The Fractal Calculus Project

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Abstract

Fractional derivatives are almost as old as their integer-order cousins. Recently, fractional derivatives have found new applications in engineering, physics, finance, and hydrology. In physics, fractional derivatives are used to model anomalous diffusion, where a cloud of particles spreads differently than the classical Brownian motion model predicts. A probability model for anomalous diffusion is based on particle jumps with power law tails. The probability of a jump length larger than r falls off like $r^{-\alpha}$ as $r \to \infty$. For $0 < \alpha < 2$ these particle jumps have infinite variance, indicating a faster than usual spreading rate. Particle traces are random fractals whose dimension α equals the power law tail exponent. A fractional diffusion equation for the concentration of particles $c(x,t)$ at time t and location x takes a form

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^\alpha c(x,t)}{\partial x^\alpha}$$

that can be solved via Fourier transforms. Fractional time derivatives model particle sticking or trapping in a porous medium. In finance, price jumps replace particle jumps, and the same models apply. In this talk, we give an introduction to this new area, starting from the beginning and ending with a look at ongoing research. The entire talk will be accessible to advanced undergraduate students.

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Fractional derivatives: An old idea gets new life

Fractional derivatives $\mathbb{D}_\alpha f(x)$ for any $\alpha > 0$ were invented by Leibnitz soon after the more familiar integer derivatives.

Some derivative formulas extended to the fractional case:

$$\mathbb{D}_\alpha \left[e^{\lambda x} \right] = \lambda^\alpha e^{\lambda x}$$

$$\mathbb{D}_\alpha \left[\sin x \right] = \sin \left(x + \frac{\pi}{2} \alpha \right)$$

$$\mathbb{D}_\alpha \left[x^p \right] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}$$

Fractional derivatives and transforms

If the Laplace transform of $f(t)$ is defined for $s > 0$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

then $\mathbb{D}_\alpha f(t)$ has Laplace transform $s^\alpha \tilde{f}(s)$.

If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{-\infty}^\infty e^{-ikx} f(x)dx$$

then $\mathbb{D}_\alpha f(x)$ has Fourier transform $(ik)^\alpha \hat{f}(k)$.

Here $(ik)^\alpha = |k|^\alpha \operatorname{sign}(k) e^{i\alpha\pi/2}$.

Probability and transforms

If the random variable X has density $f(x)$ so that

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

then $f(x)$ has Fourier transform

$$\widehat{f}(k) = \int_{-\infty}^{\infty} \left(1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots \right) f(x)dx$$

$$= 1 - ik\mu_1 - \frac{1}{2}k^2\mu_2 + \cdots$$

where the pth moment

$$\mu_p = \int_{-\infty}^{\infty} x^p f(x)dx$$

Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\widehat{f}(k) = 1 - k^2 + \cdots$

The IID sum $S_n = X_1 + \cdots + X_n$ has FT $\widehat{f}(k)^n$ and the normalized sum $S_n/\sqrt{n}$ has FT

$$\begin{aligned}
\widehat{f}(k/\sqrt{n})^n &= \left(1 - (k/\sqrt{n})^2 + \cdots\right)^n \\
&= \left(1 - \frac{k^2}{n} + \cdots\right)^n \\
&\to e^{-k^2} \equiv \widehat{g}(k) \quad \text{as } n \to \infty.
\end{aligned}$$

Inverting the Fourier transform reveals a Gaussian density

$$g(x) = \frac{1}{\sqrt{4\pi}}\, e^{-x^2/4}$$

Brownian motion

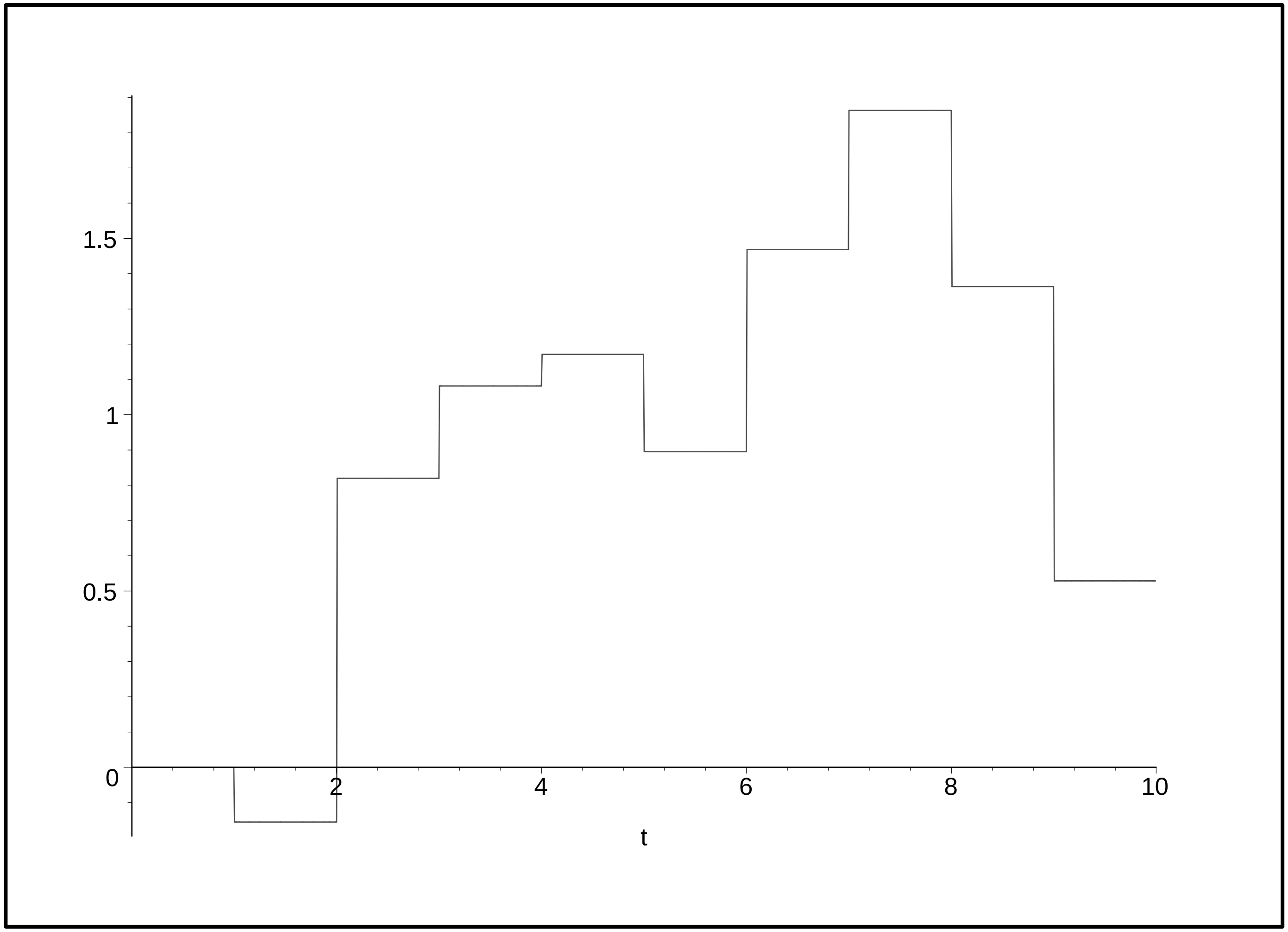
If X_n represents a particle jump at time n then $S_n = X_1 + \cdots + X_n$ is the location of the particle at time n. Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2} S_{[ct]} \Rightarrow A_t$ since

$$\begin{aligned} \hat{f}(c^{-1/2}k)^{[ct]} &= \left(1 - \frac{k^2}{c} + \cdots\right)^{[ct]} \\ &\to e^{-k^2 t} \equiv \hat{c}(k,t) \end{aligned}$$

for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process A_t is Gaussian

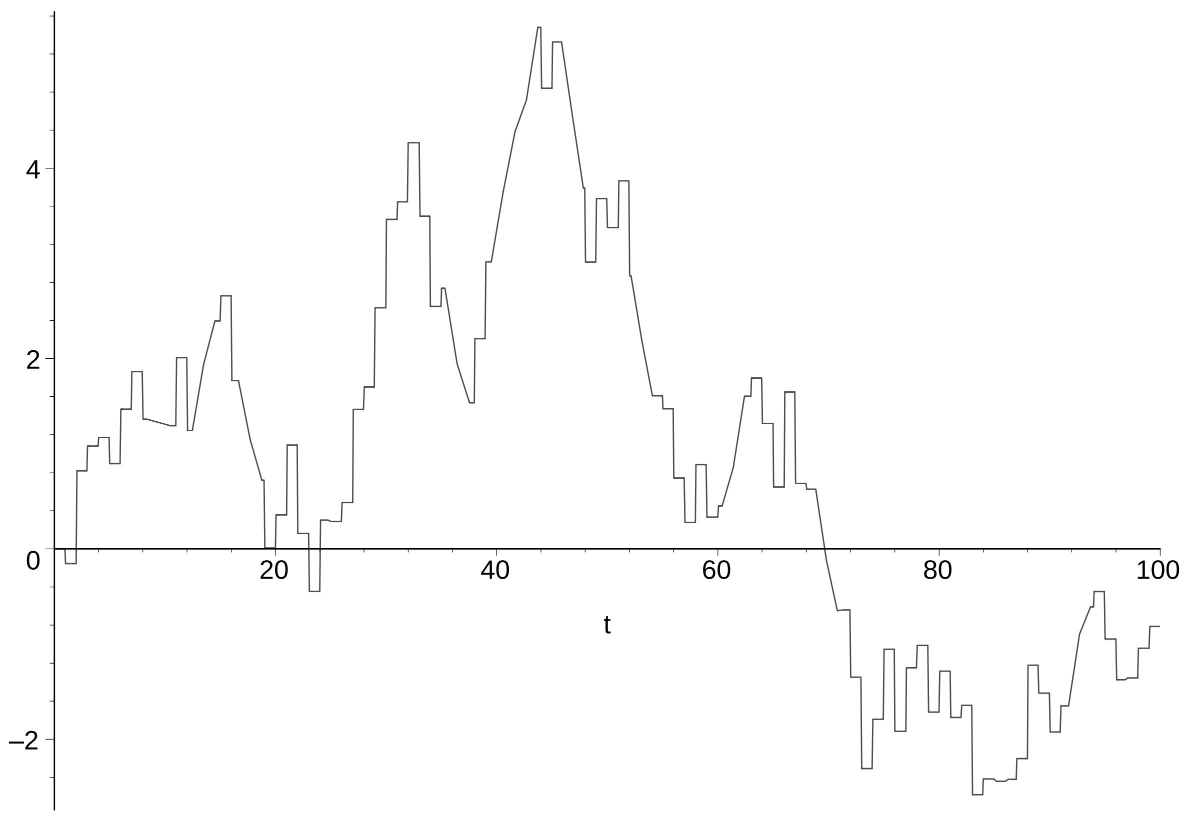
$$c(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

Simple random walk simulation

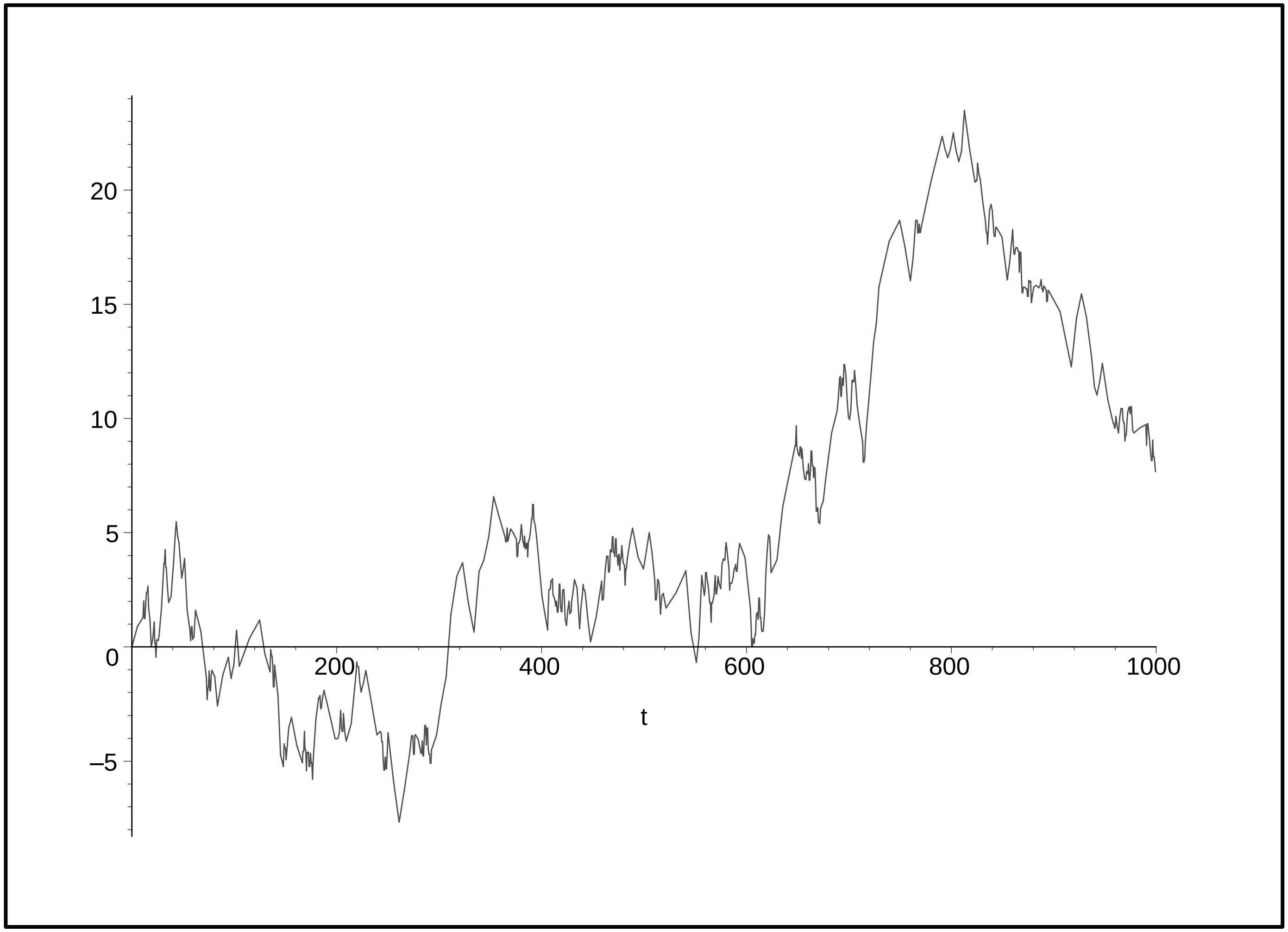


See appendix for Maple simulation code.

Longer time scale



Scaling limit: Brownian motion



The diffusion equation

Taking Fourier transforms in the classical diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^2 c(x,t)}{\partial x^2}$$

yields

$$\frac{d\widehat{c}(k,t)}{dt} = (ik)^2 \widehat{c}(k,t) = -k^2 \widehat{c}(k,t)$$

whose solution

$$\widehat{c}(k,t) = e^{-k^2 t}$$

inverts to the same limit density for the Brownian motion A_t.

For a cloud of diffusing particles $c(x,t)$ is the particle density.

Properties of Brownian motion

Since the density of $t^{1/2}A_1$ has FT

$$e^{-(\sqrt{t}k)^2} = e^{-tk^2} = \widehat{c}(k,t)$$

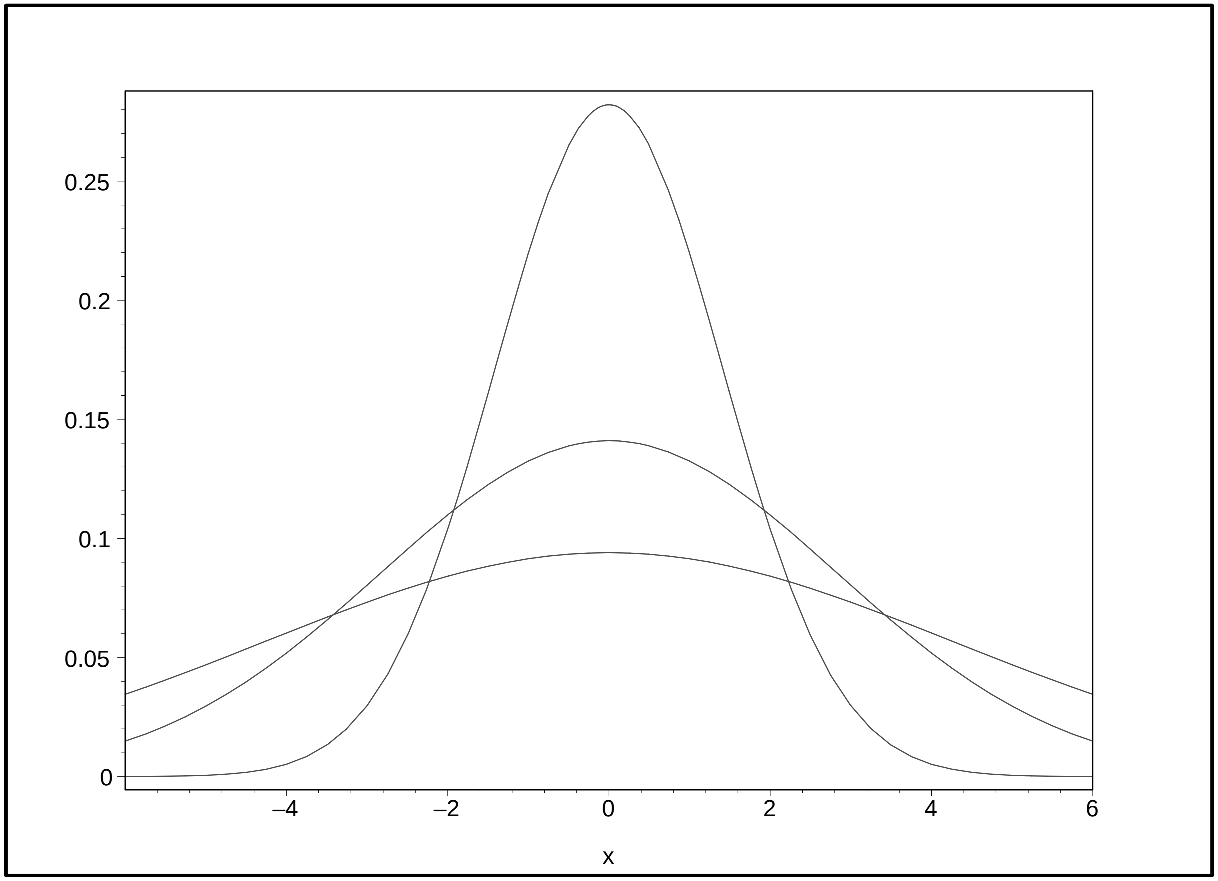
we see that $A_t \stackrel{d}{=} t^{1/2}A_1$ (same density).

A cloud of diffusing particles spreads at the rate $t^{1/2}$.

The plume tails off like e^{-x^2}.

Real plumes often spread faster, with a heavier tail.

Classical diffusion profile



Brownian motion Gaussian (Normal) density at time $t = 1, 4, 9$ showing square root spreading rate and fast tail decay.

Heavy (power law) tails

If $P(X > x) \approx x^{-\alpha}$ then $f(x) \approx \alpha x^{-\alpha-1}$ and some moments

$$\mu_k = \int_{-\infty}^{\infty} x^k f(x) dx$$

do not exist. If $1 < \alpha < 2$ and $\mu_1 = 0$ then a typical X has FT

$$\widehat{f}(k) = 1 + (ik)^\alpha + \cdots$$

and $n^{-1/\alpha}(X_1 + \cdots + X_n)$ has FT

$$\begin{aligned} \widehat{f}(n^{-1/\alpha}k)^n &= \left(1 + (n^{-1/\alpha}\, ik)^\alpha + \cdots\right)^n \\ &= \left(1 + \frac{(ik)^\alpha}{n} + \cdots\right)^n \\ &\to e^{(ik)^\alpha} \equiv \widehat{g}(k) \quad \text{as } n \to \infty. \end{aligned}$$

The inverse Fourier transform $g(x)$ is called a stable density.

Lévy motion

If $S_n = X_1 + \cdots + X_n$ is particle location at time n then the scaling limit $c^{-1/\alpha} S_{[ct]} \Rightarrow A_t$ since

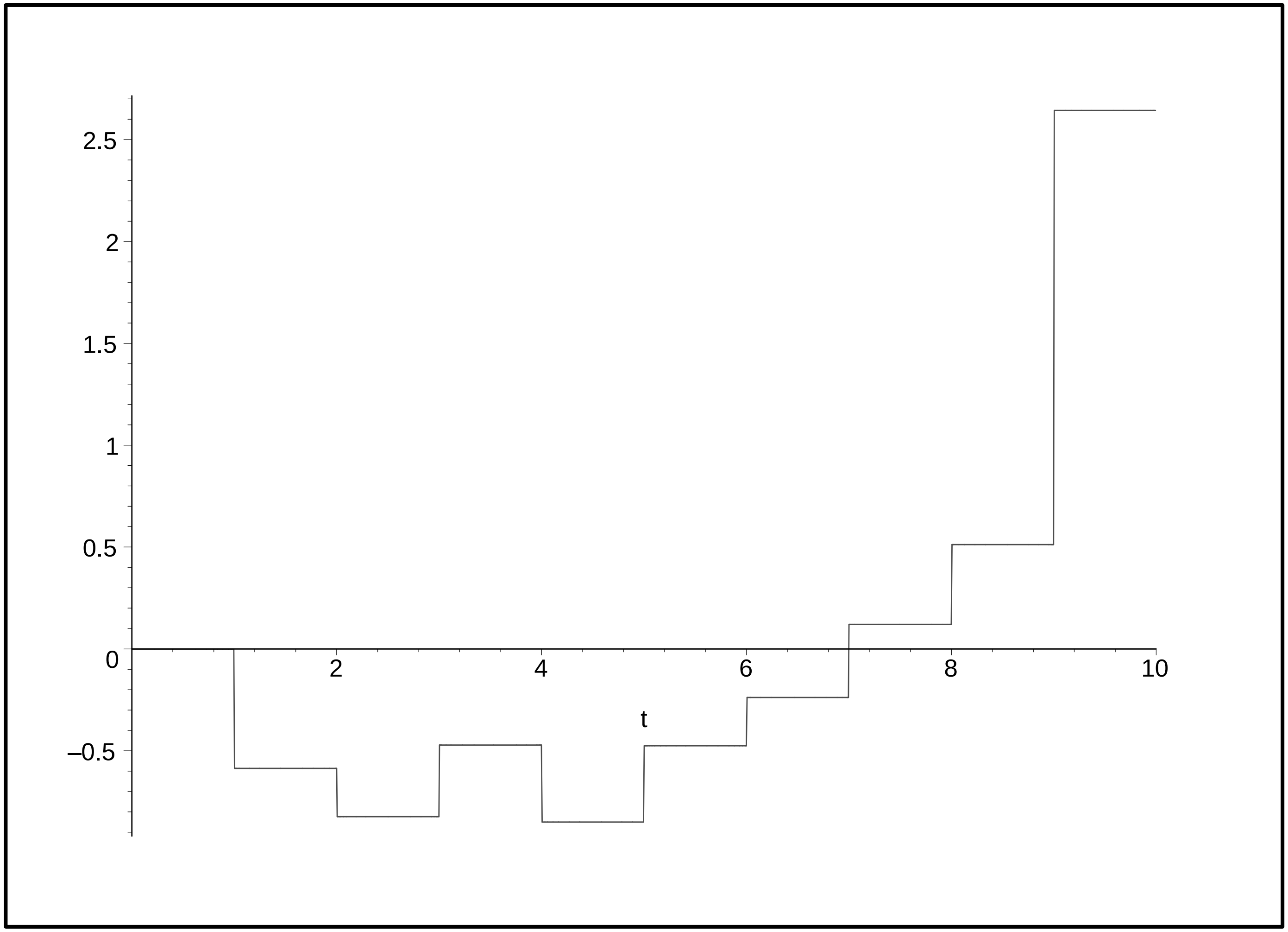
$$\begin{aligned} \hat{f}(c^{-1/\alpha} k)^{[ct]} &= \left(1 + \frac{(ik)^\alpha}{c} + \cdots\right)^{[ct]} \\ &\to e^{t(ik)^\alpha} \equiv \hat{c}(k,t). \end{aligned}$$

Now $A_t \stackrel{d}{=} t^{1/\alpha} A_1$ since $t^{1/\alpha} A_1$ has FT $e^{(ikt^{1/\alpha})^\alpha} = e^{t(ik)^\alpha}$.

The limit process A_t is called a Lévy motion.

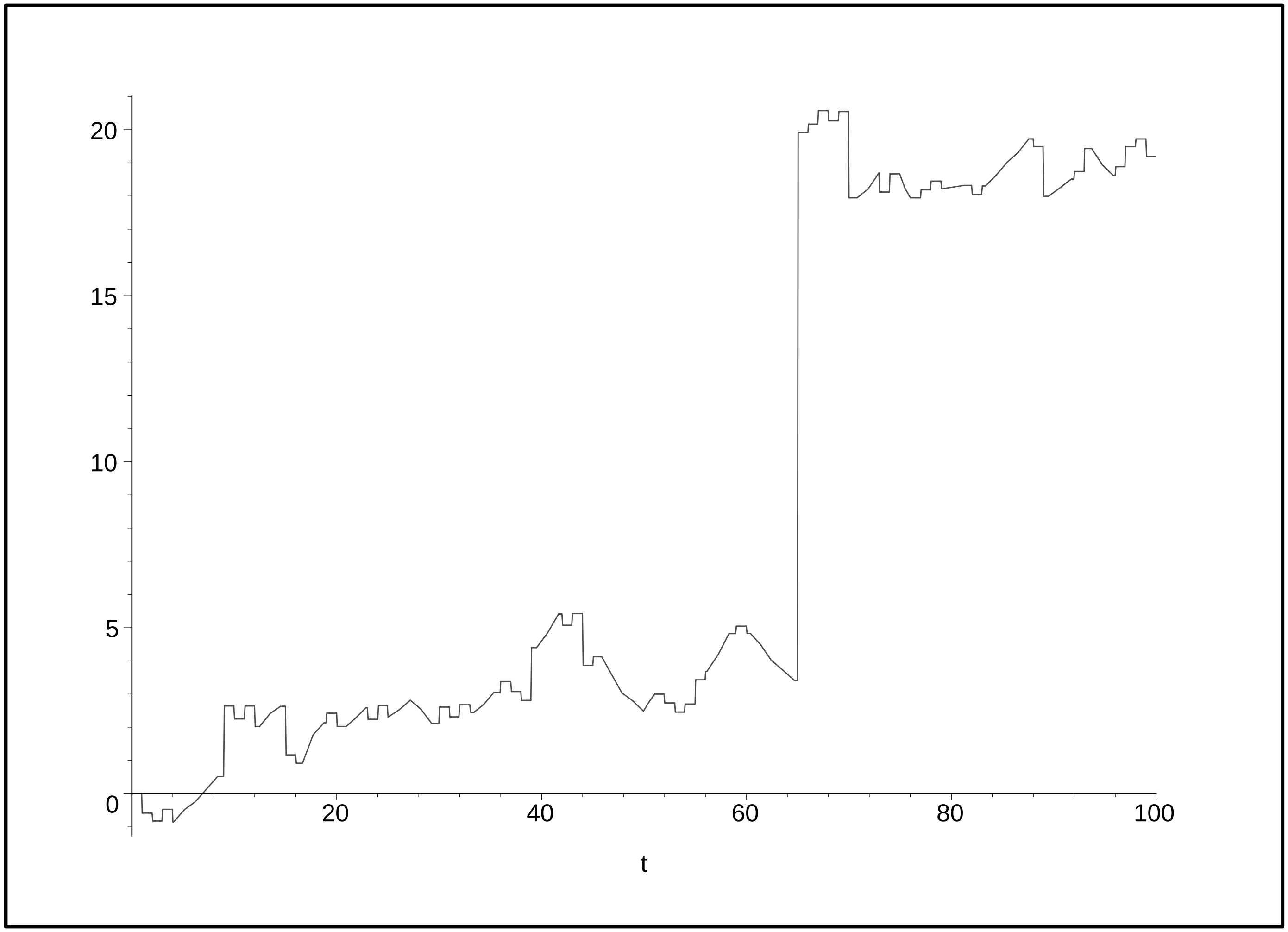
Physicists use Lévy motion to model superdiffusion, where a plume spreads like $t^{1/\alpha}$ for $\alpha < 2$, faster then Brownian motion.

Heavy tail random walk simulation

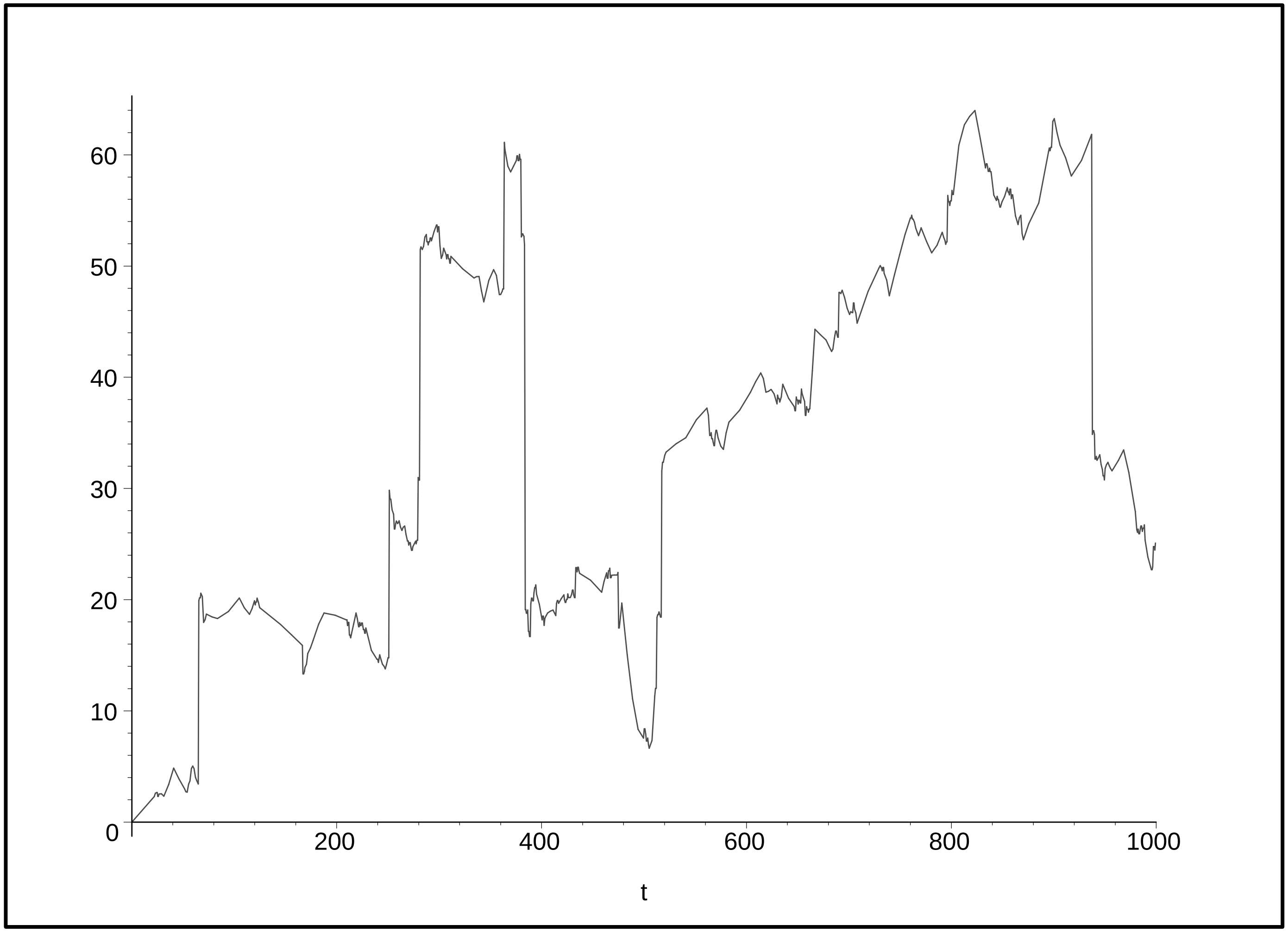


See appendix for Maple simulation code.

Longer time scale



Scaling limit: Stable Lévy motion



Fractional diffusion equation

To solve the fractional diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial^\alpha c(x,t)}{\partial x^\alpha}$$

take Fourier transforms to get

$$\frac{d\widehat{c}(k,t)}{dt} = (ik)^\alpha \widehat{c}(k,t)$$

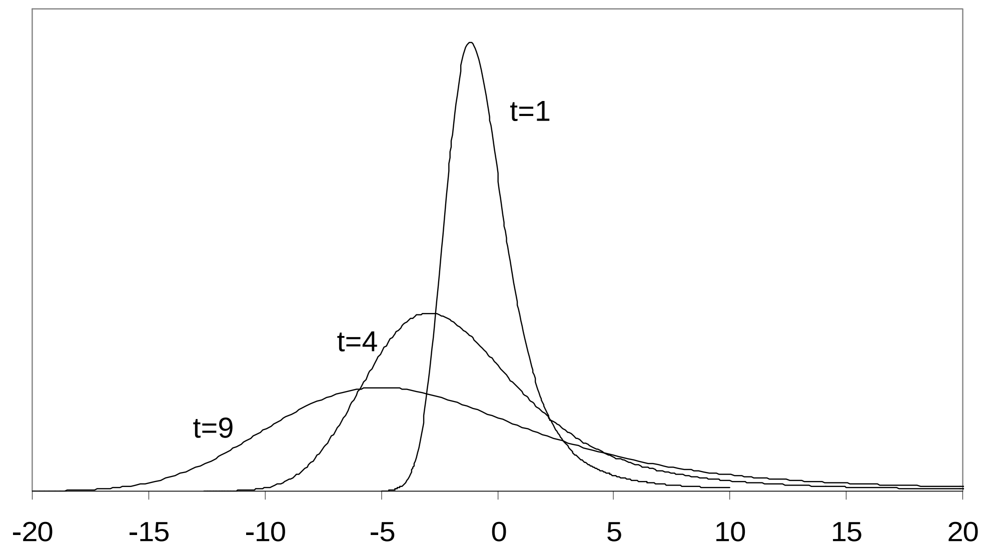
and solve to obtain

$$\widehat{c}(k,t) = e^{t(ik)^\alpha}$$

so the density of the Lévy motion solves this fractional PDE.

In this case $c(x,t)$ falls off like $x^{-\alpha-1}$ as $x \to \infty$.

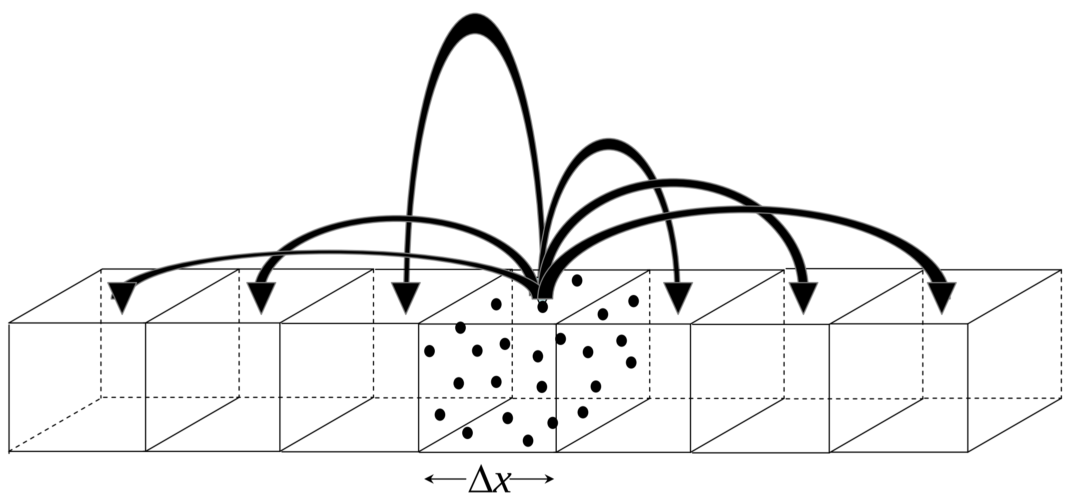
Fractional diffusion profile



Stable $\alpha = 1.5$ Lévy motion density at time $t = 1, 4, 9$ showing super-diffusive spreading rate, skewness, and power law tail.

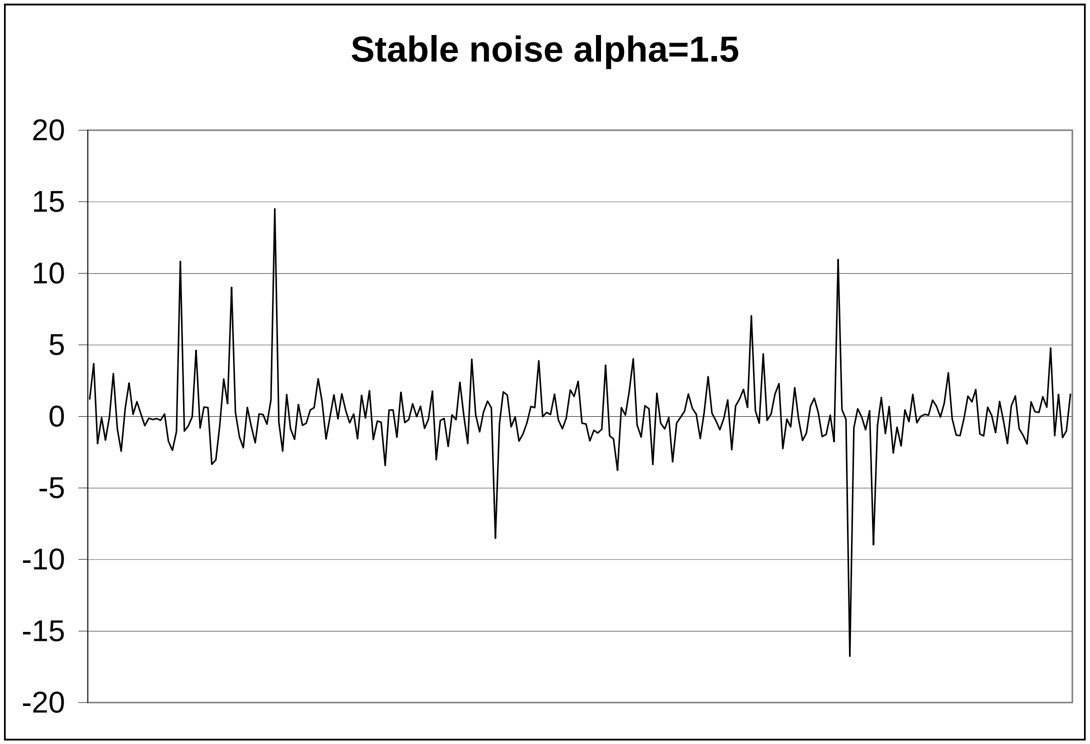
Eulerian view

Fractional derivatives are nonlocal. Particles diffuse via long jumps.



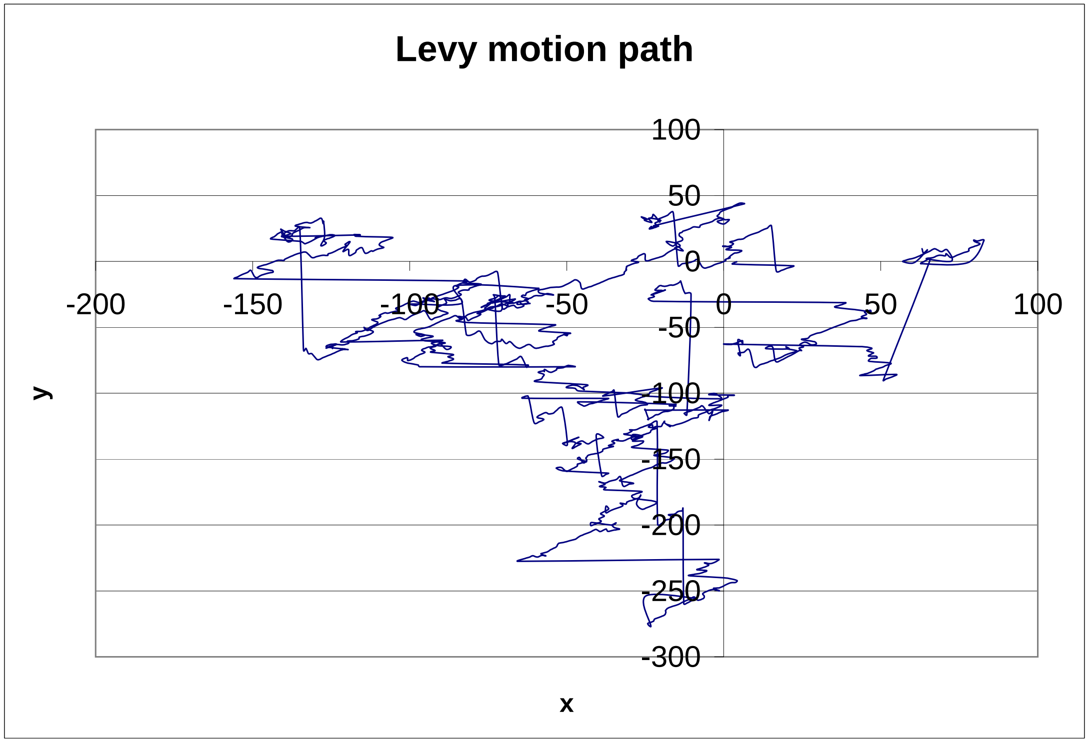
Heavy tails in electrical engineering

Electrical engineers use Lévy motion to model impulsive noise.



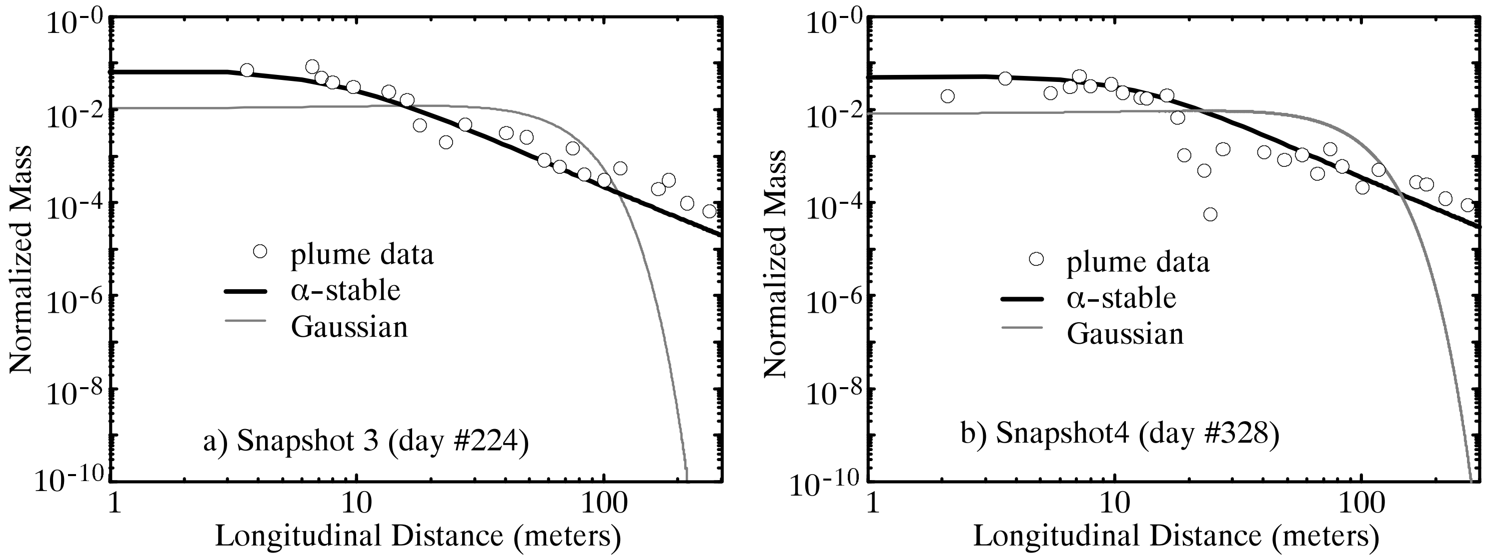
Random fractals

Lévy particle traces are random fractals with dimension α, so it takes $\approx n^\alpha$ disks of size $1/n$ to cover the path. Here $\alpha = 1.5$.



Tracer test in an underground aquifer

The Lévy motion density $c(x,t)$ with $\alpha = 1.1$ gives a good fit. Brownian motion badly underestimates tail concentrations.



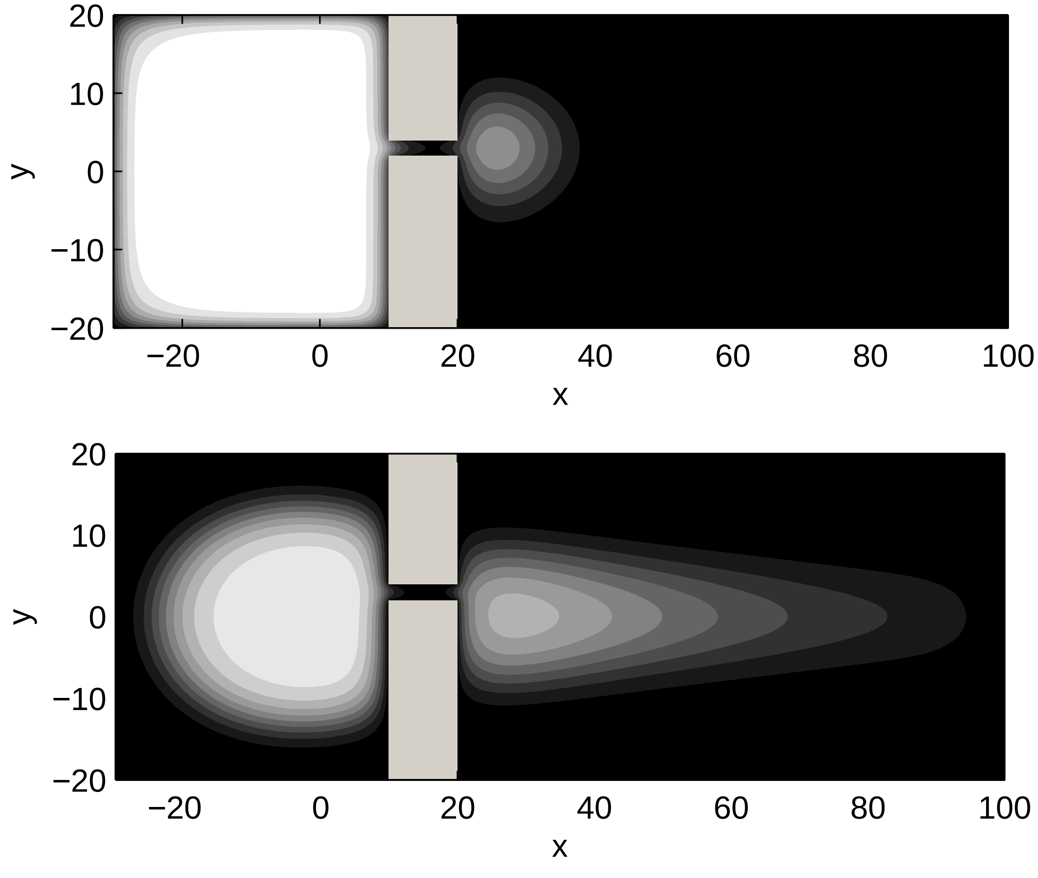
Tracer plume has heavy power law tails and spreads like $t^{1/\alpha}$.

Biological species growth and dispersion

Fractional derivatives model fast spreading via long movements.

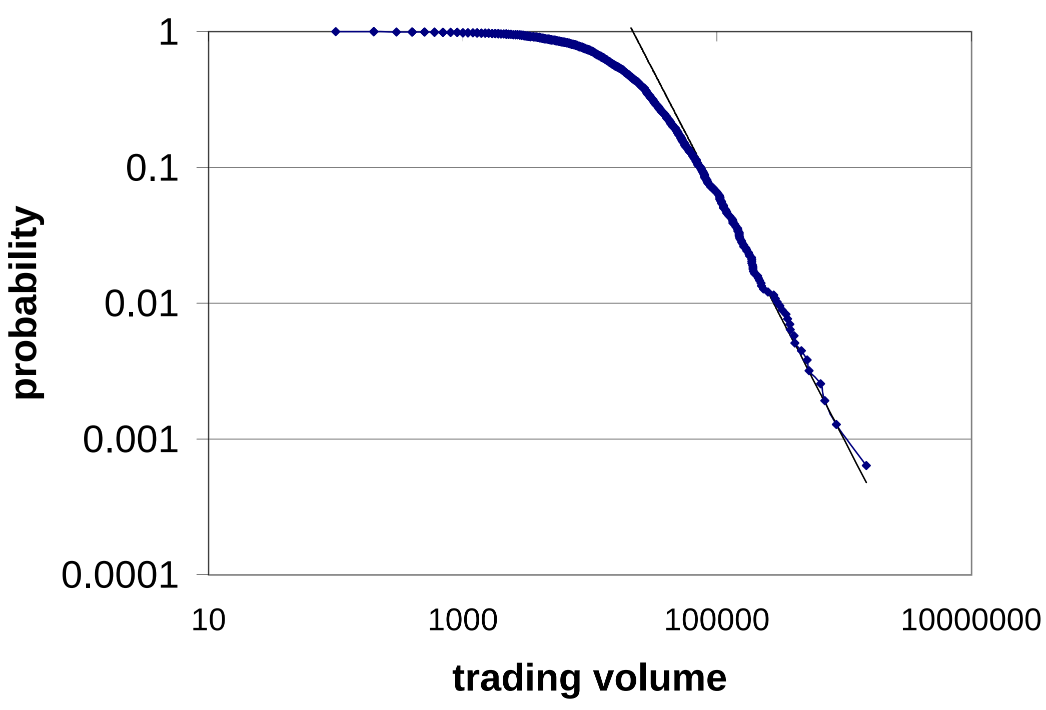
$$\frac{\partial P}{\partial t} = C\frac{\partial^\alpha P}{\partial x^\alpha} + D\frac{\partial^2 P}{\partial y^2} + rP\left(1 - \frac{P}{K}\right)$$

Compare $\alpha = 2$ (top) to $\alpha = 1.7$ (bottom).



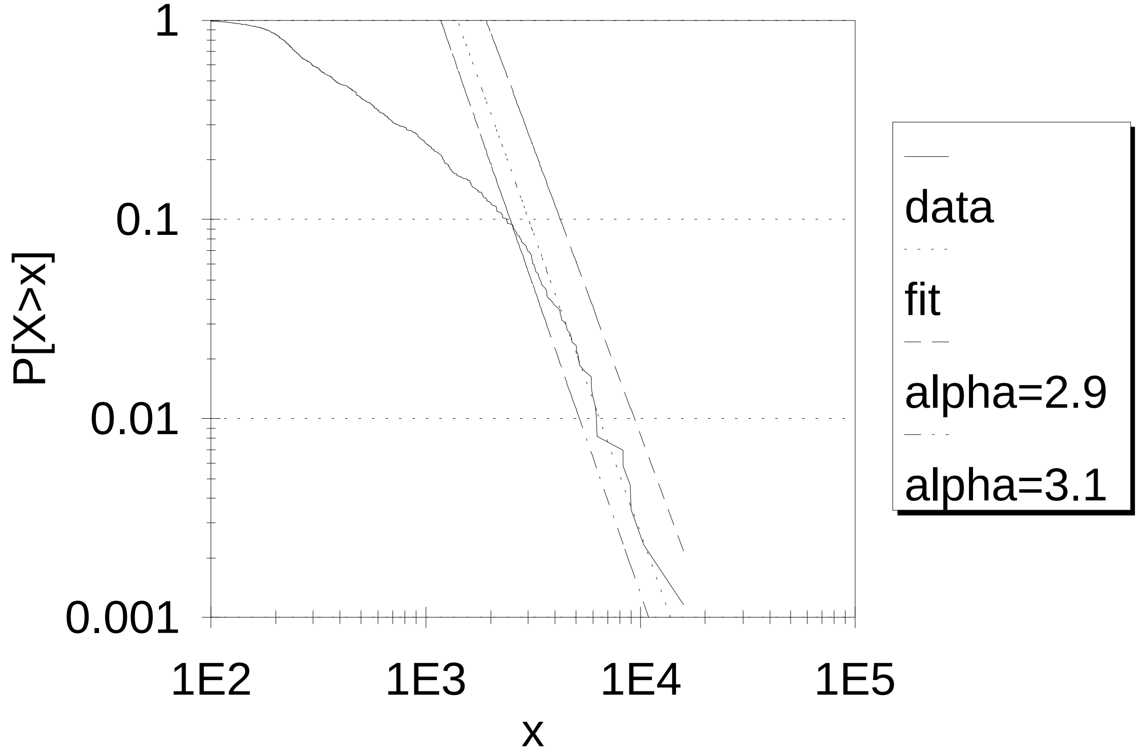
Heavy tails in finance

Trading volume for Amazon, Inc. has a heavy tail with $\alpha \approx 2.7$. Heavy tails in finance were observed by Mandelbrot around 1960.



Heavy tails in hydrology

Monthly average flows for the Salt river in Roosevelt AZ have a heavy upper tail with $\alpha \approx 3$.



Fractional time derivatives

A time-fractional diffusion equation

$$\frac{\partial^\beta c(x,t)}{\partial t^\beta} = \frac{\partial^\alpha c(x,t)}{\partial x^\alpha} + c_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

models particles that wait for a time T_n before the nth jump, where $P(T_n > t) \approx t^{-\beta}$ for some $0 < \beta < 1$ and $c(x,0) = c_0(x)$.

Plume spreads like $t^{\beta/\alpha}$. Physicists use this equation to model subdiffusion, where a plume grows slower than $t^{1/2}$.

Limit process is $A_{E(t)}$ where the random time $E(t)$ grows like t^β.

In finance, $E(t)$ represents the number of trades by time t.

Iterated Brownian motion

Iterated Brownian motion $A_1(|A_2(t)|)$ models diffusion in a crack. Its governing equation

$$\frac{\partial}{\partial t}c(t,x) = \frac{\partial^4}{\partial x^4}c(t,x) + \frac{\partial^2}{\partial x^2}\frac{c_0(x)}{\sqrt{\pi t}}$$

is mathematically equivalent to the time-fractional diffusion equation with $\beta = 1/2$:

$$\frac{\partial^{1/2}c(x,t)}{\partial t^{1/2}} = \frac{\partial^2 c(x,t)}{\partial x^2} + \frac{c_0(x)}{\sqrt{\pi t}}$$

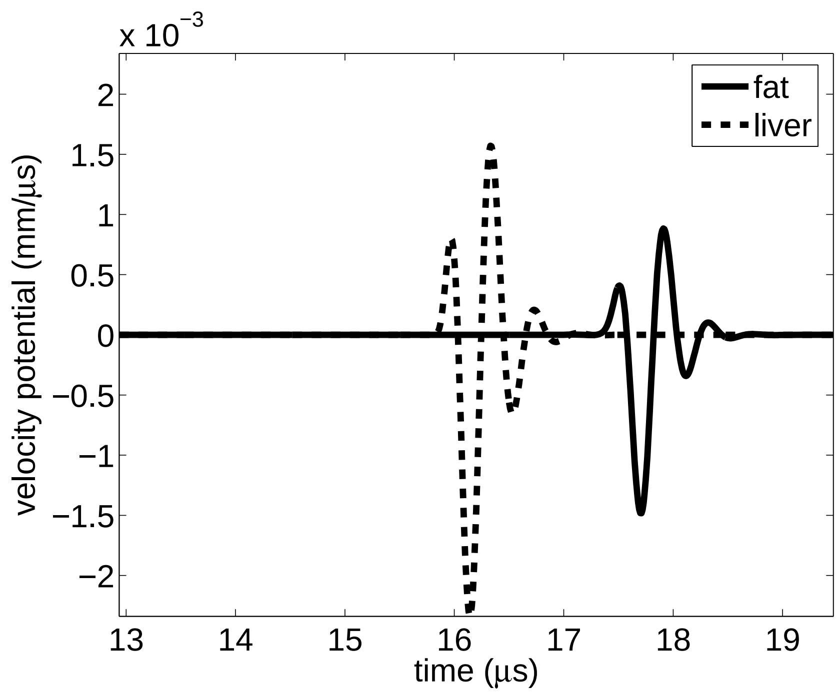
The inner process $|A_2(t)|$ is a "reflected" Brownian motion.

The process $|A_2(t)| = E(t)$ in distribution when $\beta = 1/2$.

Sound wave propogation

We use $\beta = 2.5$ for human fat tissue and $\beta = 2.1$ for liver tissue.

$$\frac{\partial^2}{\partial t^2}c(t,x) + C\frac{\partial^\beta}{\partial t^\beta}c(t,x) = D\frac{\partial^2}{\partial x^2}c(t,x)$$



Coupled fractional derivatives

If the waiting time T_n and jump size X_n are dependent, the limit density solves a coupled space-time diffusion equation like

$$\left(\frac{\partial}{\partial t} - \frac{\partial^\alpha}{\partial x^\alpha}\right)^\beta c(x,t) = c_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \qquad (*)$$

If the Laplace-Fourier transform

$$\bar{c}(k,s) = \int_0^\infty \int_{-\infty}^\infty e^{-st} e^{-ikx} c(x,t)\,dx\,dt$$

then the LHS of $(*)$ has LFT $(s-(ik)^\alpha)^\beta\ \bar{c}(k,s)$.

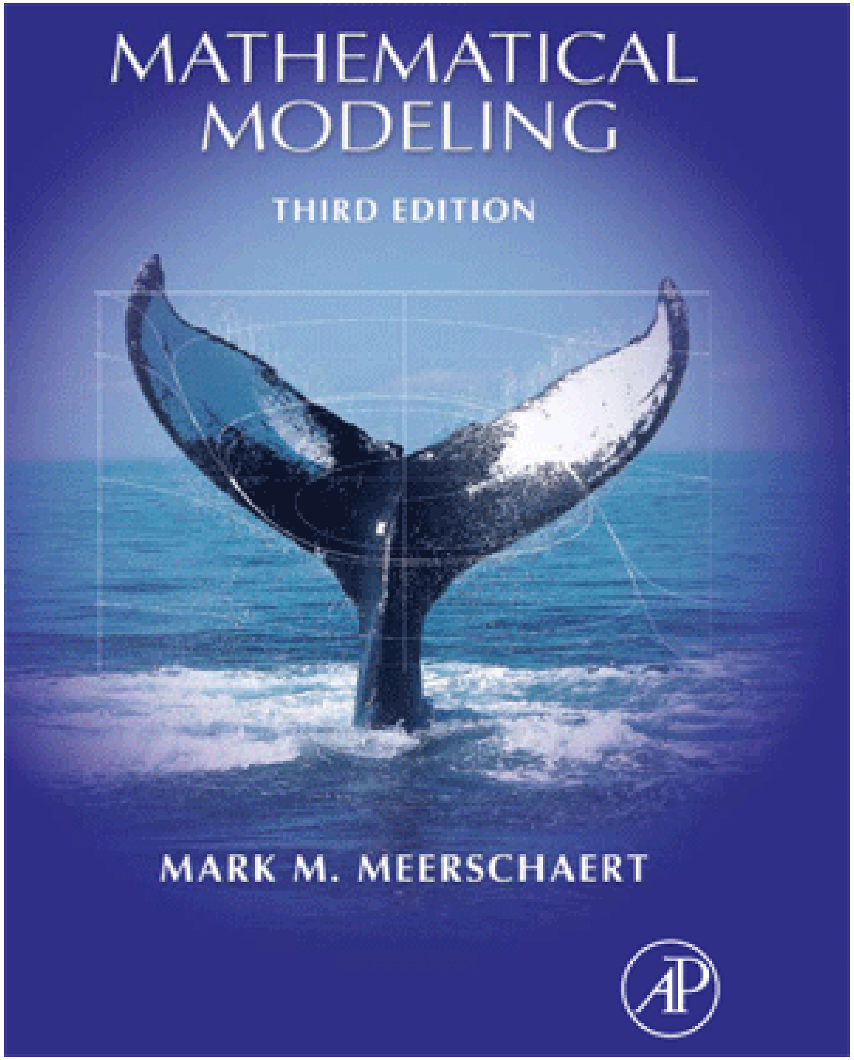
Particles that stick longer jump farther in groundwater flow.

Financial waiting times and price jumps are also dependent.

Some open problems

- Fractional boundary value problems
- Extension to $\beta > 1$ and $\alpha > 2$
- **Truncated or tempered extensions**
- Space-time coupled PDEs
- Applications – interdisciplinary research

Conclusion: Math solves real world problems



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Derivatives of power laws

If both p and α are integers then

$$\begin{aligned}
\mathbb{D}_1 [x^p] &= px^{p-1} \\
\mathbb{D}_2 [x^p] &= p(p-1)x^{p-2} \\
&\vdots \\
\mathbb{D}_\alpha [x^p] &= \frac{p!}{(p-\alpha)!} x^{p-\alpha}
\end{aligned}$$

For $p > 0$ the Gamma function extends $p! = \Gamma(p+1)$ via

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx.$$

Use the property $\Gamma(p+1) = p\Gamma(p)$ to get

$$\mathbb{D}_\alpha [x^p] = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}.$$

Fractional derivatives of power laws

If $p > 0$ then the Laplace transform

$$\text{LT}\{x^p\} = \int_0^\infty e^{-sx} x^p dx \qquad \boxed{\text{substitute } y = sx}$$

$$= \int_0^\infty e^{-y}(y/s)^p dy/s = s^{-p-1}\Gamma(p+1).$$

Then

$$\text{LT}\{\mathbb{D}_\alpha x^p\} = s^\alpha s^{-p-1}\Gamma(p+1)$$

$$= s^{-(p-\alpha)-1}\Gamma(p-\alpha+1) \cdot \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}$$

$$= \text{LT}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}\right\}$$

and the uniqueness of the LT yields

$$\mathbb{D}_\alpha[x^p] = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}.$$

Difference quotients

The derivative $\mathbb{D}_1 f(x) = \lim_{h\to 0} h^{-1}\Delta f(x)$ where

$$\Delta f(x) = f(x) - f(x-h).$$

For positive integers α, $\mathbb{D}_\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\begin{aligned}
\Delta^2 f(x) &= (f(x) - f(x-h)) - (f(x-h) - f(x-2h)) \\
&= f(x) - 2f(x-h) + f(x-2h), \\
\Delta^3 f(x) &= f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h) \\
&\vdots \\
\Delta^\alpha f(x) &= \sum_{m=0}^{\alpha} \binom{\alpha}{m} (-1)^m f(x-mh). \quad \text{Here } \binom{\alpha}{m} = \frac{\alpha!}{m!(\alpha-m)!}
\end{aligned}$$

Fractional difference quotients

For $\alpha > 0$ define $\mathbb{D}_\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\Delta^\alpha f(x) = \sum_{m=0}^{\infty}\binom{\alpha}{m}(-1)^m f(x-mh), \qquad \binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)}$$

Since $f(x-h)$ has FT $e^{-ikh}\widehat{f}(k)$, and using the Binomial formula

$$(1+z)^\alpha = \sum_{m=0}^{\infty}\binom{\alpha}{m} z^m \quad \text{for any complex } |z| \le 1$$

we see that $\Delta^\alpha f(x)$ has FT

$$\sum_{m=0}^{\infty}\binom{\alpha}{m}(-1)^m e^{-ikmh}\widehat{f}(k) = (1-e^{-ikh})^\alpha \widehat{f}(k)$$

and then the FT of $h^{-\alpha}\Delta^\alpha f(x)$ is

$$h^{-\alpha}(ikh)^\alpha\left(\frac{1-e^{-ikh}}{ikh}\right)^\alpha \widehat{f}(k) \to (ik)^\alpha \widehat{f}(k) \quad \text{as } h\to 0.$$

Random walk simulation code (Maple)

```
> N:=1000:
> J:=random[uniform[-1,1]](N): # jump distribution
> n:='n':T:=0:
> for n from 1 to N do
>   T:=T+1;
>   S[n]:=T;
> od:n:='n':
> plot(sum(J[n]*Heaviside(t-S[n]),n=1..1000),t=0..10);
```

See `http://www.maplesoft.on.ca/`

Heavy tail random walk simulation code (Maple)

```
> lambda:=1:N:=1000:alpha:=1.5:C:=.1:
> P:=random[uniform[0,1]](N):
> J:=random[uniform[0,1]](N):
> n:='n':T:=0:
> for n from 1 to N do
>   T:=T+1;
>   S[n]:=T;
> od:n:='n':
> plot(sum((2*floor(2*P[n])-1)*(C/J[n])^(1/alpha)
   *Heaviside(t-S[n]),n=1..1000),t=0..1000);
```

See `http://www.maplesoft.on.ca/`

Heavy tailed jumps $U^{-1/\alpha}$ where $U \sim \text{Uniform}[0,1]$.