Tempered stable models for anomalous diffusion

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Abstract

The fractional diffusion equation replaces the usual first order time derivative, and second order spatial derivative, by their fractional order analogues. Stable processes with non-Markovian inverse stable subordinators are governed by the fractional diffusion equation, in the same way that Brownian motion is governed by the classical diffusion equation. In certain applications to hydrology and finance, plume evolution is intermediate between the classical and fractional case. For example, daily price returns are heavy tailed, but annual returns are essentially Gaussian. Then a subordinated process involving tempered stable laws is effective at capturing the real world dynamics. The governing equation involves tempered fractional derivatives. Applications to ground water pollution migration illustrate the practical utility of the results.

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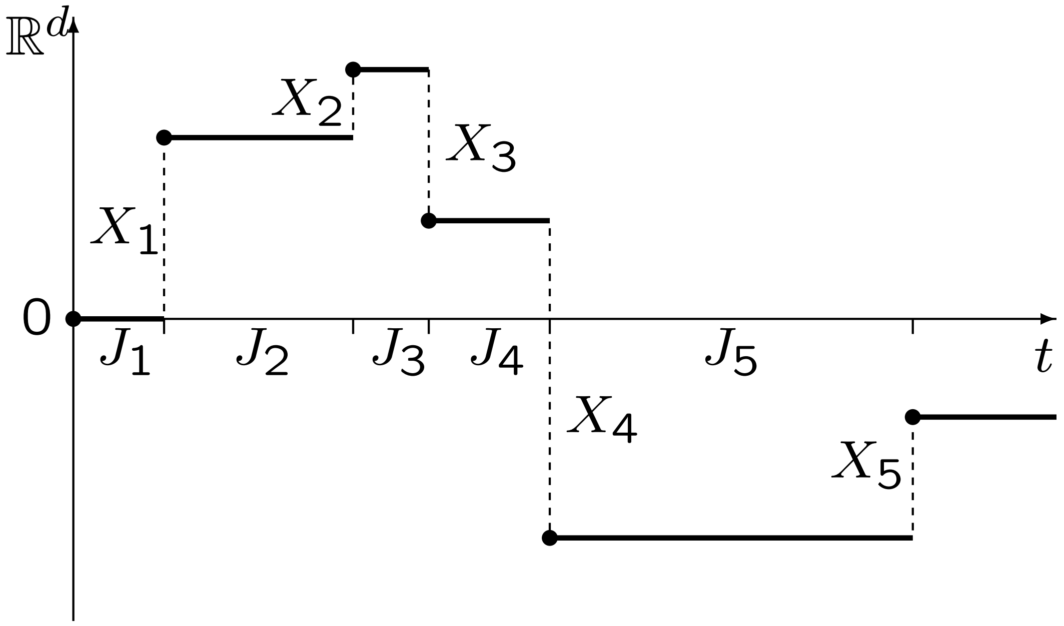
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

CTRW triangular arrays

Consider a sequence of CTRW at each scale $c > 0$

$S^{(c)}(n) = X_1^{(c)} + \cdots + X_n^{(c)}$ particle location after n jumps

$T^{(c)}(n) = J_1^{(c)} + \cdots + J_n^{(c)}$ time of the nth jump

$N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$ number of jumps by time $t > 0$

$S^{(c)}(N_t^{(c)})$ particle location at time $t > 0$ (CTRW)

Note $\{T^{(c)}(n) \leq t\} = \{N_t^{(c)} \geq n\}$ inverse processes

CTRW scaling limits

Assume $(S^{(c)}(cu), T^{(c)}(cu)) \Rightarrow (A(u), D(u))$ infinitely divisible

Write $\mathbb{E}(e^{-ik\cdot A(u)-sD(u)}) = e^{-u\psi(k,s)}$

$$\psi(k,s) = ia\cdot k + k\cdot Qk + \int \left(1 - e^{-ik\cdot x}e^{-st} - \frac{ik\cdot x}{1+\|x\|^2}\right)\phi(dx,dt)$$

$\mathbb{E}(e^{ik\cdot A(u)}) = e^{-u\psi_A(k)}$ with $\psi_A(k) = \psi(k,0)$
$\mathbb{E}(e^{-sD(u)}) = e^{-u\psi_D(s)}$ with $\psi_D(s) = \psi(0,s)$

Inverse mapping yields $N_{ct}^{(i)} \Rightarrow E(t)$
$E(t) = \inf\{u > 0 : D(u) > t\}$ inverse process.

CTRW scaling limit $S^{(c)}(N_t^{(c)}) \Rightarrow A(E(t))$

Semigroups and generators

The CTRW scaling limit defines a semigroup

$$T(u)f(x,t) = \int_0^t \int_{\mathbb{R}^d} f(x-y,t-r)\, P_{(A(u),D(u))}(dy,dr)$$

with generator

$$\begin{aligned}\psi(-iD_x,\partial_t)f(x,t) &= a\cdot\nabla f(x,t) - \nabla\cdot Q\nabla f(x,t)\\ &\quad - \int\left(f(x-y,t-u) - f(x,t) + \frac{\nabla f(x,t)\cdot y}{1+\|y\|^2}\right)\phi(dy,du)\end{aligned}$$

The pseudodifferential operator $\psi(-iD_x,\partial_t)$ has symbol $\psi(k,s)$

Inverse subordinators

Let $g(t,u)$ be Lebesgue density of $t = D(u)$

Assume $\phi_D(0,\infty) = \infty$ and $\int_0^1 y|\ln y|\,\phi_D(dy) < \infty$ (technical).

Theorem $E(t) = \inf\{u > 0 : D(u) > t\}$ has Lebesgue density

$$f(u,t) = \int_0^t \phi_D(t-y,\infty) g(y,u)\,dy.$$

Moreover, the mapping $(u,t) \mapsto f(u,t)$ is measurable.

Idea: $f(u,t) = \frac{d}{du}P(E(t) \le u) = \frac{d}{du}P(D(u) \ge t)$
Compute with Laplace transforms

Governing equation

Suppose A, D are independent $\psi(k,s) = \psi_A(k) + \psi_D(s)$

Suppose $x = A(u)$ has Lebesgue density $p(x,u)$

CTRW scaling limit $A(E(t))$ has a density

$$m(x,t) = \int_0^\infty p(x,u) f(u,t)\, du$$

that solves the pseudodifferential equation

$$\psi_D(\partial_t) m(x,t) = -\psi_A(-iD_x) m(x,t) + \delta(x)\phi_D(t,\infty)$$

Fractional derivatives

In the simplest case $0 < \alpha < 1$

$$D_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y) \Big) y^{-\alpha-1} dy$$

Apply the Fourier transform $\widehat{f}(k) = \int e^{-ikx} f(x)\, dx$

Since $f(x-y)$ has FT $e^{-iky}\widehat{f}(k)$, $D_x^\alpha f(x)$ has FT

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-iky} \right) \widehat{f}(k) y^{-\alpha-1} dy$$

For $\lambda > 0$ it is not hard to compute

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-(\lambda+ik)y} \right) y^{-\alpha-1} dy = (\lambda + ik)^\alpha$$

Let $\lambda \to 0$: $D_x^\alpha f(x)$ has FT $(ik)^\alpha \widehat{f}(k)$

Space-time fractional diffusion

Power laws: $P(J > t) \approx t^{-\beta}$ and $P(X > x) \approx x^{-\alpha}$

Then $\psi_A(k) = (ik)^\alpha$ stable jump limit

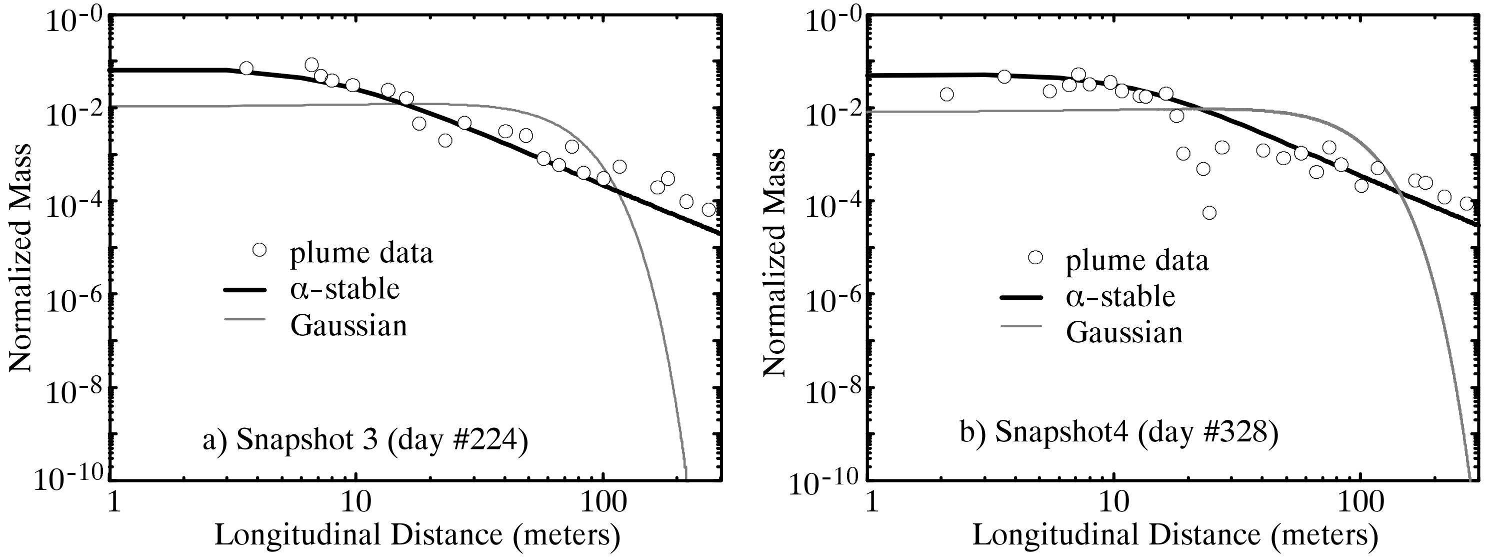
And $\psi_D(s) = s^\beta$ stable waiting time limit

Now $\psi_A(-iD_x) = D_x^\alpha$ and $\psi_D(\partial_t) = \partial_t^\beta$ so

$$\partial_t^\beta m(x,t) = -D_x^\alpha m(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

Tracer test in an underground aquifer

Space-fractional diffusion model captures early arrivals at the MADE experimental site.

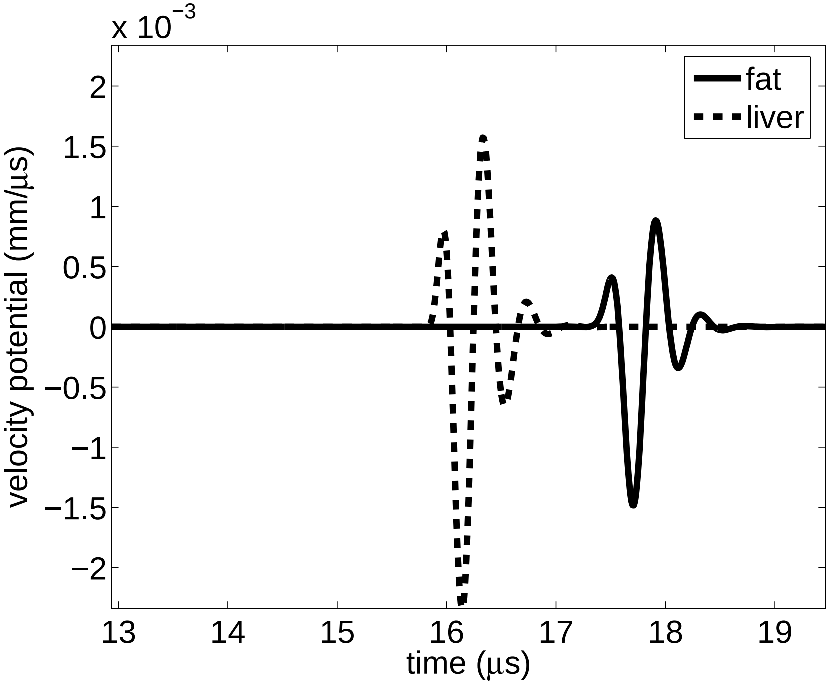


CTRW model has power-law jumps in the direction of flow

Sound wave propogation

We use $\beta = 2.5$ for human fat tissue and $\beta = 2.1$ for liver tissue.

$$\frac{\partial^2}{\partial t^2}c(t,x) + C\frac{\partial^\beta}{\partial t^\beta}c(t,x) = D\frac{\partial^2}{\partial x^2}c(t,x)$$



Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (tempered)

Tempered stable waiting times

Rosiński (2007): Tempered stable laws $0 < \beta < 1$

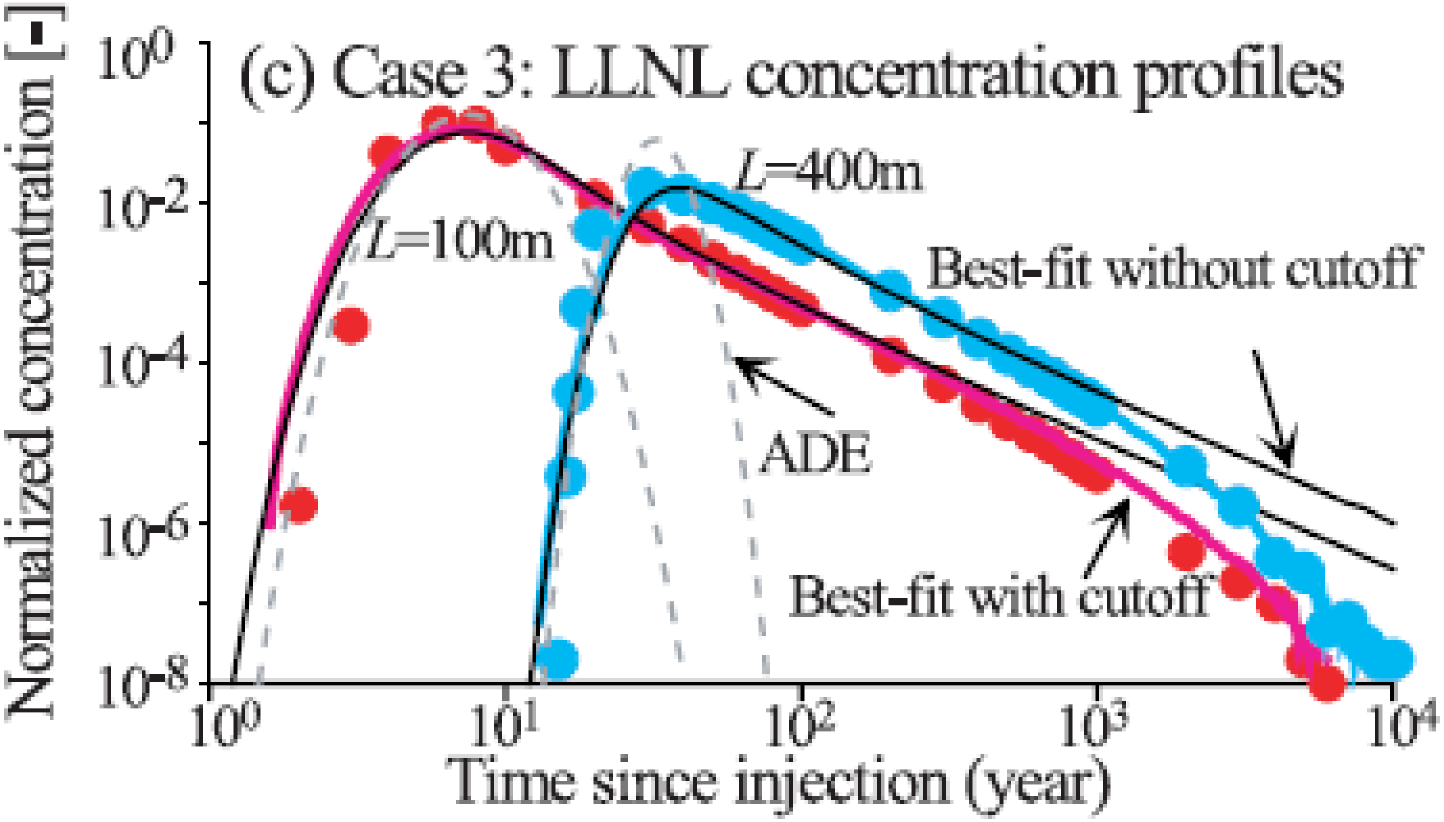
Lévy measure is modified to $e^{-\lambda t}\phi_D(dt)$

Results in tapered density $Ce^{-\lambda t}g(t,u)$

$\partial_t^{\beta} m$ becomes $\partial_t^{\beta,\lambda} m = e^{-\lambda t}\partial_t^{\beta}(e^{\lambda t}m) - \lambda^{\beta} m$

Laplace symbol s^{β} becomes $(\lambda + s)^{\beta} - \lambda^{\beta}$

Tempered waiting times



Tempered waiting times with $\beta = 0.61$ and $\lambda = 0.0006$ fit simulated groundwater plume.

Tempered stable jumps

Start with $\psi_A(k) = (ik)^\alpha$ for $1 < \alpha < 2$

Jump intensity modified to $e^{-\lambda x}\phi_A(dx)$

Results in tapered density $Ce^{-\lambda x}p(x,t)$

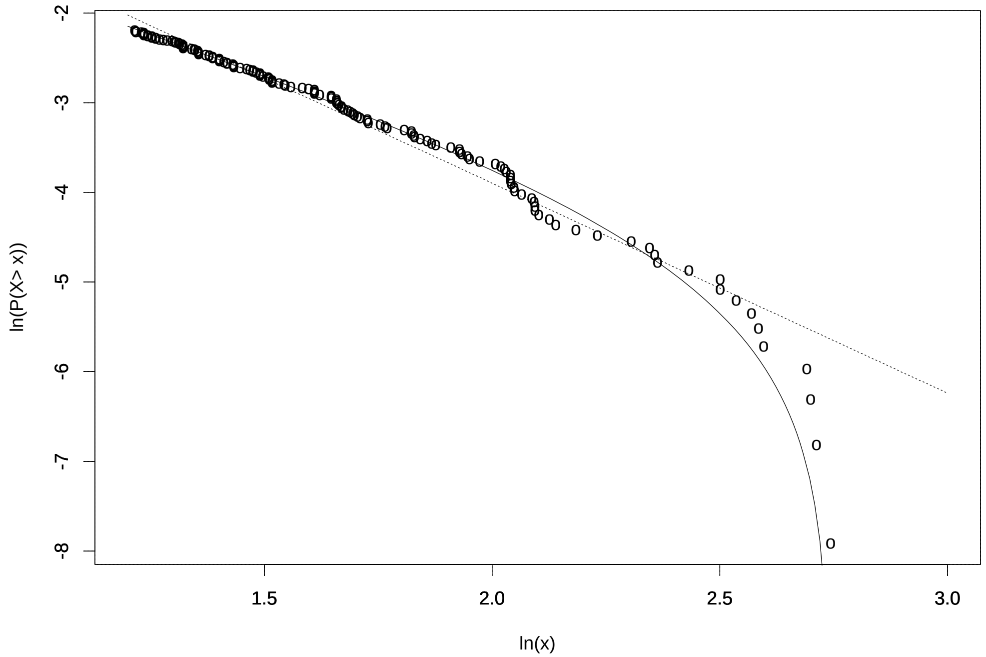
Skewness assures $\int_{-\infty}^{\infty} e^{-\lambda x}p(x,t)\,dx < \infty$

Fourier symbol $(ik)^\alpha$ becomes $(\lambda + ik)^\beta - \lambda^\beta$

Scaling limit $A(t) \approx$ Brownian motion as $t \to \infty$

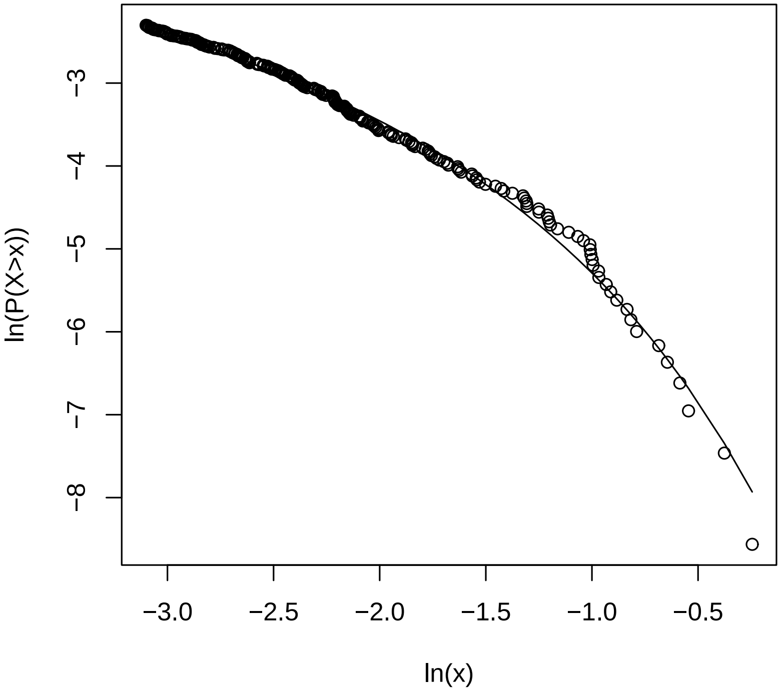
Tempered stables in finance

AMZN stock price changes fit a tempered power law model $P(X > x) \approx x^{-0.6} e^{-0.3x}$ for x large



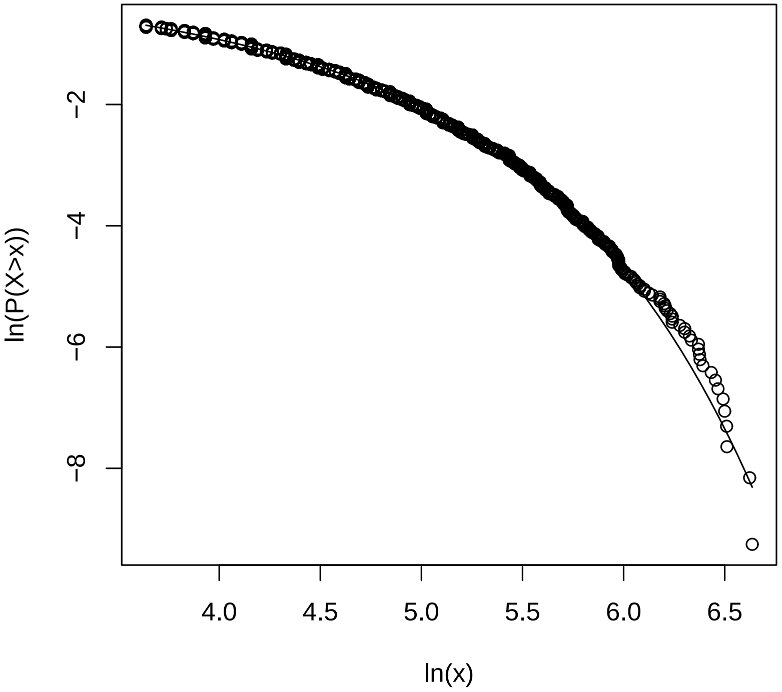
Tempered stables in hydrology

Tempered power law model $P(X > x) \approx x^{-0.6}e^{-5.2x}$ for increments in hydraulic conductivity at the MADE site.



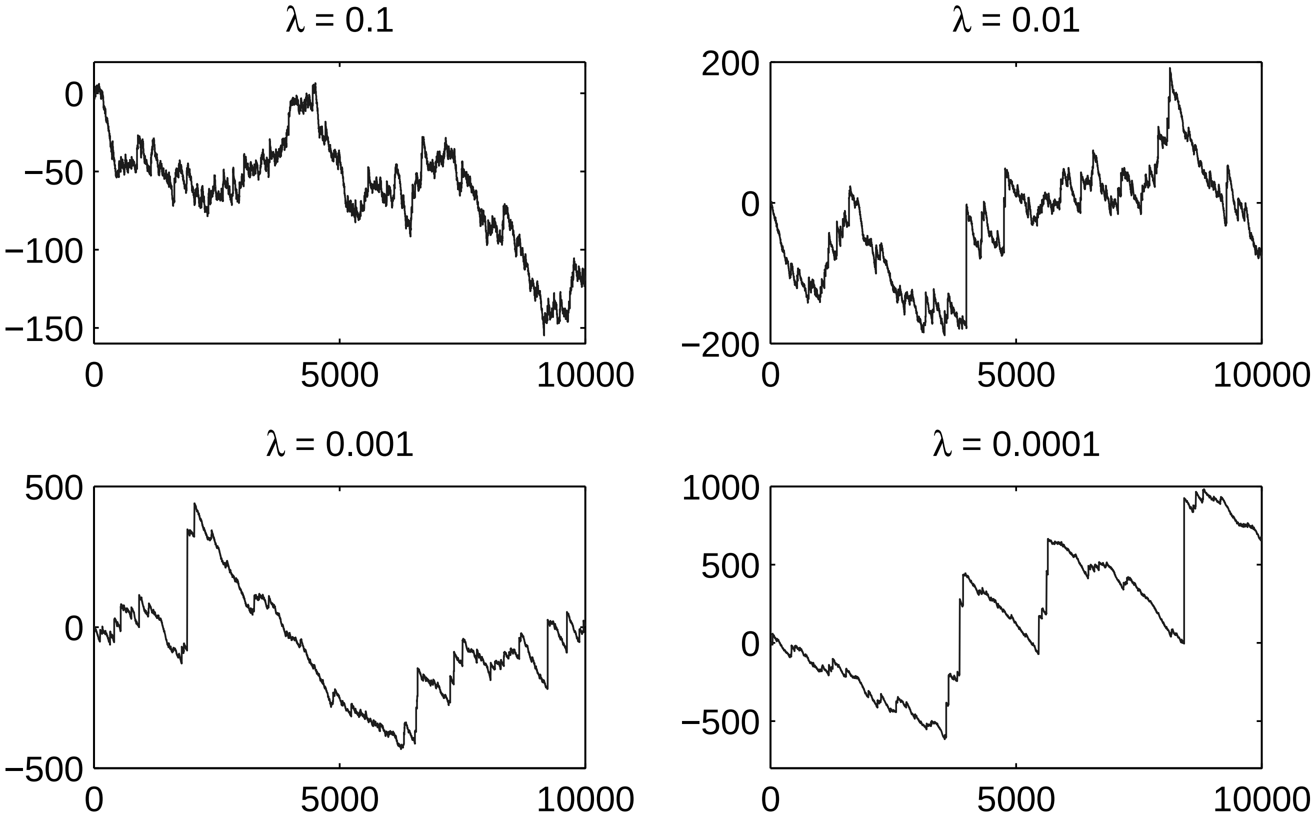
Tempered stables in atmospheric science

Tempered power law model $P(X > x) \approx x^{-0.2}e^{-0.01x}$ for daily precipitation data at Tombstone AZ.



Tempered Lévy motion

Tempered stable Lévy motion with $\alpha = 1.2$

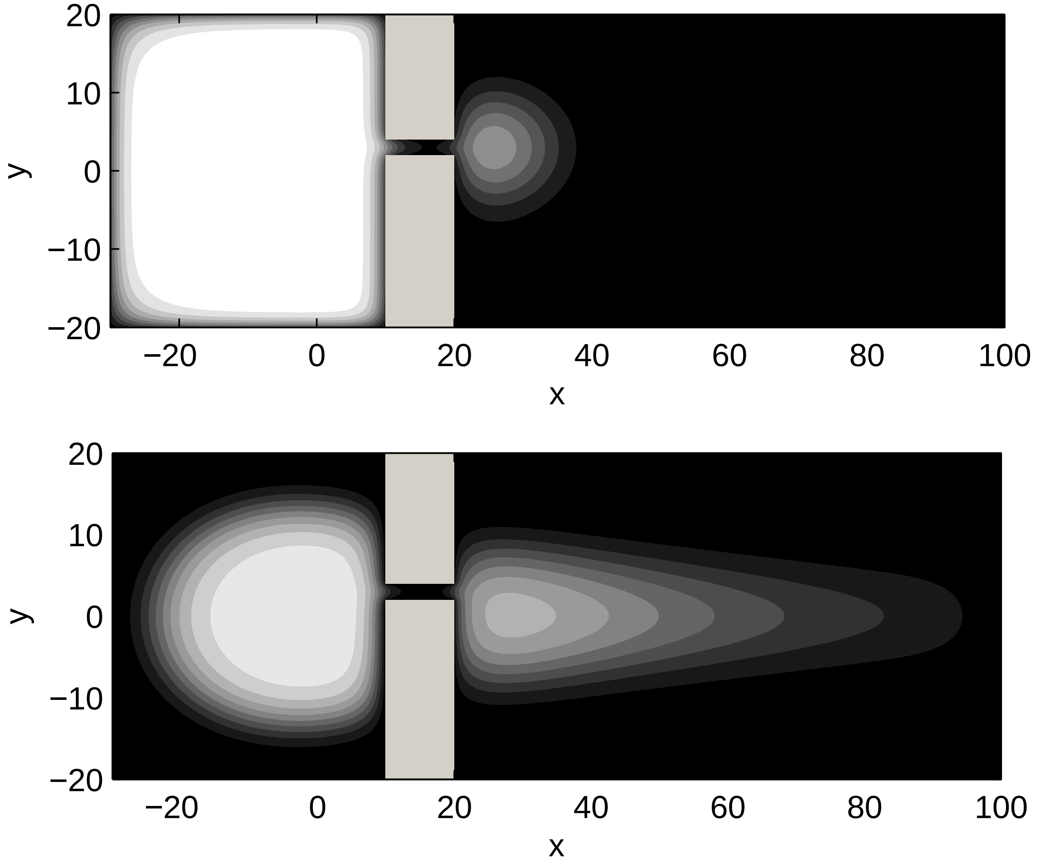


Operator stable jumps

Cartea and del Castillo-Negrete (2007): Accelerating fronts.

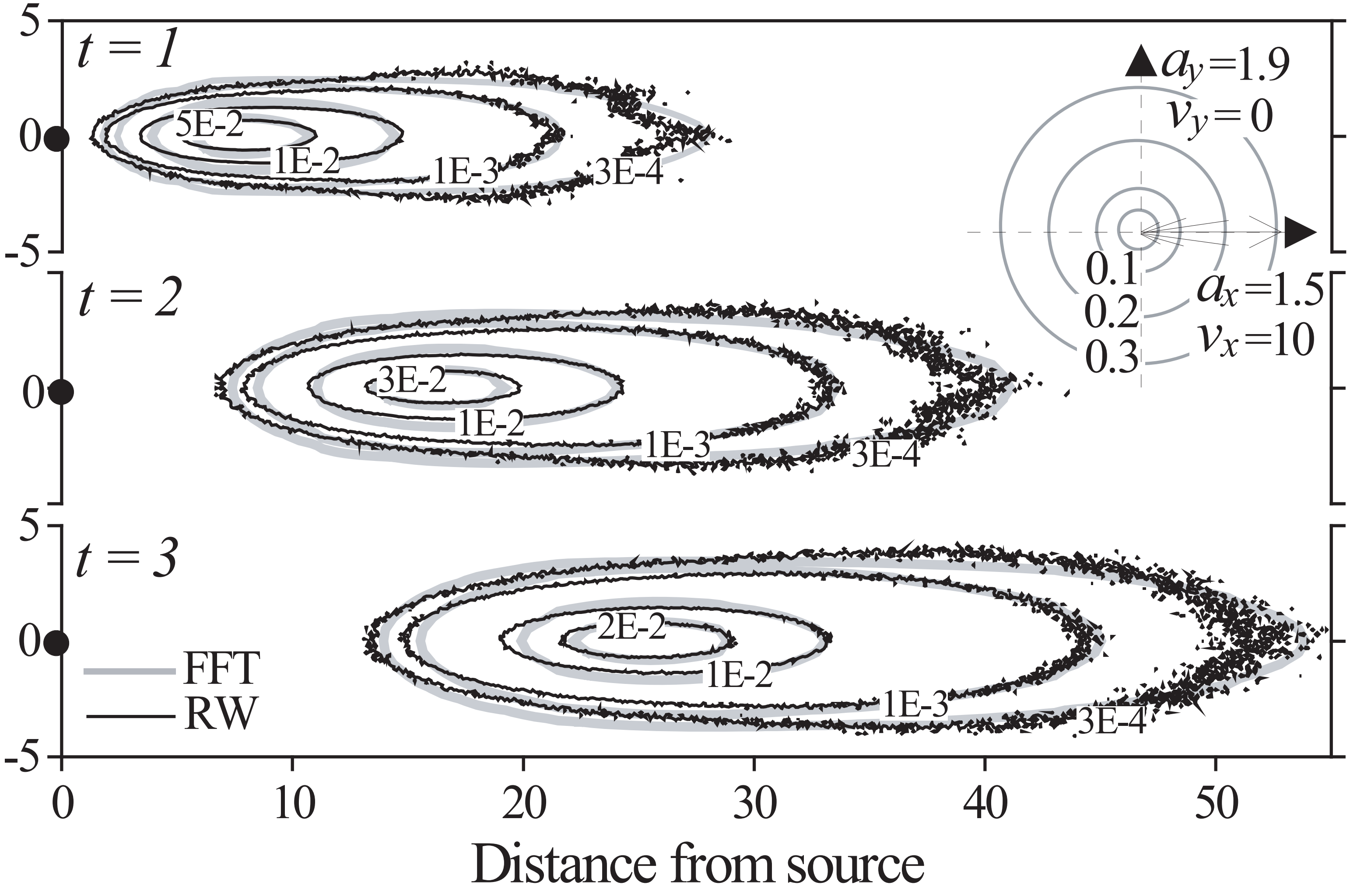
$$\frac{\partial P}{\partial t} = C\frac{\partial^\alpha P}{\partial x^\alpha} + D\frac{\partial^2 P}{\partial y^2} + rP\left(1 - \frac{P}{K}\right)$$

Compare $\alpha = 2$ (top) to $\alpha = 1.7$ (bottom).

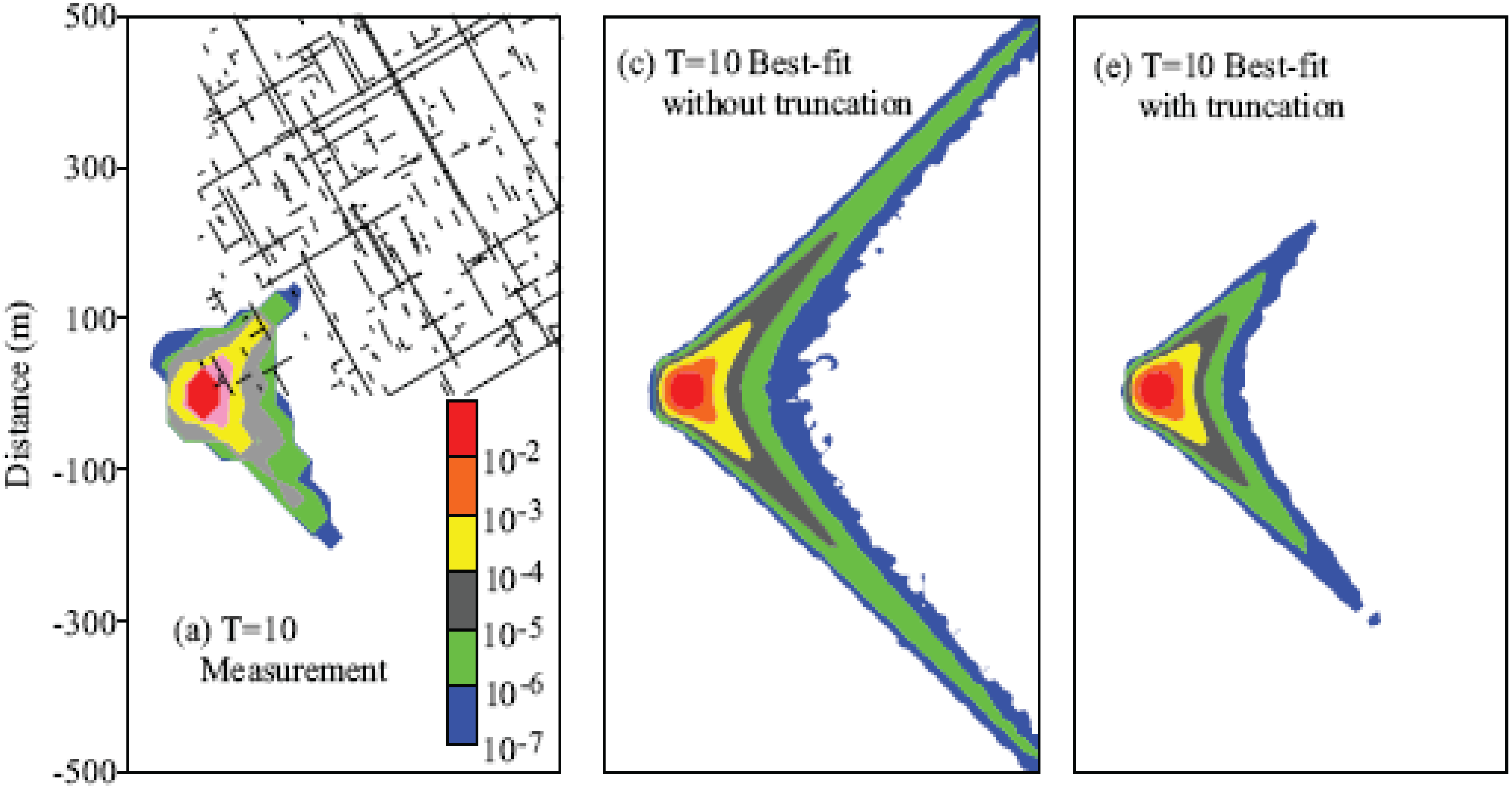


Particle tracking

Here $\partial_t p = -v \cdot \nabla p + a\partial_x^{1.5} p + b\partial_y^{1.9} p$



Tempered operator stable



Tempered operator stable model for flow in fractured rock (nuclear waste).

Bounded domains

Let $\tau_D(X) = \inf\{t \geq 0 : \ X_t \notin D\}$ where D is a nice bounded domain. For $0 < \alpha < 1$ and $T_D(t)f(x) = E_x[f(X_t)I(t < \tau_D(X))]$, under some technical conditions

$$\begin{aligned} u(t,x) &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\ &= \frac{t}{\beta}\int_0^\infty T_D(r)f(x)g_\beta(tr^{-1/\beta})r^{-1/\beta-1}dr \end{aligned}$$

is the unique (classical) solution to

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D,\ t > 0 \\ u(t,x) &= 0,\ x \in \partial D,\ t > 0, \\ u(0,x) &= f(x),\ x \in D. \end{aligned}$$

PROOF: Eigenfunction expansion, Mittag-Leffler solutions

Some open problems

- Triangular arrays with tempered limits
- Tempered operator stable models
- Numerical methods for tempered diffusion
- Tempered boundary value problems
- Applications – interdisciplinary research

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Numerical methods

Variable coefficient fractional PDEs require numerical solution.

The Grünwald finite difference approximation

$$D_x^\alpha f(x) = \lim_{h\to 0} \frac{1}{h} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(x-kh)$$

comes from the generator formula with a discrete Lévy measure.

Convergence proof uses Fourier methods:
Lévy representation, convergence criteria infinitely divisible laws

Proof of $O(h)$ convergence does not extend to bounded domains.
A new approach is required.

Numerical methods for tempered stable diffusion

Grünwald finite difference approximation

$$D_x^{\alpha,\lambda} f(x) \approx \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \Big(w_j e^{-(j-p)h\lambda} f(x-(j-p)h) \Big) - e^{ph\lambda} \frac{\left(1-e^{-h\lambda}\right)^\alpha}{h^\alpha} f(x)$$

where $w_j = \dfrac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = (-1)^j \binom{\alpha}{j}$ provides a convergent $O(h)$ estimate.

A stable and consistent 2nd order method via Richardson extrapolation:

Δt	Δx	Max Error	Error rate
$1/10$	$1/50$	2.8514×10^{-6}	–
$1/20$	$1/100$	7.2120×10^{-7}	3.95
$1/40$	$1/200$	1.8157×10^{-7}	3.97
$1/80$	$1/400$	4.5555×10^{-8}	3.99

Numerical solutions by particle tracking

Compute adjoint to get backward equation and generator

Explicitly identify the underlying Markov process

Simulate many particles to estimate the transition density.

CTRW scaling limit $x = A(u)$ Markov but $u = E(t)$ is not:
Simulate Markov process $t = D(u)$ instead

Particle trajectories are (x_n, t_n) where $x_i = A(u_i)$, $t_i = D(u_i)$.

Theory recently established for space-fractional diffusion

Simulating tempered stable laws

Simulation codes for stable random variates are widely available.

If $X > 0$ has stable density density $f(x)$, TS density is

$$f_\lambda(x) = \frac{e^{-\lambda x} f(x)}{\int_0^\infty e^{-\lambda u} f(u)\, du}$$

Take $Y \sim \exp(\lambda)$ independent of X, (X_i, Y_i) IID with (X, Y)

Let $N = \min\{n : X_n \leq Y_n\}$. Then $X_N \sim f_\lambda(x)$

Proof: Compute $P(X_N \leq x) = P(X \leq x | X \leq Y)$ by conditioning, then take d/dx to verify.

Tempered power law fit to tempered stable

A tempered power law fits the upper tail of a tempered stable.