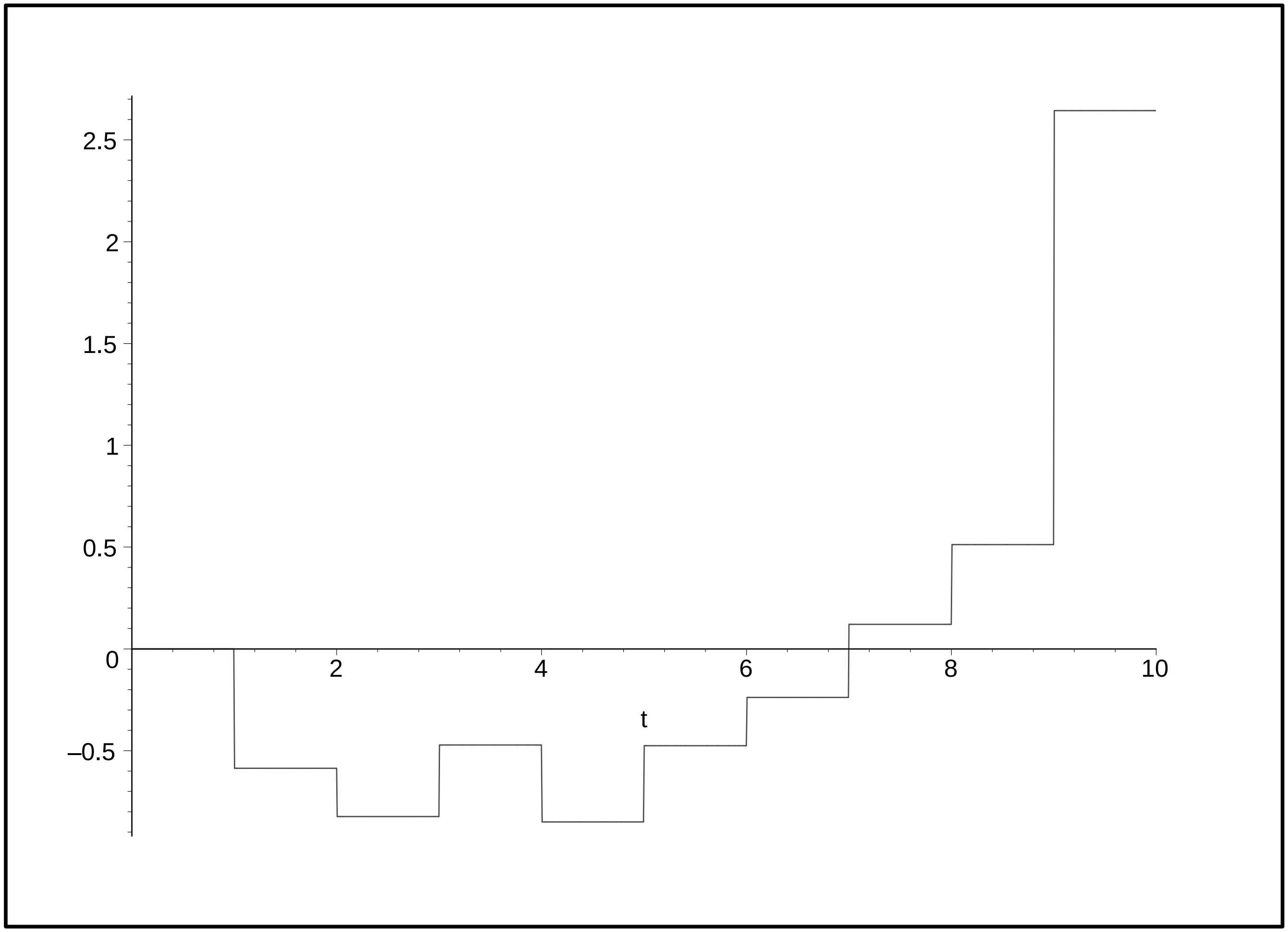
Longer time scale



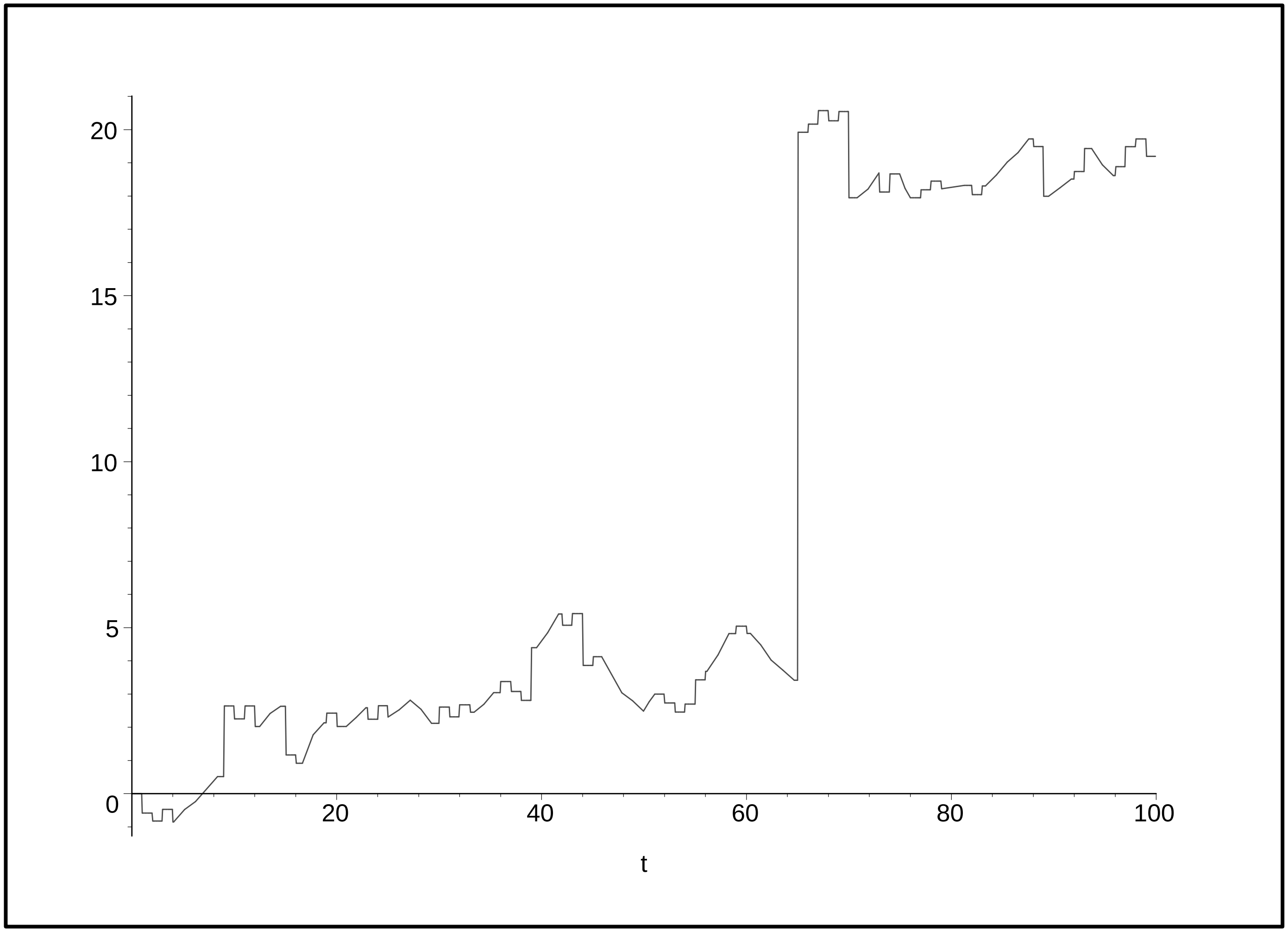
Scaling limit: Brownian motion



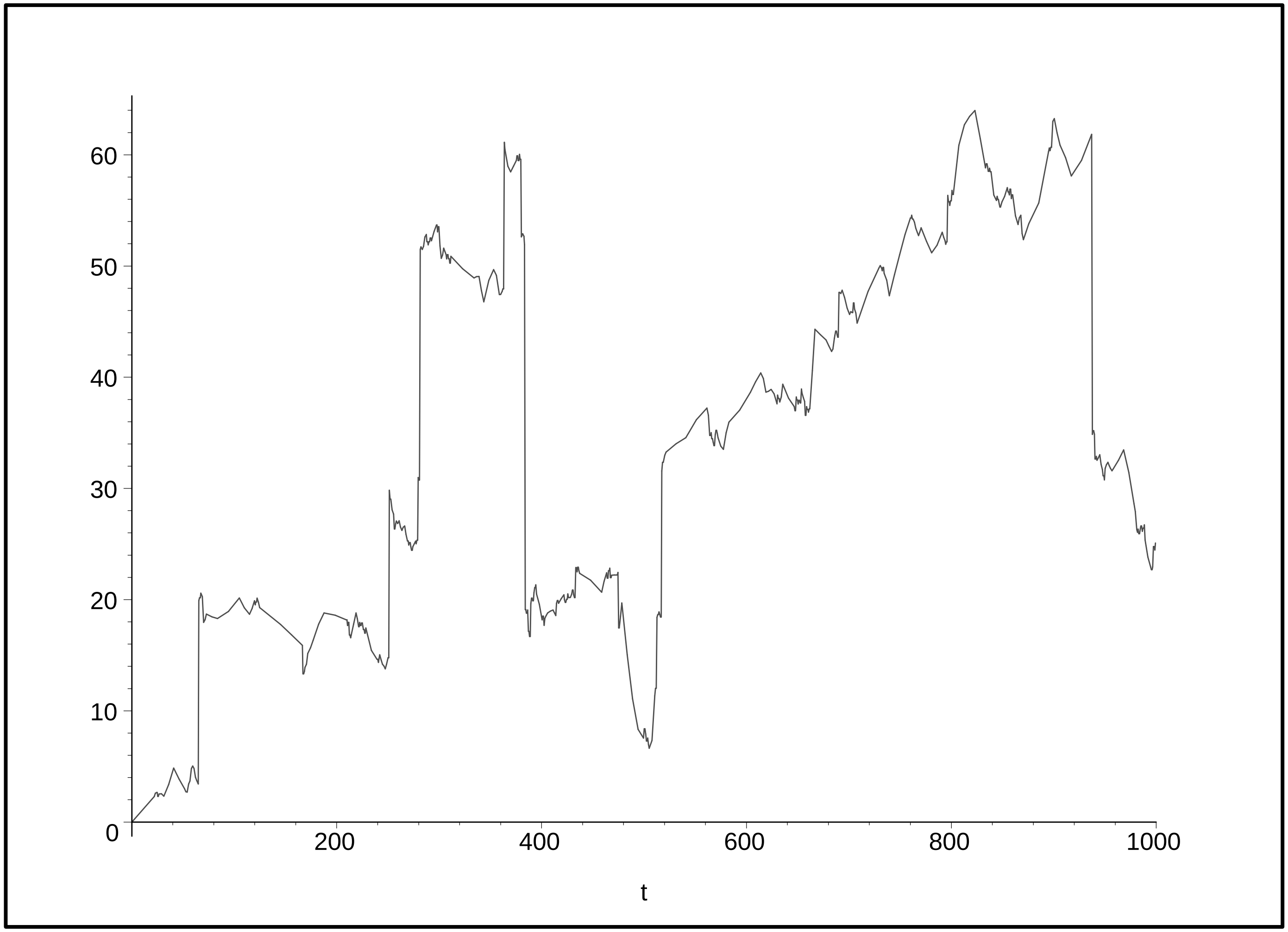
Random walk simulation: Heavy tails in space



Longer time scale

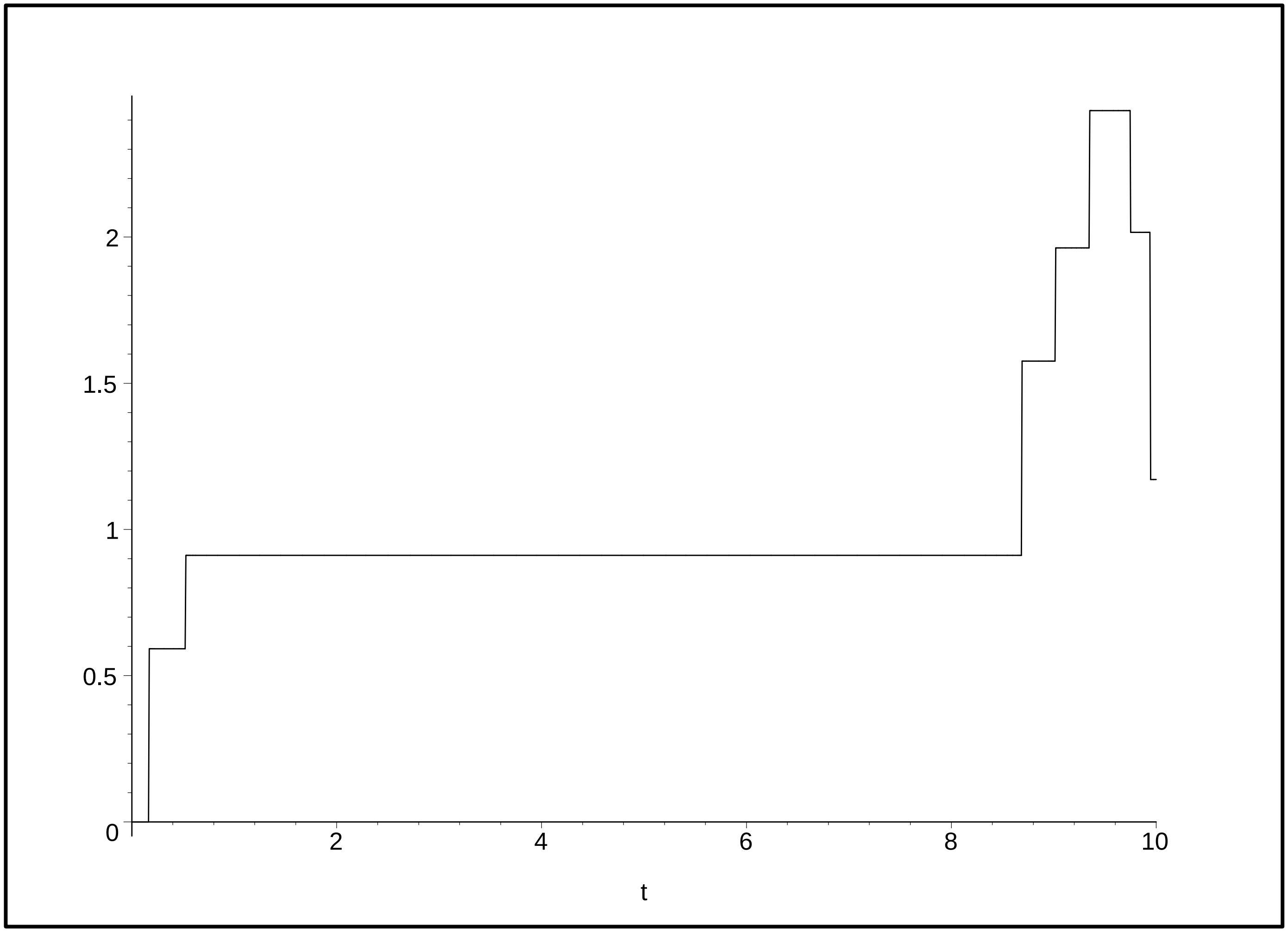


Scaling limit: Stable Lévy motion

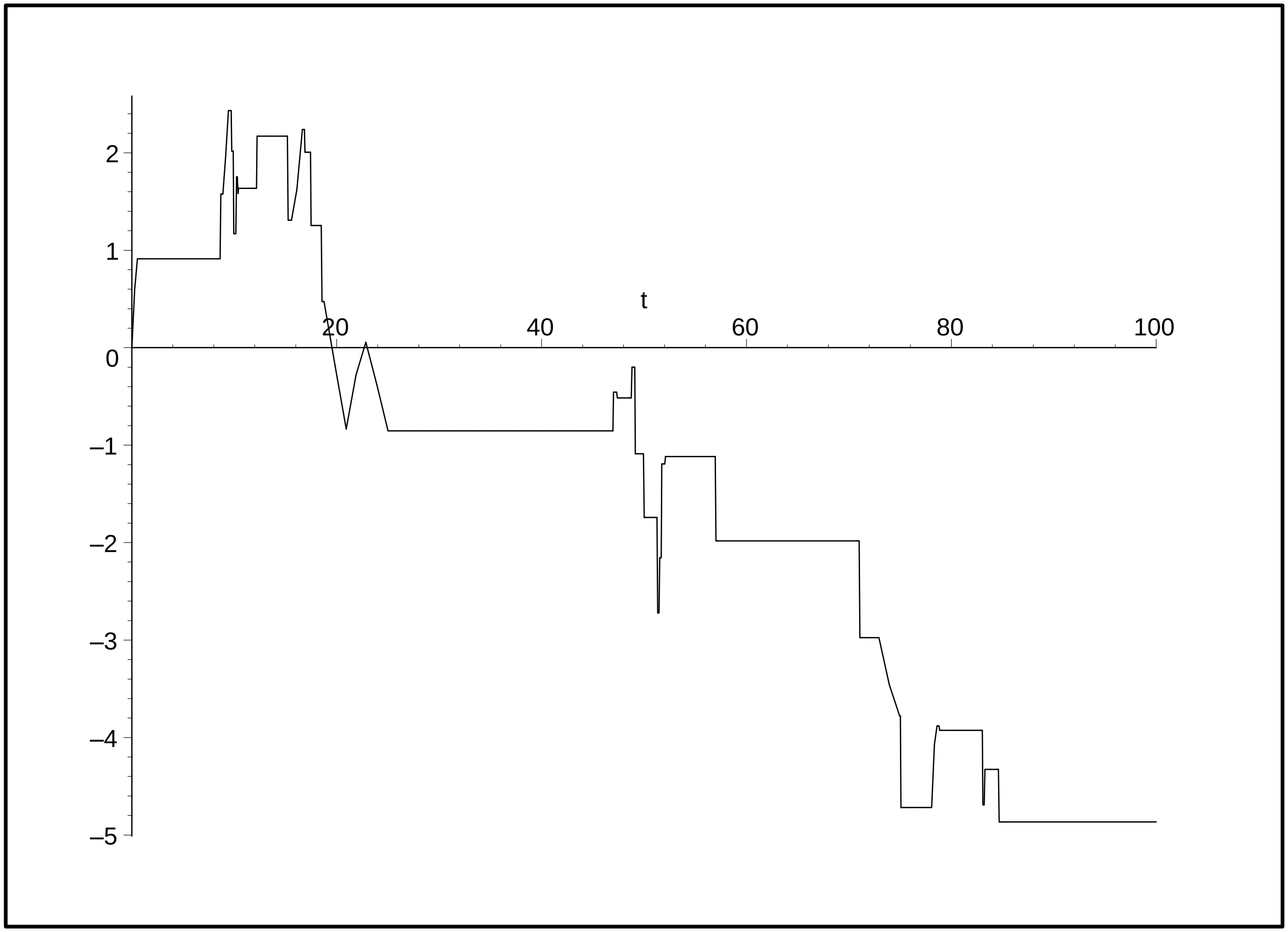


Limit process retains large jumps in space.

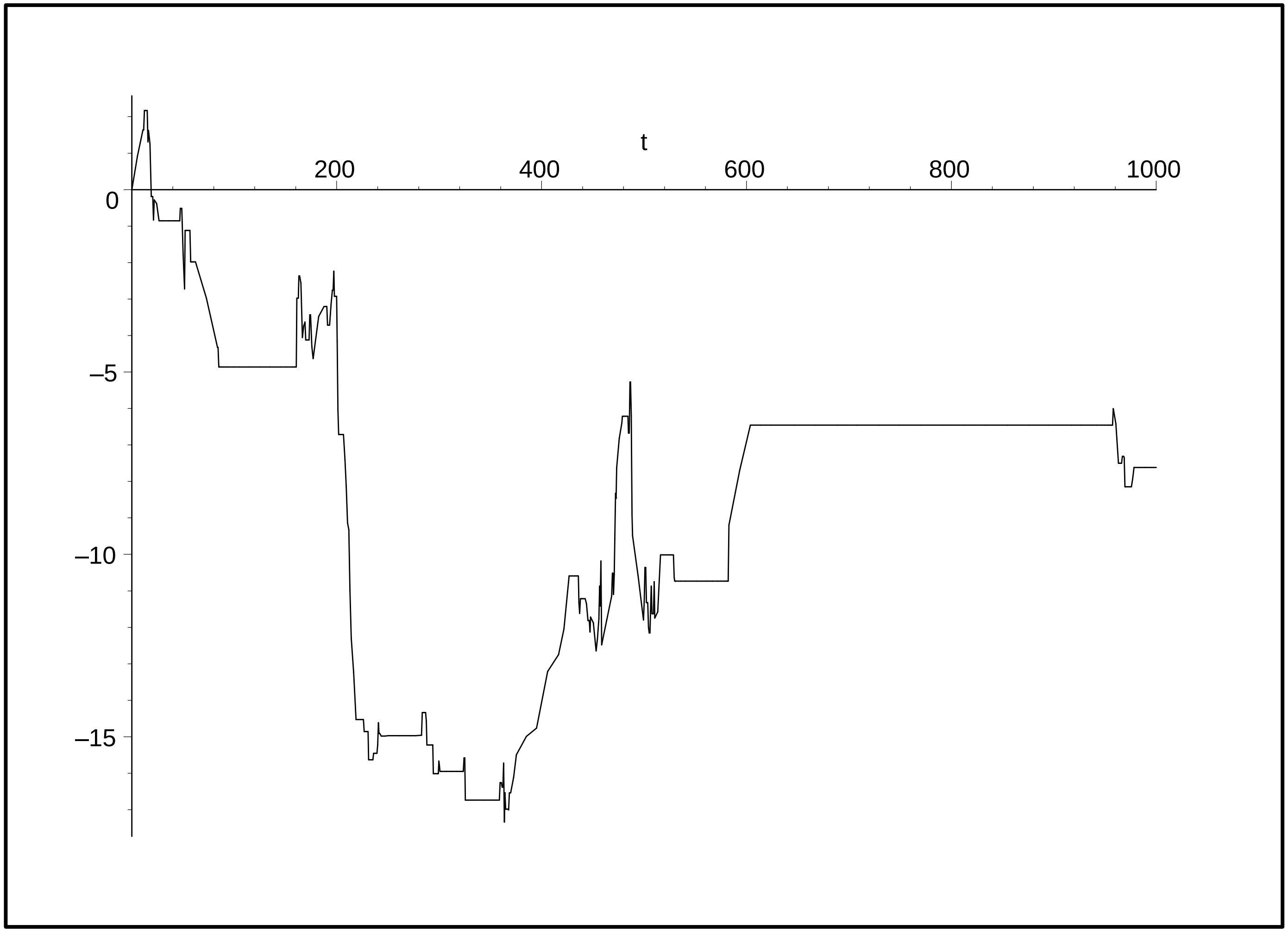
Heavy tail waiting times



Longer time scale



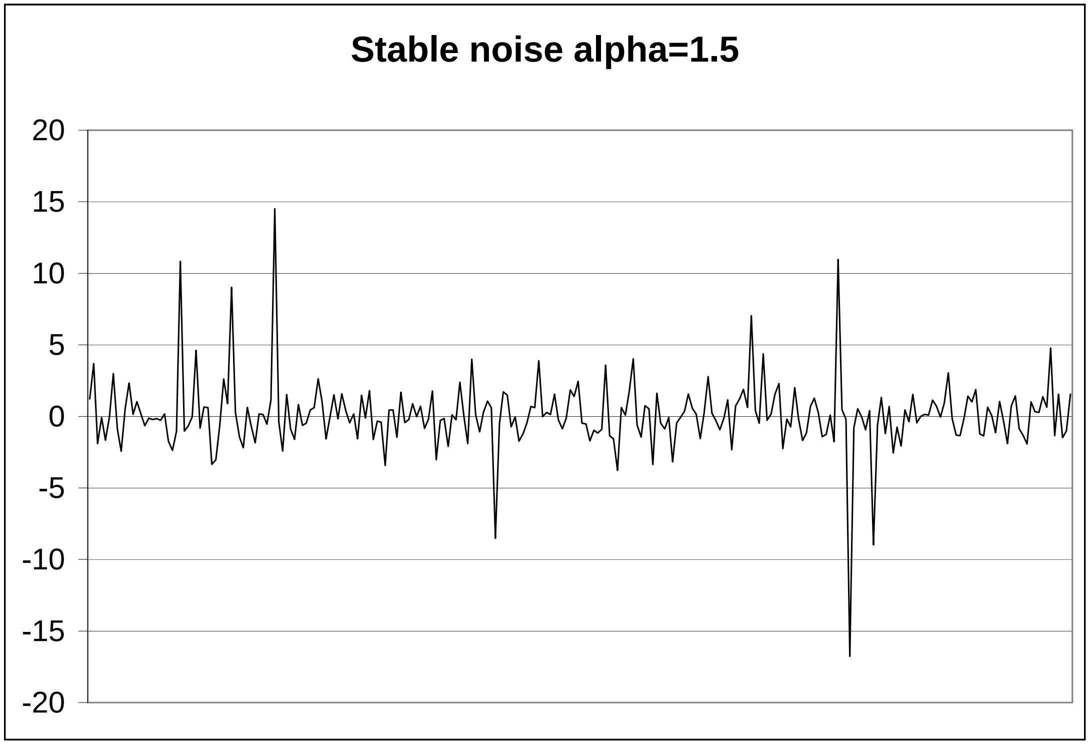
Scaling limit: Subordinated Lévy motion



Limit process retains large jumps in time.

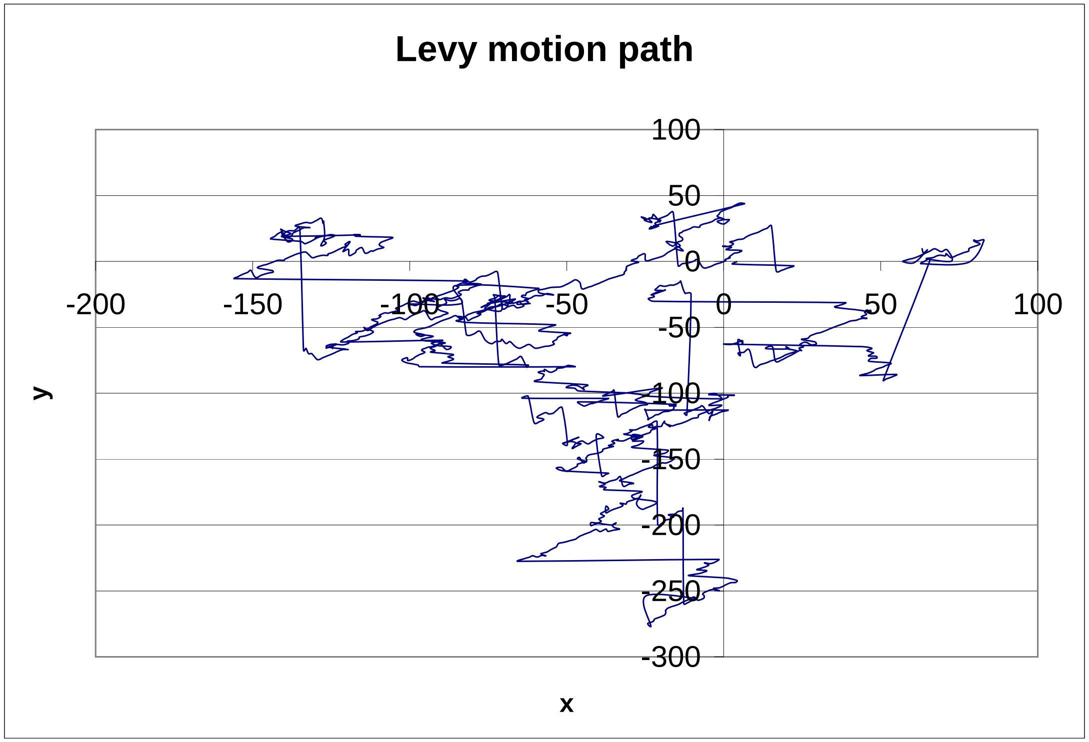
Heavy tails in electrical engineering

Electrical engineers use Lévy motion to model impulsive noise.



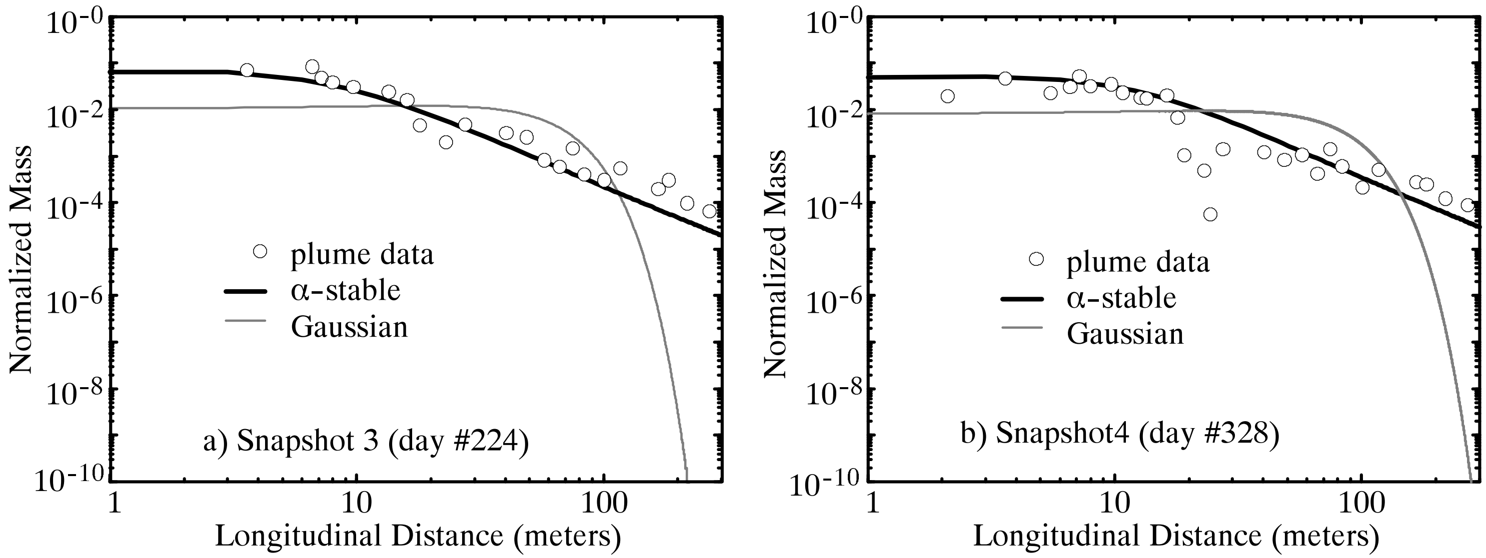
Random fractals

Lévy particle traces are random fractals with dimension α, so it takes $\approx n^\alpha$ disks of size $1/n$ to cover the path. Here $\alpha = 1.5$.



Tracer test in an underground aquifer

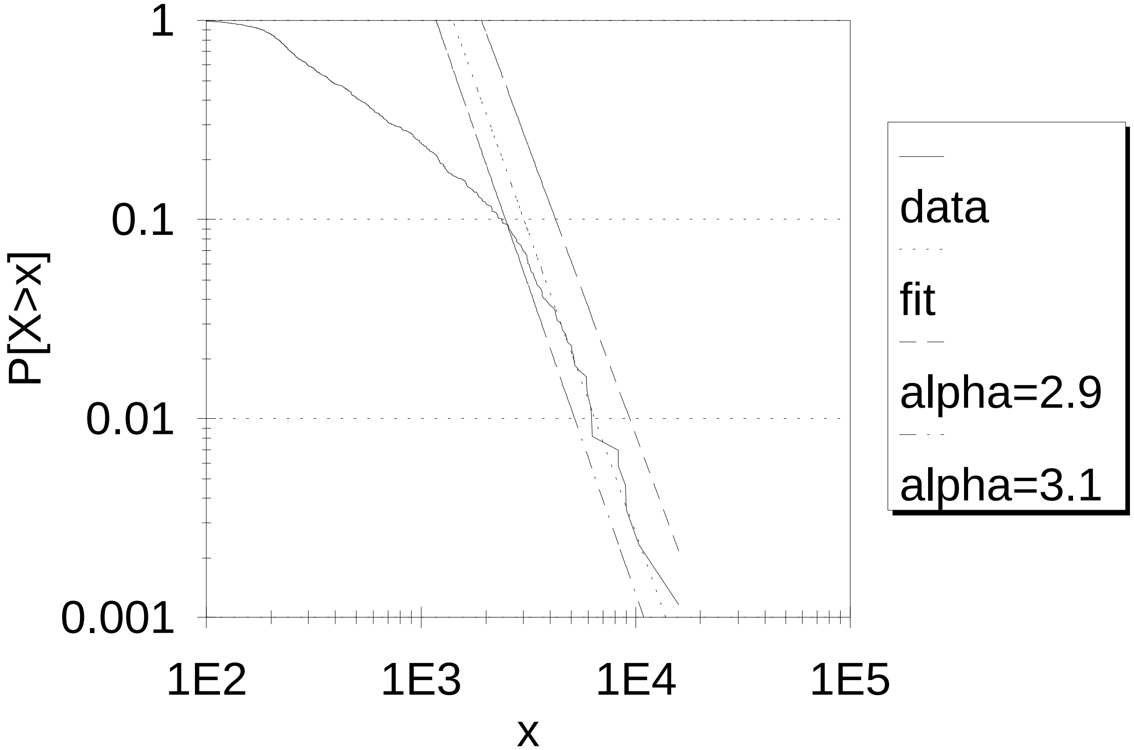
Stable Lévy motion density $p(x,t)$ with $\alpha = 1.1$ gives a good fit. Brownian motion badly underestimates tail concentrations.



Tracer plume has heavy power law tails and spreads like $t^{1/\alpha}$.

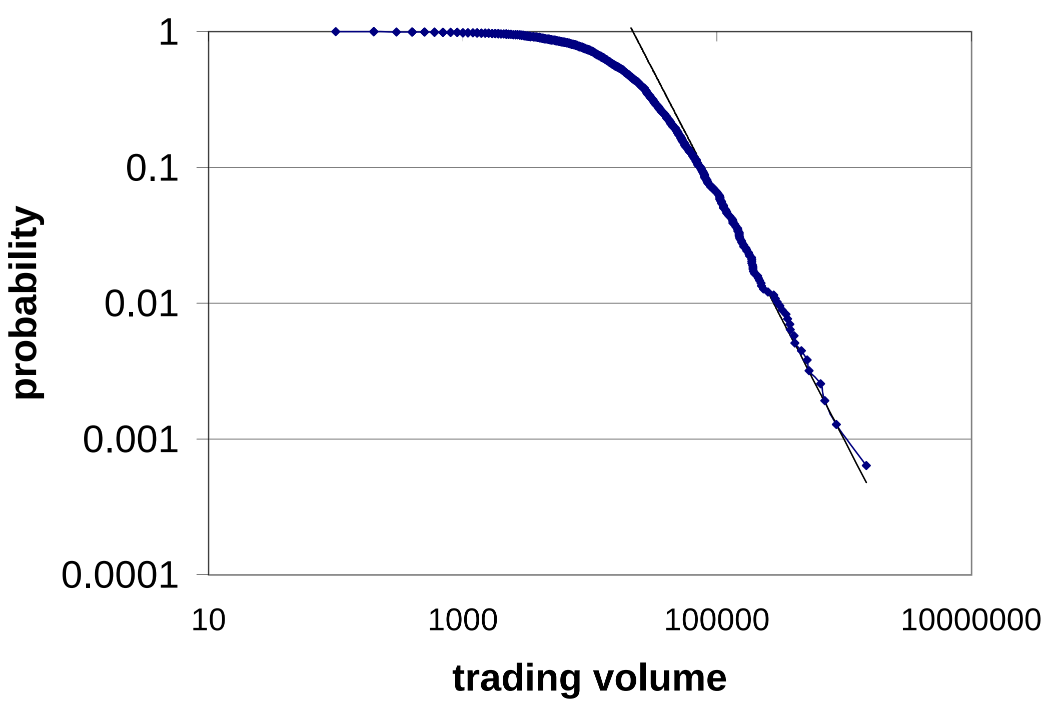
Heavy tail river flows

Monthly average flows for the Salt river in Roosevelt AZ have a heavy upper tail with $\alpha \approx 3$.



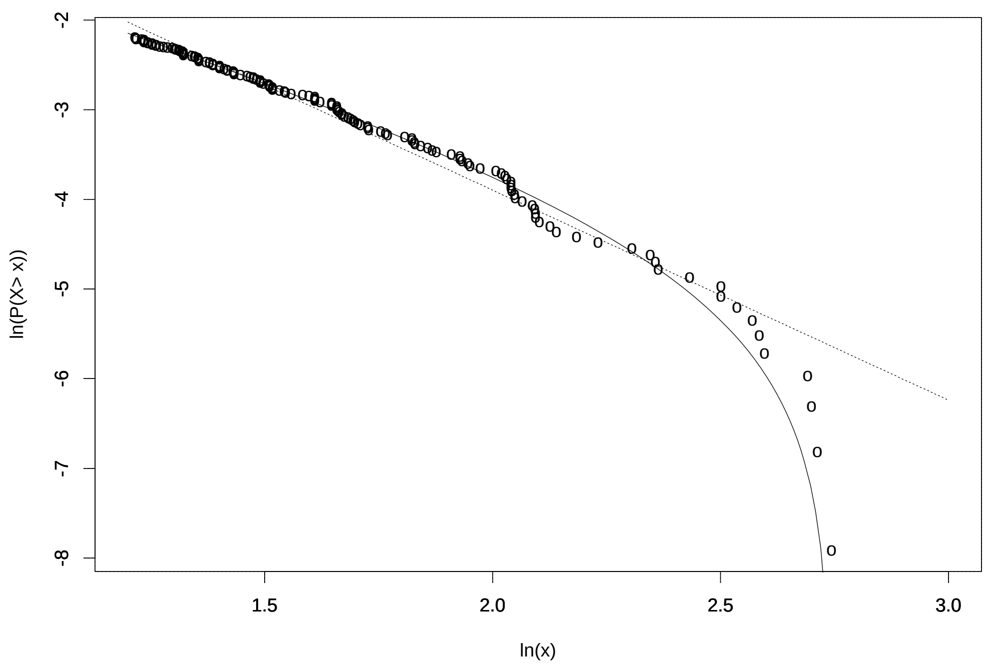
Heavy tails in finance

Trading volume for Amazon, Inc. has a heavy tail with $\alpha \approx 2.7$.
Heavy tails in finance were observed by Mandelbrot around 1960.



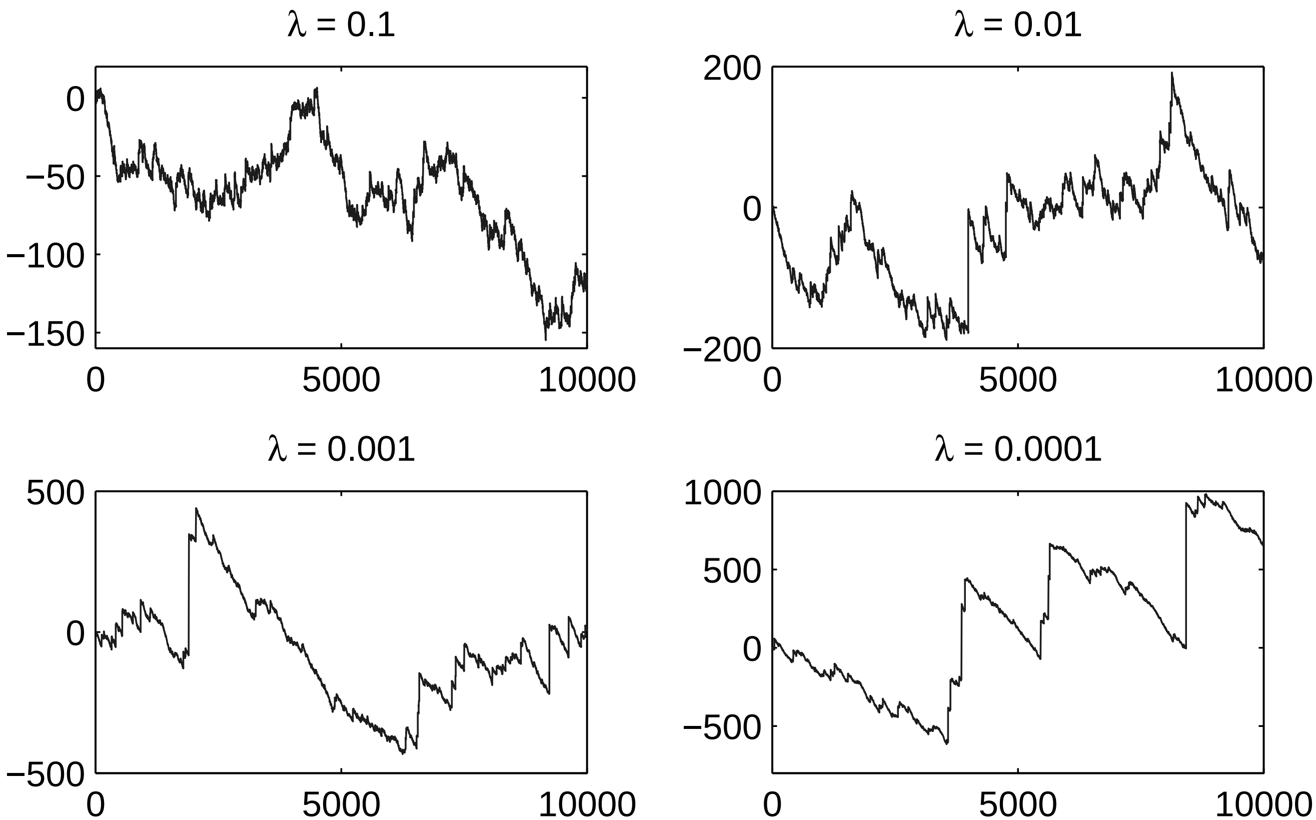
Tempered power laws in finance

AMZN stock price changes fit a tempered power law model $P(X > x) \approx x^{-\alpha} e^{-\lambda x}$ for x large: $\alpha = 0.6$, $\lambda = 0.3$.



Tempered Lévy motion

Tempered stable Lévy motion with $\alpha = 1.2$

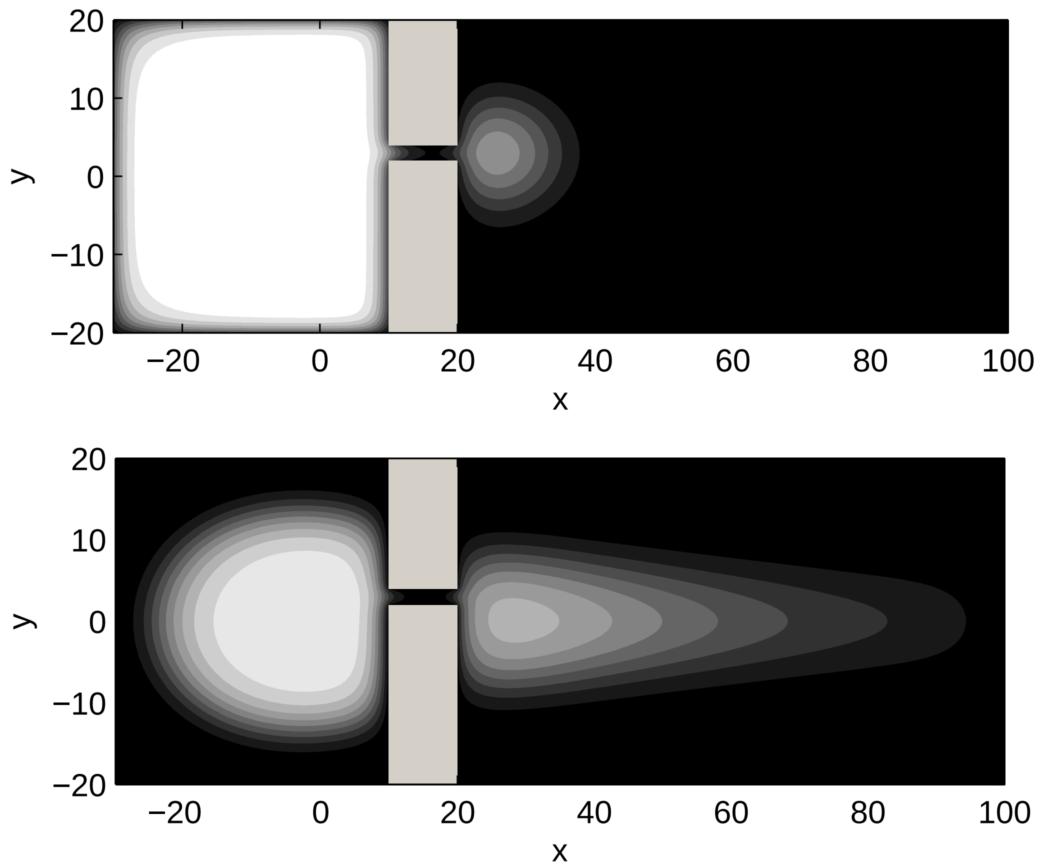


Biological species growth and dispersion

Fractional derivatives model fast spreading via long movements.

$$\frac{\partial p}{\partial t} = C \frac{\partial^\alpha p}{\partial x^\alpha} + D \frac{\partial^2 p}{\partial y^2} + rp\left(1 - \frac{p}{K}\right)$$

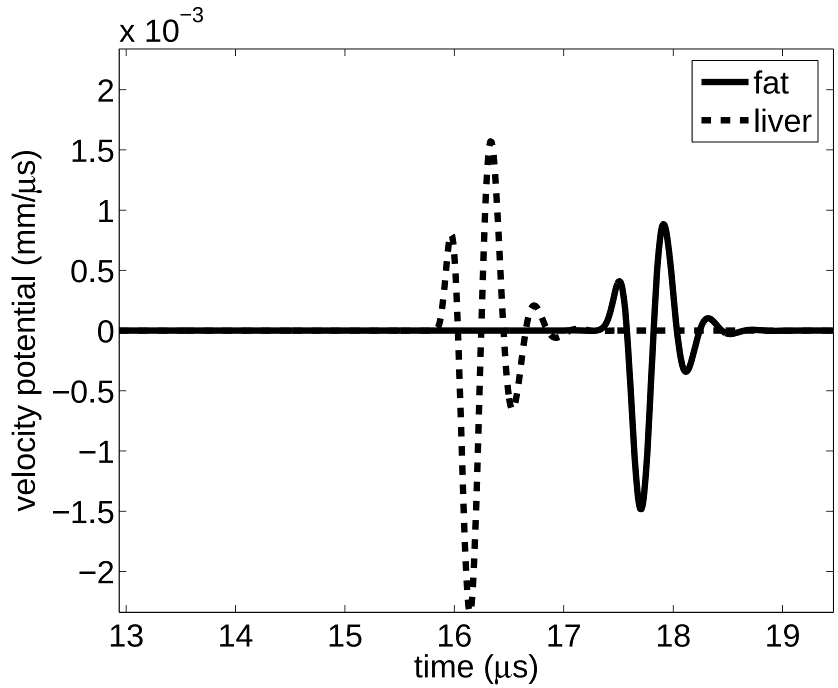
Compare $\alpha = 2$ (top) to $\alpha = 1.7$ (bottom).



Sound wave propogation

We use $\beta = 2.5$ for human fat tissue and $\beta = 2.1$ for liver tissue.

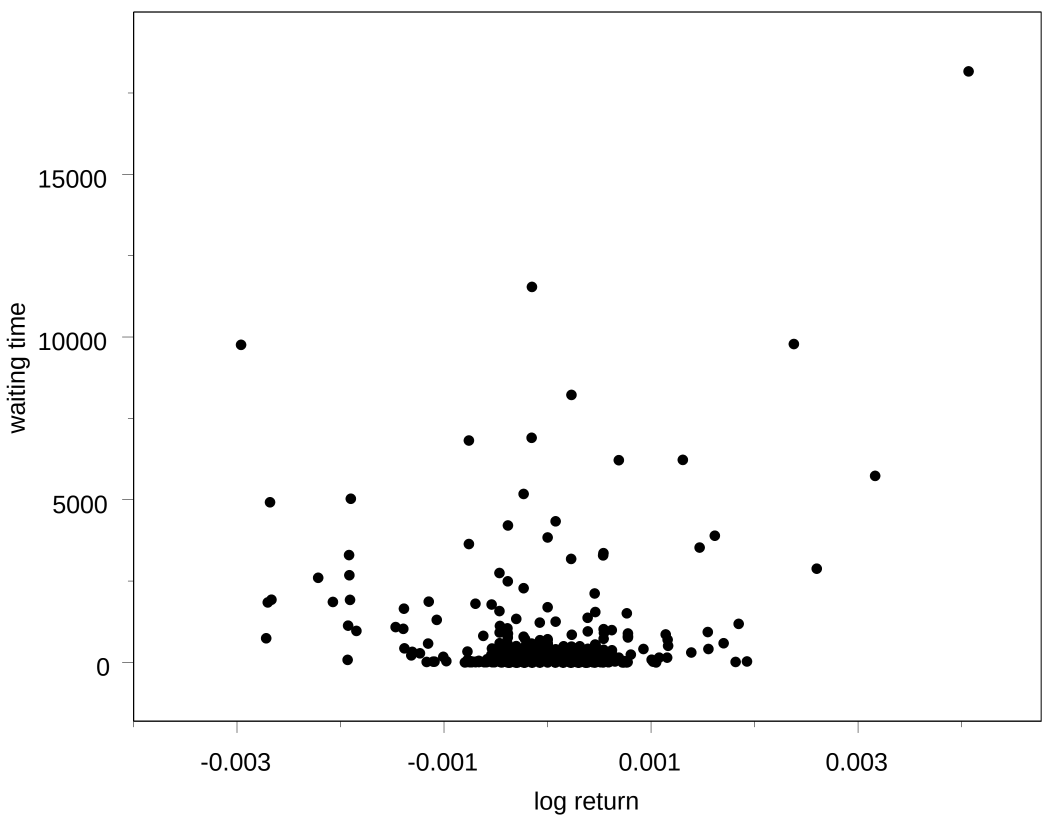
$$\frac{\partial^2}{\partial t^2} m(x,t) + C \frac{\partial^\beta}{\partial t^\beta} m(x,t) = D \frac{\partial^2}{\partial x^2} m(x,t)$$



LIFFE BTP bond futures Sept 1997 delivery

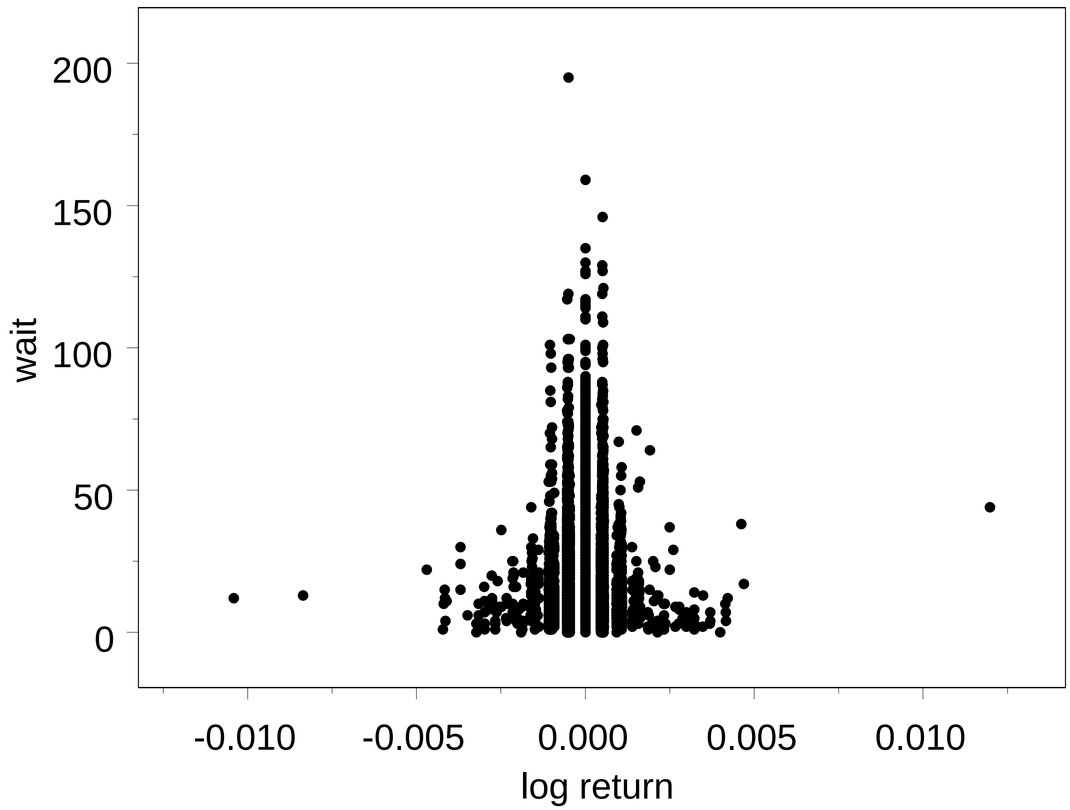
Log returns and waiting times (sec) are dependent.

$$(\partial_t - \mathbb{D}_x^2)^{0.95} m(x,t) = \delta(x) t^{-0.95} / \Gamma(0.05)$$



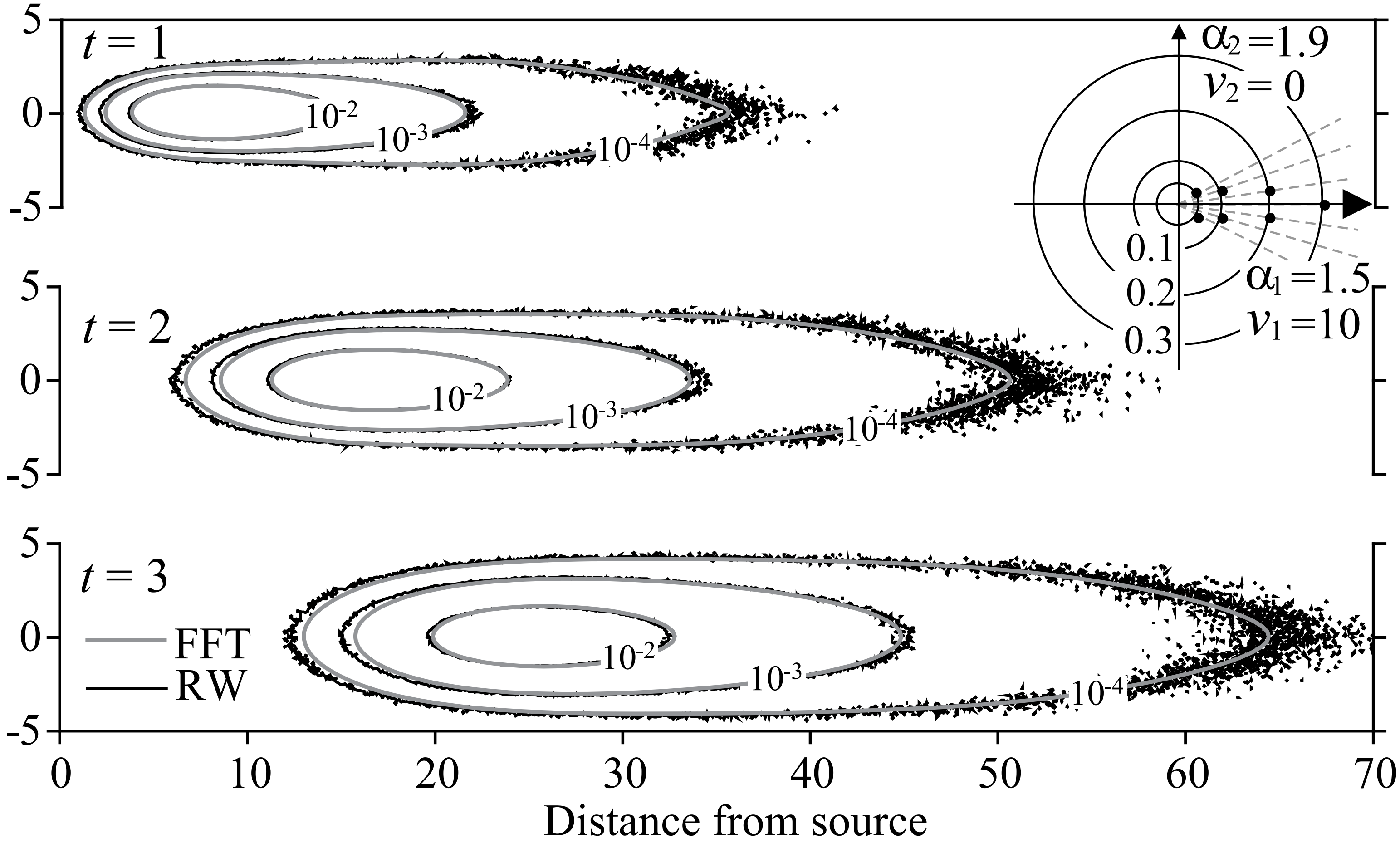
General Electric stock October 1999

Long waiting times and large returns appear asymptotically independent $\Rightarrow$ uncoupled space-time diffusion equation.



Particle tracking

Here $\partial_t p = -v \cdot \nabla p + a\partial_x^{1.5} p + b\partial_y^{1.9} p$



Iterated Brownian motion

IBM $A_{|B_t|}$ models diffusion in a crack. Essentially, the sample path A_t models the (fractal) crack. The density $c(x,t)$ of IBM is the solution to

$$\frac{\partial c(x,t)}{\partial t} = \frac{L\,f(x)}{\sqrt{\pi t}} + L^2 c(x,t); \quad u(0,x) = f(x)$$

where $L = \Delta = \sum_j \partial^2/\partial x_j^2$ is the generator of the semigroup associated with the Brownian motion $A(t)$.

Taking $\beta = 1/2$ in the time-fractional diffusion equation yields exactly the same 1-D distributions $A_{|B_t|} \stackrel{d}{=} A_{E_t}$.

The result extends to Markov process generators.

Bounded domains

For $0 < \alpha < 1$ and $T_D(t)f(x) = E_x[f(X_t)I(t < \tau_D(X))]$, under some technical conditions

$$\begin{aligned} u(t,x) &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\ &= \frac{t}{\beta}\int_0^\infty T_D(l)f(x)g_\beta(tl^{-1/\beta})l^{-1/\beta-1}dl \end{aligned}$$

is the unique (classical) solution to the fractional Cauchy problem

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D,\ t > 0 \\ u(t,x) &= 0,\ x \in \partial D,\ t > 0, \\ u(0,x) &= f(x),\ x \in D. \end{aligned}$$

Hahn, Kobayashi, Umarov (2009) extend to $\int_0^1 \partial_t^\beta p(\beta)\, d\beta$

Uncoupled CTRW with serial dependence

Particle jumps $Y_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ with $\{Z_n\}$ IID.

Short range dependence $\Rightarrow$ the usual limit and PDE.

Long range dependence: If Z_n has light tails, subordinated fractional Brownian motion limit $B_H(E_t)$.

For heavy tails, subordinated linear fractional stable motion $L_{\alpha,H}(E_t)$.

Open problems: Governing equations, dependent waiting times.

Many open problems

- Extension to $\alpha > 2$ and $\beta > 1$
- Fractional reaction diffusion equations
- Coupled CTRW limits
- Distributed order fractional models
- Fractional boundary value problems
- Applications – interdisciplinary research

References

1. I.B. Aban and M.M. Meerschaert (2004) Generalized least squares estimators for the thickness of heavy tails, *J. Statist. Plann. Inf.* **119**, 341–352.

2. I.B. Aban, M.M. Meerschaert, and A.K. Panorska, Parameter Estimation for the Truncated Pareto Distribution, Journal of the American Statistical Association: Theory and Methods, Volume 101 (2006), Number 473, pp.270-277.

3. P.L. Anderson and M.M. Meerschaert, Periodic moving averages of random variables with regularly varying tails, The Annals of Statistics, Vol. 25 (1997), No. 2, pp. 771-185.

4. P.L. Anderson and M.M. Meerschaert, Modeling river flows with heavy tails, Water Resources Research, Vol. 34 (1998) No. 9, pp. 2271-2280.

5. B. Baeumer and M.M. Meerschaert (2001) Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis* **4**, 481–500.

6. B. Baeumer, M. Kovács, and M.M. Meerschaert (2007) Fractional reproduction-dispersal equations and heavy tail dispersal kernels. *Bull. Math. Biology* **69**, 2281–2297.

7. B. Baeumer, M. Kovács, and M.M. Meerschaert (2008) Numerical solutions for fractional reaction-diffusion equations. *Comput. Math. Appl.* **55**, 2212-2226.

8. B. Baeumer, M.M. Meerschaert and E. Nane (2009) Space-time duality for fractional diffusion. *J. Applied Probab.*, to appear. `www.stt.msu.edu/~mcubed/duality.pdf`

9. B. Baeumer and M.M. Meerschaert (2009) Tempered stable Levy motion and transient super-diffusion. Preprint at `www.stt.msu.edu/~mcubed/temperedLM.pdf`

10. P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2003) Hausdorff dimension of operator stable sample paths. *Monatshefte fur Mathematik*, **140**, No. 2, 90–101.

11. P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for coupled continuous time random walks. *The Annals of Probability* **32**, No. 1B, 730–756.

12. D.A. Benson, R. Schumer, M.M. Meerschaert and S. W. Wheatcraft (2001) Fractional dispersion, Lévy motion, and the MADE tracer tests. *Trans. Por. Media* **42**, 211–240.

13. P. Chakraborty (2009) Stochastic Differential Equation Model with Jumps for Fractional Advection and Dispersion. Journal of Statistical Physics, 136: 527-551.

14. P. Chakraborty, M.M. Meerschaert and C.Y. Lim (2009) Parameter Estimation for Fractional Transport: A particle tracking approach. *Water Resources Research*, to appear. Preprint at `www.stt.msu.edu/~mcubed/fADEfit.pdf`

15. M.G. Hahn, K. Kobayashi and S. Umarov (2009) SDEs driven by a time-changed Lévy process and their associated time-fractional order pseudo-differential equations, `arXiv:0907.0253v1`

16. J.F. Kelly and R.J. McGough, M.M. Meerschaert (2008) Time-Domain 3D Greens Functions for Power Law Media. *J. Acoustical Soc. Amer.* **124**, 2861-2872.

17. M.M. Meerschaert and H.P. Scheffler (2001) *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Wiley, New York.

18. M.M. Meerschaert, D.A. Benson, H.P. Scheffler and P. Becker-Kern (2002) Governing equations and solutions of anomalous random walk limits. *Phys. Rev. E* **66**, 102–105.

19. M.M. Meerschaert and H.P. Scheffler, Portfolio modeling with heavy tailed random vectors, Handbook of Heavy-Tailed Distributions in Finance, pp. 595-640, S. T. Rachev, Ed., Elsevier North-Holland, New York (2003) ISBN 0-444-50896-1.

20. M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab.* **41**(3), 623–638.

21. M.M. Meerschaert and Y. Xiao (2005) Dimension results for sample paths of operator stable Lévy processes. *Stochastic Processes and Their Applications* **115**, 55–75.

22. M.M. Meerschaert and E. Scalas (2006) Coupled continuous time random walks in finance. *Physica A*, **370**, 114–118.

23. M.M. Meerschaert and H.P. Scheffler (2008) Triangular array limits for continuous time random walks. *Stochastic Processes and their Applications* **118**, No. 9, 1606-1633.

24. M.M. Meerschaert, Y. Zhang, B. Baeumer, Tempered anomalous diffusion in heterogeneous systems, Geophysical Research Letters, Vol. 35 (2008), p. L17403, doi:10.1029/2008GL034899.

25. M.M. Meerschaert, E. Nane and P. Vellaisamy (2009) Fractional Cauchy problems on bounded domains. *Ann. Probab.* **37**, No. 3, 979-1007.

26. M.M. Meerschaert, E. Nane, Y. Xiao, Correlated continuous time random walks, Statistics and Probability Letters, Vol. 79 (2009), pp. 1194-1202.

27. M.M. Meerschaert, P. Roy and Q, Shao (2009) Parameter estimation for tempered power law distributions. Preprint at `www.stt.msu.edu/~mcubed/TempPareto.pdf`

28. C. Nikias and M. Shao (1995) *Signal Processing with Alpha Stable Distributions and Applications*. Wiley, New York.

29. S. Samko, A. Kilbas and O. Marichev (1993) *Fractional Integrals and derivatives: Theory and Applications*. Gordon and Breach, London.

30. G. Samorodnitsky and M. Taqqu (1994) *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, London.

31. Y. Zhang, D.A. Benson, M.M. Meerschaert, H.P. Scheffler, On using random walks to solve the space-fractional advection-dispersion equations, Journal of Statistical Physics, Vol. 123 (2006), No.1, pp. 89-110.

32. Y. Zhang, D.A. Benson, M.M. Meerschaert, E. M. LaBolle, and H.P. Scheffler, Random walk approximation of fractional-order multiscaling anomalous diffusion, Physical Review E, Vol. 74 (2006), 026706 (10 pp).

Derivatives of power laws

If both p and α are integers then

$$\begin{aligned}
\mathbb{D}_1 [x^p] &= p x^{p-1} \\
\mathbb{D}_2 [x^p] &= p(p-1) x^{p-2} \\
&\vdots \\
\mathbb{D}_\alpha [x^p] &= \frac{p!}{(p-\alpha)!} x^{p-\alpha}
\end{aligned}$$

For $p > 0$ the Gamma function extends $p! = \Gamma(p+1)$ via

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx.$$

Use the property $\Gamma(p+1) = p\Gamma(p)$ to get

$$\mathbb{D}_\alpha [x^p] = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}.$$

Fractional derivatives of power laws

If $p > 0$ then the Laplace transform

$$\mathsf{LT}\{x^p\} = \int_0^\infty e^{-sx} x^p dx \qquad \boxed{\text{substitute } y = sx}$$
$$= \int_0^\infty e^{-y} (y/s)^p dy/s = s^{-p-1}\Gamma(p+1).$$

Then

$$\mathsf{LT}\{\mathbb{D}_\alpha x^p\} = s^\alpha s^{-p-1}\Gamma(p+1)$$
$$= s^{-(p-\alpha)-1}\Gamma(p-\alpha+1) \cdot \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}$$
$$= \mathsf{LT}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}\right\}$$

and the uniqueness of the LT yields

$$\mathbb{D}_\alpha [x^p] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}.$$

Difference quotients

The derivative $\mathbb{D}_1 f(x) = \lim_{h\to 0} h^{-1}\Delta f(x)$ where

$$\Delta f(x) = f(x) - f(x-h).$$

For positive integers α, $\mathbb{D}_\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\begin{aligned}
\Delta^2 f(x) &= (f(x) - f(x-h)) - (f(x-h) - f(x-2h)) \\
&= f(x) - 2f(x-h) + f(x-2h), \\
\Delta^3 f(x) &= f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h) \\
&\vdots \\
\Delta^\alpha f(x) &= \sum_{m=0}^{\alpha} \binom{\alpha}{m} (-1)^m f(x-mh). \quad \text{Here } \binom{\alpha}{m} = \frac{\alpha!}{m!(\alpha-m)!}
\end{aligned}$$

Fractional difference quotients

For $\alpha > 0$ define $\mathbb{D}_\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\Delta^\alpha f(x) = \sum_{m=0}^{\infty} \binom{\alpha}{m} (-1)^m f(x - mh), \qquad \binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha - m + 1)}$$

Since $f(x-h)$ has FT $e^{-ikh}\widehat{f}(k)$, and using the Binomial formula

$$(1+z)^\alpha = \sum_{m=0}^{\infty} \binom{\alpha}{m} z^m \quad \text{for any complex } |z| \le 1$$

we see that $\Delta^\alpha f(x)$ has FT

$$\sum_{m=0}^{\infty} \binom{\alpha}{m} (-1)^m e^{-ikmh} \widehat{f}(k) = (1 - e^{-ikh})^\alpha \widehat{f}(k)$$

and then the FT of $h^{-\alpha}\Delta^\alpha f(x)$ is

$$h^{-\alpha}(ikh)^\alpha \left(\frac{1 - e^{-ikh}}{ikh}\right)^\alpha \widehat{f}(k) \to (ik)^\alpha \widehat{f}(k) \quad \text{as } h \to 0.$$

Random walk simulation code (Maple)

```
> N:=1000:
> J:=random[uniform[-1,1]](N): # jump distribution
> n:='n':T:=0:
> for n from 1 to N do
>   T:=T+1;
>   S[n]:=T;
> od:n:='n':
> plot(sum(J[n]*Heaviside(t-S[n]),n=1..1000),t=0..10);
```

See http://www.maplesoft.on.ca/

Heavy tail random walk simulation code (Maple)

```
> lambda:=1:N:=1000:alpha:=1.5:C:=.1:
> P:=random[uniform[0,1]](N):
> J:=random[uniform[0,1]](N):
> n:='n':T:=0:
> for n from 1 to N do
>   T:=T+1;
>   S[n]:=T;
> od:n:='n':
> plot(sum((2*floor(2*P[n])-1)*(C/J[n])^(1/alpha)
   *Heaviside(t-S[n]),n=1..1000),t=0..1000);
```

See `http://www.maplesoft.on.ca/`

Heavy tailed jumps $U^{-1/\alpha}$ where $U \sim \text{Uniform}[0,1]$.