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Correlated continuous time random walks

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ABSTRACT

Continuous time random walks impose a random waiting time before each particle jump. Scaling limits of heavy-tailed continuous time random walks are governed by fractional evolution equations. Space-fractional derivatives describe heavy-tailed jumps, and the time-fractional version codes heavy-tailed waiting times. This paper develops scaling limits and governing equations in the case of correlated jumps. For long-range dependent jumps, this leads to fractional Brownian motion or linear fractional stable motion, with the time parameter replaced by an inverse stable subordinator in the case of heavy-tailed waiting times. These scaling limits provide an interesting class of non-Markovian, non-Gaussian self-similar processes.

1. Introduction

Continuous time random walks (CTRW) separate IID particle jumps $\{Y_n\}$ by IID waiting times $\{J_n\}$. CTRW models are important in applications to geology, physics and finance; see, for example, Berkowitz et al. (2006), Metzler and Klafter (2004) and Scalas (2004) for more information. If $\{Y_n\}$ and $\{J_n\}$ are independent, then the CTRW is called decoupled. Otherwise it is called coupled. Throughout this paper we will only consider decoupled CTRWs with values in $\mathbb{R}$.

In the case of heavy-tailed waiting times, Meerschaert and Scheffler (2004) proved that CTRW scaling limits are subordinated processes that are self-similar but non-Markovian, and their transition densities are governed by fractional diffusion equations (see also Meerschaert et al. (2002)). Fractional diffusion equations replace the usual integer-order derivatives in the diffusion equation by their fractional-order analogues (Miller and Ross, 1993; Samko et al., 1993). Just as the diffusion equation $\partial_t u = a\partial_x^2 u$ governs the scaling limit of a simple random walk, the fractional diffusion equation $\partial_t^\beta u = a\partial_x^\alpha u$ governs the scaling limit of a CTRW with heavy tail jumps $\mathbb{P}(Y_n > r) \sim r^{-\alpha}$ for $0 < \alpha < 2$ and waiting times $\mathbb{P}(J_n > t) \sim t^{-\beta}$ for $0 < \beta < 1$.

This paper develops limit theorems and governing equations for CTRW with correlated (or dependent) jumps $Y_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$, where $\{Z_n\}$ are IID and $\{c_n\}$ are real numbers (see Section 2 for precise conditions). These CTRW models are useful for correlated observations separated by random waiting times, which are common, for example, in finance (Scalas et al., 2000). Scaling limits of the partial sum process $S(t) = Y_1 + \cdots + Y_{[t]}$ in the case of long-range dependence include fractional Brownian motion (FBM) for light-tailed jumps (Davydov, 1970; Whitt, 2002) and linear fractional stable motion (LFSM) for heavy-tailed jumps (Astrauskas, 1983; Kasahara and Maejima, 1988; Whitt, 2002). Letting $T_0 = 0, T_n = J_1 + \cdots + J_n$ the time of the nth jump, and $N_t = \max\{n : T_n \le t\}$ the number of jumps by time $t > 0$, the scaling limit of the CTRW $S(N_t)$ is a FBM or LFSM subordinated to an inverse stable subordinator, which is connected to the local time of a strictly stable

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Lévy process (Meerschaert et al., 2008), or the supremum process of a spectrally negative stable Lévy process (Bingham, 1973). This extends the results of Meerschaert and Scheffler (2004) and Becker-Kern et al. (2004) to the case of dependent jumps. We also discuss some interesting properties of these self-similar limit processes, and governing equations for their probability densities.

2. Results

Let $\{Z_n, -\infty < n < \infty\}$ denote IID random variables that belong to the strict domain of attraction of some strictly stable law A with index $0 < \alpha \le 2$. This means that the sequence of partial sums $P(n) = Z_1 + \cdots + Z_n$ satisfies $a_n P(n) \Rightarrow A$ for some $a_n > 0$, see Feller (1971, p. 312–313) or Whitt (2002, p. 114–115). Here $\Rightarrow$ denotes convergence in distribution.

The particle jumps that we consider in this paper are given by the stationary linear process $\{Y_n, -\infty < n < \infty\}$ defined by $Y_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$, where c_j are real constants such that $\sum_{j=0}^{\infty} |c_j|^\rho < \infty$ for some $\rho \in (0, \alpha)$. This condition ensures that the series $\sum_{j=0}^{\infty} c_j Z_{n-j}$ converges in $L^\rho(\mathbb{P})$ and almost surely (see Avram and Taqqu (1992)). The dependence structure of the linear process $\{Y_n, -\infty < n < \infty\}$ relies on the sequence $\{c_j\}$. For example, if $\mathbb{E}(Z_n) = 0$ and $\mathbb{E}(Z_n^2) < \infty$, then it can be verified that $\sum_{n=1}^{\infty} |\mathbb{E}(Y_0 Y_n)| < \infty$ if $\sum_{j=0}^{\infty} |c_j| < \infty$; and $\sum_{n=1}^{\infty} |\mathbb{E}(Y_0 Y_n)| = \infty$ if the real numbers c_j eventually have the same sign and $\sum_{j=0}^{\infty} |c_j| = \infty$. In the literature, a second-order stationary process $\{Y_n, -\infty < n < \infty\}$ with mean 0 is said to be short-range dependent if $\sum_{n=1}^{\infty} |\mathbb{E}(Y_0 Y_n)| < \infty$ and long-range dependent otherwise. Even though in this paper we are primarily interested in particle jumps with heavy-tailed distributions and typically $\mathbb{E}(Y_n^2) = \infty$, we will, by analogy, call the linear process $\{Y_n, -\infty < n < \infty\}$ short-range dependent if $\sum_{j=0}^{\infty} |c_j| < \infty$, and long-range dependent if $\sum_{j=0}^{\infty} |c_j| = \infty$.

Let $J_n > 0$ be IID waiting times that are independent of $\{Z_n\}$, $T_n = J_1 + \cdots + J_n$ the time of the nth particle jump, and $N_t = \max\{n : T_n \le t\}$ the number of jumps by time $t > 0$. Let $S(0) = 0$ and $S(n) = Y_1 + \cdots + Y_n$ denote the location of the particle after n jumps, so that the continuous time random walk (CTRW) $S(N_t)$ gives the location of the particle at time $t > 0$. Suppose that J_n belongs to the domain of attraction of some stable law D with index $0 < \beta < 1$ and $D > 0$ almost surely. Hence $b_n T_n \Rightarrow D$ for some norming constants $b_n > 0$. Let $b(t) = b_{[t]}$ and take $\tilde{b}(t)$ an asymptotic inverse of the regularly varying function $1/b(t)$, so that $t b(\tilde{b}(t)) \to 1$ as $t \to \infty$ (Meerschaert and Scheffler, 2004).

Let $\{A(t), t \ge 0\}$ and $\{D(t), t \ge 0\}$ be strictly stable Lévy processes with $A(1) = A$, $D(1) = D$, respectively. Note that $\{D(t), t \ge 0\}$ is a stable subordinator of index β, hence its sample functions are almost surely strictly increasing (Bertoin, 1996, p. 75). Therefore, the inverse or hitting time process of $\{D(t), t \ge 0\}$,

$$E_t = \inf\{x > 0 : D(x) > t\}, \quad \forall t \ge 0$$

is well defined and the sample function $t \mapsto E_t$ is strictly increasing almost surely.

Our first result shows that the CTRW scaling limit in the case of short-range dependence is quite similar to the case of independent jumps studied by Meerschaert and Scheffler (2004).

Theorem 2.1. *Under the conditions of this section, suppose that $0 < \alpha < 2$, $c_j \ge 0$ and $\sum_{j=0}^{\infty} c_j^\rho < \infty$ for some $\rho \in (0, \alpha)$ with $\rho \le 1$, and that one of the following holds:*

(a) $0 < \alpha \le 1$; *or*
(b) $c_j = 0$ *for all but finitely many* j; *or*
(c) $1 < \alpha < 2$, c_j *is monotone and* $\sum_{j=0}^{\infty} c_j^\rho < \infty$ *for some* $\rho < 1$.

Then we have

$$w^{-1} a_{[\tilde{b}(c)]} S(N_{ct}) \Rightarrow A(E_t) \tag{2.1}$$

as $c \to \infty$ in the M_1 topology on $D([0, \infty), \mathbb{R})$, where $w = \sum_j c_j$.

In view of Theorem 1 in Avram and Taqqu (1992), the convergence in (2.1) cannot be strengthened to the J_1 topology. Note that the processes $\{A(t), t \ge 0\}$ and $\{E_t, t \ge 0\}$ are independent and self-similar. The latter means that, for every constant $c > 0$

$$\{A(ct), t \ge 0\} \stackrel{d}{=} \{c^{1/\alpha} A(t), t \ge 0\}$$

and

$$\{E_{ct}, t \ge 0\} \stackrel{d}{=} \{c^\beta E_t, t \ge 0\},$$

where $\stackrel{d}{=}$ means equality in all finite-dimensional distributions. It follows immediately that the scaling limit $\{A(E_t), t \ge 0\}$ in (2.1) is self-similar with index β/α. When $0 < \beta \le 1/2$, the inner process E_t in (2.1) is also the local time at zero of a strictly stable Lévy motion (Meerschaert et al., 2008). When $1/2 \le \beta < 1$, the inner process E_t is also the supremum process of a stable Lévy motion with index $1/\beta$ and no negative jumps (Bingham, 1973).

Let $\partial_t^\beta g(t)$ denote the Caputo fractional derivative, the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ where $\tilde{g}(s) = \int_0^\infty e^{-st} g(t)\,dt$ is the usual Laplace transform of g. Let $\partial_{\pm x}^\alpha f(x)$ denote the Liouville fractional derivative, the inverse Fourier transform of $(\pm ik)^\alpha \hat{f}(k)$, where $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,ds$ is the usual Fourier transform. The stable random variable $A(t)$ has a smooth density with Fourier transform $e^{-t\psi(k)}$ where $\psi(k) = a[p(ik)^\alpha + q(-ik)^\alpha]$ with $0 \le p, q \le 1$ and $p + q = 1$ (Meerschaert and Scheffler, 2001). Then the limit $A(E_t)$ in (2.1) has a density $h(x, t)$ that solves the fractional diffusion equation $\partial_t^\beta h = ap\partial_x^\alpha h + aq\partial_{-x}^\alpha h$, see Meerschaert and Scheffler (2004).

Next we consider the CTRW scaling limit for heavy-tailed particle jumps with long-range dependence. To simplify the presentation, we assume $a_n = n^{-1/\alpha}$ (domain of normal attraction) and power-law weights; namely $c_j \sim c_0 j^{H-1-1/\alpha}$ as $j \to \infty$, for some $c_0 > 0$. Consequently, we have $\sum_{j=0}^\infty |c_j| < \infty$ if and only if $0 < H < 1/\alpha$. Hence, the case $0 < H < 1/\alpha$ means the stationary sequence $\{Y_n\}$ has short-range dependence, while the case $1/\alpha < H < 1$ means $\{Y_n\}$ has long-range dependence. The scaling limit of CTRW with short-range dependence has been partially covered by Theorem 2.1. The rest of the cases are treated in Theorems 2.2 and 2.3.

We will make use of the following definition. Given constants $\alpha \in (0, 2)$ and $H \in (0, 1)$, the α-stable process $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$ defined by

$$L_{\alpha,H}(t) = \int_{\mathbb{R}} \left[(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right] A(\mathrm{d}s) \tag{2.2}$$

is called a linear fractional stable motion (LFSM) with indices α and H. In the above, $a_+ = \max\{0, a\}$ for all $a \in \mathbb{R}$, $0^{H-1/\alpha} = 0$ and $\{A(t), t \in \mathbb{R}\}$ is a two-sided strictly stable Lévy process of index α with $A(1) = A$ given at the beginning of Section 2 (namely, $n^{-1/\alpha}P(n) \Rightarrow A$ as $n \to \infty$). Because of this, $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$ defined by (2.2) differs from the LFSM in Theorem 4.7.2 in Whitt (2002) by a constant factor. Note that, when $H = 1/\alpha$, $L_{\alpha,H}(t) = A(t)$ for all $t \ge 0$. When $H \ne 1/\alpha$, the stochastic integral in (2.2) is well defined because

$$\int_{\mathbb{R}} \left| (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right|^\alpha \mathrm{d}s < \infty.$$

See Example 3.6.5 or Section 7.4 in Samorodnitsky and Taqqu (1994).

By (2.2), it can be verified that $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$ is H-self-similar with stationary increments (Samorodnitsky and Taqqu, 1994, Proposition 7.4.2). However, for $H \ne 1/\alpha$, it does not have independent increments. LFSM is an α-stable analogue of fractional Brownian motion and its probabilistic and statistical properties have been investigated by several authors. In particular, it is known that

(i) If $1/\alpha < H < 1$ (this is possible only when $1 < \alpha < 2$), then the sample paths of $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$ are almost surely continuous.
(ii) If $0 < H < 1/\alpha$, then the sample paths of $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$ are almost surely unbounded on every interval of positive length.

We refer the reader to Theorem 12.4.1 and Example 10.2.5 in Samorodnitsky and Taqqu (1994) for more information.

Theorem 2.2. *We assume the setting of this section. If* $1 < \alpha < 2$, $1/\alpha < H < 1$, *and* $c_j \sim c_0 j^{H-1-1/\alpha}$ *as* $j \to \infty$ *for some* $c_0 > 0$, *then as* $c \to \infty$

$$[\tilde{b}(c)]^{-H} S(N_{ct}) \Rightarrow K_1 L_{\alpha,H}(E_t) \tag{2.3}$$

in the J_1 *topology on* $D([0, \infty), \mathbb{R})$, *where* $K_1 = c_0\alpha/(H\alpha - 1)$.

The topology on $D([0, \infty), \mathbb{R})$ in Theorem 2.2 is stronger than that in Theorem 2.1, thanks to the fact that $L_{\alpha,H}(t)$ is a.s. continuous whenever $1/\alpha < H < 1$.

Observe that the case when $0 < H < 1/\alpha$ and the constants c_j $(j \ge 0)$ are not all non-negative is left uncovered by Theorems 2.1 and 2.2. Because of Property (ii) of $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$, the limiting process does not belong to the function space $D([0, \infty), \mathbb{R})$. Nevertheless, we have the following theorem.

Theorem 2.3. *We assume the setting of this section. If* $0 < \alpha < 2$, $0 < H < 1/\alpha$, $c_j \sim c_0 j^{H-1-1/\alpha}$ *as* $j \to \infty$ *for some* $c_0 > 0$, *and* $\sum_{j=0}^\infty c_j = 0$, *then*

$$[\tilde{b}(c)]^{-H} S(N_{ct}) \stackrel{f.d.}{\longrightarrow} K_1 L_{\alpha,H}(E_t) \tag{2.4}$$

as $c \to \infty$, *where* $\stackrel{f.d.}{\longrightarrow}$ *means convergence of all finite-dimensional distributions and* $K_1 = c_0\alpha/(H\alpha - 1)$.

It is interesting to note that the constants in Theorems 2.2 and 2.3 are determined by c_0, α and H in the same way. But K_1 is positive when $1/\alpha < H < 1$, and is negative when $0 < H < 1/\alpha$.

It follows from the self-similarity and the independence of $\{L_{\alpha,H}(t), t \in \mathbb{R}\}$ and $\{E(t), t \ge 0\}$ that the scaling limits in (2.3) and (2.4) are self-similar with index $H\beta$. When $1/\alpha < H < 1$, it can be seen that $\{L_{\alpha,H}(E_t), t \ge 0\}$ has continuous sample functions almost surely. However, if $0 < H < 1/\alpha$, then $\{L_{\alpha,H}(E_t), t \ge 0\}$ is almost surely unbounded on every interval of positive length. It would be interesting to further study the properties of the process $\{L_{\alpha,H}(E_t), t \ge 0\}$.

We mention that both Theorems 2.2 and 2.3 can be extended to $\{Z_n\}$ in the strict domain of attraction of A and $\{c_j\}$ regularly varying at ∞ with index $H - 1 - 1/\alpha$, using a slightly different normalization in (2.3) depending on a_n and the probability tail of Z_n, compare Astrauskas (1983).

Finally we consider the case $\alpha = 2$. If $\{A(t), t \in \mathbb{R}\}$ in (2.2) is replaced by ordinary two-sided Brownian motion, then (2.2) defines a fractional Brownian motion $W_H = \{W_H(t), t \in \mathbb{R}\}$ on $\mathbb{R}$ of index H, which is a Gaussian process with mean zero and covariance function

$$\mathbb{E}[W_H(t)W_H(s)] = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right].$$

Theorem 2.4 gives the CTRW scaling limit for light-tailed particle jumps with long-range dependence.

Theorem 2.4. *We assume the setting of this section. If $\alpha = 2$, $\mathbb{E}(Z_n) = 0$, $\mathbb{E}(Z_n^2) < \infty$, $\sum_{j=0}^{\infty} c_j^2 < \infty$, $\mathrm{Var}(S(n)) = \sigma_n^2$ varies regularly at ∞ with index $2H$ for some $0 < H < 1$, and $\mathbb{E}\left(S(n)^{2\rho}\right) \leq K_2 \left[\mathbb{E}(S(n)^2)\right]^{\rho}$ for some constants $K_2 > 0$ and $\rho > 1/H$, then as $c \to \infty$*

$$\sigma^{-1}_{[\tilde{b}(c)]} S(N_{ct}) \Rightarrow W_H(E_t) \tag{2.5}$$

in the J_1 topology on $D([0,\infty),\mathbb{R})$.

Note that it is not difficult to provide examples of sequences of IID random variables $\{Z_n\}$ and real numbers $\{c_j\}$ that satisfy the conditions of Theorem 2.4, see Davydov (1970) and Giraitis et al. (2003). It follows from the results of Taqqu (1975) that the conclusion of Theorem 2.4 still holds if the linear process $\{Y_n\}$ is replaced by the stationary sequence $\{g(\xi_n)\}$, where $\{\xi_n\}$ is a stationary Gaussian sequence with mean 0, variance 1 and long-range dependence, and $g \in L^2(\mathrm{e}^{-x^2/2}\mathrm{d}x)$ is a function with Hermite rank 1.

Theorem 2.4 contains the case $H = 1/2$ where $W_H(t) = A(t)$ is a standard Brownian motion. This includes the situation of mean zero finite variance particle jumps, and heavy-tailed waiting times between jumps. In this case, the CTRW scaling limit $A(E_t)$ has a density $h(x,t)$ that solves the time-fractional diffusion equation $\partial_t^{\beta} h = a\,\partial_x^2 h$, see Meerschaert and Scheffler (2004). Since $\{W_H(ct), t \geq 0\} \overset{d}{=} \{c^H W_H(t), t \geq 0\}$, the scaling limit in (2.5) is self-similar with index $H\beta$. Some results on large deviation and sample path regularity have recently been obtained for $\{W_H(E_t), t \geq 0\}$ in Meerschaert et al. (2008).

In the case of finite mean waiting times, the CTRW scaling limit is essentially the same as for the underlying random walk. If $\mu = \mathbb{E}(J_n) < \infty$, then $\mu N_t/t \to 1$ almost surely as $t \to \infty$, and a simple argument along the lines of the proof of Theorem 2.1 shows that $w^{-1}a_{[c]}S(N_{ct}) \Rightarrow A(t/\mu)$ in the M_1 topology on $D([0,\infty),\mathbb{R})$. Theorems 2.2–2.4 can be extended similarly.

An easy argument with Fourier transforms shows that the density $h(x,t)$ of $L_{\alpha,H}(t)$ solves $\partial_t h = \alpha H t^{\alpha H-1}[ap\partial_x^{\alpha} h + aq\partial_{-x}^{\alpha} h]$. A similar argument shows that the density of $W_H(t)$ solves $\partial_t h = 2Ht^{2H-1}a\partial_x^2 h$. An interesting open question is to establish the governing equation for the CTRW scaling limit in (2.3) and (2.5). This is not as simple as replacing the first time derivative by a fractional derivative in the governing equation for the outer process, since the t variable also appears on the right-hand side, so that Theorem 3.1 of Baeumer and Meerschaert (2001) does not apply.

3. Proofs

The proofs in this section are based on invariance principles for stationary sequences with short- or long-range dependence (see, for example, Whitt (2002)) and the CTRW limit theory developed in Meerschaert and Scheffler (2004). Due to the non-Markovian nature of the CTRW scaling limits in this paper, standard subordination methods cannot be applied directly. Instead we apply continuous mapping-type arguments to prove Theorems 2.1, 2.2 and 2.4. The proof of Theorem 2.3 is quite different and relies on a criterion for the convergence of all finite-dimensional distributions of composite processes established by Becker-Kern et al. (2004).

Recall that $J_n > 0$ are IID waiting times, $T_n = J_1 + \cdots + J_n$ the time of the nth particle jump, and $N_t = \max\{n : T_n \leq t\}$ the number of jumps by time $t > 0$. Since J_n belongs to the domain of attraction of some stable law D with index $0 < \beta < 1$ and $D > 0$ almost surely, with $b_n T_n \Rightarrow D$ for some norming constants $b_n > 0$, the sequence b_n varies regularly with index $-1/\beta$ (see, e.g., Feller (1971)). Then the asymptotic inverse $\tilde{b}(t)$ of $1/b$ varies regularly with index β, see Seneta (1976). Recall that the stable Lévy motion $\{D(x), x \geq 0\}$ with $D(1) = D$ is a stable subordinator of index β and thus is almost surely strictly increasing (Bertoin, 1996, p. 75). Its inverse or hitting time process $E_t = \inf\{x > 0 : D(x) > t\}$ is almost surely strictly increasing with continuous sample paths, has moments of all orders, and its increments are neither stationary nor independent (Meerschaert and Scheffler, 2004). Bingham (1971) shows that E_t has a Mittag-Leffler distribution, and gives a differential equation that governs its finite-dimensional distributions.

Proof of Theorem 2.1. Corollary 3.4 in Meerschaert and Scheffler (2004) shows that $\tilde{b}(c)^{-1}N_{ct} \Rightarrow E_t$ as $c \to \infty$ in the J_1 topology on $D([0,\infty),[0,\infty))$. Note that $\tilde{b}(c) \to \infty$ as $c \to \infty$ since this function is regularly varying at ∞ with index $\beta > 0$. Theorem 4.7.1 in Whitt (2002) shows that $a_n S(nt) \Rightarrow wA(t)$ in the M_1 topology on $D([0,\infty),\mathbb{R})$. Since the J_1 topology is stronger, and since the waiting times $\{J_n\}$ are independent of $\{Y_n\}$, we have

$$\left(a_{[\tilde{b}(c)]}S(\tilde{b}(c)t), \tilde{b}(c)^{-1}N_{ct}\right) \Rightarrow (A(t), E_t)$$

in the M_1 topology of the product space $D([0,\infty),\mathbb{R}\times[0,\infty))$. Note that this last statement also follows from Theorem 3.2 in Billingsley (1968).

Since the process $\{E_t, t \geq 0\}$ is almost surely strictly increasing and continuous, Theorem 13.2.4 in Whitt (2002) yields

$$a_{[\tilde{b}(c)]} S\left(\tilde{b}(c) \cdot \tilde{b}(c)^{-1} N_{ct}\right) \Rightarrow A(E_t)$$

in the M_1 topology on $D([0, \infty), \mathbb{R})$, which completes the proof. □

Proof of Theorem 2.2. Recall that $\tilde{b}(c)^{-1} N_{ct} \Rightarrow E_t$ in the J_1 topology on $D([0, \infty), [0, \infty))$ (Meerschaert and Scheffler, 2004, Corollary 3.4). Theorem 4.7.2 in Whitt (2002), originally due to Astrauskas (1983), shows that $n^{-H} S(nt) \Rightarrow K_1 L_{\alpha,H}(t)$ in the J_1 topology on $D([0, \infty), \mathbb{R})$, where $K_1 = c_0\alpha/(H\alpha - 1)$.

Since $\{N_t, t \geq 0\}$ is independent of $\{S(n), n \geq 1\}$, we have

$$([\tilde{b}(c)]^{-H} S(\tilde{b}(c)t), \tilde{b}(c)^{-1} N_{ct}) \Rightarrow (K_1 L_{\alpha,H}(t), E_t)$$

in the product space. Combining this with Theorem 13.2.4 in Whitt (2002) yields (2.5) in the M_1 topology. Since both processes $\{L_{\alpha,H}(t), t \geq 0\}$ and $\{E_t, t \geq 0\}$ are continuous, and the latter is strictly increasing, one can apply Theorem 13.3.1 in Whitt (2002) to strengthen the conclusion to convergence in the J_1 topology. This proves Theorem 2.2. □

Proof of Theorem 2.3. It is sufficient to show that for all integers $m \geq 1$, $0 < t_1 < \cdots < t_m$, we have

$$\tilde{b}(c)^{-H} \left(S(N_{ct_1}), \ldots, S(N_{ct_m})\right) \Rightarrow K_1 \left(L_{\alpha,H}(E_{t_1}), \ldots, L_{\alpha,H}(E_{t_m})\right) \tag{3.1}$$

as $c \to \infty$. For this purpose, we will make use of Proposition 4.1 in Becker-Kern et al. (2004), which provides a useful criterion for the convergence of all finite-dimensional distributions of composite processes, and Corollary 3.3 in Kasahara and Maejima (1988) which is concerned with convergence of finite-dimensional distributions of weighted partial sums of IID random variables.

We will adopt some notation from Becker-Kern et al. (2004). For $\mathbf{t} = (t_1, \ldots, t_m)$ and $c > 0$, let $\rho_c := \rho_c^{\mathbf{t}}$ denote the distribution of the random vector $\tilde{b}(c)^{-1} (N_{ct_1}, \ldots, N_{ct_m})$, and let $\rho := \rho^{\mathbf{t}}$ be the distribution of $(E_{t_1}, \ldots, E_{t_m})$. Since $\tilde{b}(c)^{-1} N_{ct} \Rightarrow E_t$ in the J_1 topology on $D([0, \infty), [0, \infty))$ (Meerschaert and Scheffler, 2004, Corollary 3.4), we have $\rho_c \Rightarrow \rho$ as $c \to \infty$.

It follows from the definition of $\{Y_n\}$ that, for every $x \geq 0$, $S(nx)$ can be rewritten as

$$S(nx) = \sum_{j=-\infty}^{\infty} \left(\sum_{k=1-j}^{[nx]-j} \widetilde{c}_k\right) Z_j, \tag{3.2}$$

where $\widetilde{c}_k = c_k$ if $k \geq 0$ and $\widetilde{c}_k = 0$ if $k < 0$. Under the assumptions of Theorem 2.3, we have $\sum_{k=-\infty}^{\infty} |\widetilde{c}_k| < \infty$, $\sum_{k=-\infty}^{\infty} \widetilde{c}_k = 0$ and

$$\sum_{k=n}^{\infty} \widetilde{c}_k \sim c_0 \sum_{k=n}^{\infty} k^{H-1/\alpha-1} \sim -\frac{c_0\alpha}{H\alpha - 1} n^{H-1/\alpha}$$

as $n \to \infty$. Thus, the conditions of Theorem 5.2 in Kasahara and Maejima (1988) are satisfied with $\psi(n) = n^{H-1/\alpha}$, $a = -K_1$ (recall that $K_1 = c_0\alpha/(H\alpha - 1)$), $b = 0$ and $A = 0$. It follows that

$$n^{-H} S(nx) \xrightarrow{f.d.} K_1 L_{\alpha,H}(x) \quad \text{as } n \to \infty. \tag{3.3}$$

For any $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}_+^m$, let $\mu_c(\mathbf{x})$ be the distribution of $\tilde{b}(c)^{-H} \left(S(\tilde{b}(c)x_1), \ldots, S(\tilde{b}(c)x_m)\right)$ and let $\nu(\mathbf{x})$ be the distribution of $K_1 \left(L_{\alpha,H}(x_1), \ldots, L_{\alpha,H}(x_m)\right)$. Then for every $c > 0$, the mapping $\mathbf{x} \mapsto \mu_c(\mathbf{x})$ is weakly measurable. Since the linear fractional stable motion $\{L_{\alpha,H}(t), t \geq 0\}$ is stochastically continuous, the mapping $\mathbf{x} \mapsto \nu(\mathbf{x})$ is weakly continuous. Moreover, it follows from (3.3) that, for every $\mathbf{x} \in \mathbb{R}_+^m$, $\mu_c(\mathbf{x}) \Rightarrow \nu(\mathbf{x})$ as $c \to \infty$.

As in Becker-Kern et al. (2004), we apply a conditioning argument and the independence between the sequences $\{Y_n\}$ and $\{J_n\}$ to derive that the distribution of $\tilde{b}(c)^{-H} \left(S(N_{ct_1}), \ldots, S(N_{ct_m})\right)$ can be written as $\int_{\mathbb{R}_+^m} \mu_c(\mathbf{x})\, \mathrm{d}\rho_c(\mathbf{x})$, which is a probability measure on $\mathbb{R}^m$. Similarly, the distribution of the random vector $K_1 \left(L_{\alpha,H}(E_{t_1}), \ldots, L_{\alpha,H}(E_{t_m})\right)$ can be written as $\int_{\mathbb{R}_+^m} \nu(\mathbf{x})\, \mathrm{d}\rho(\mathbf{x})$.

Therefore, (3.1) follows from Proposition 4.1 in Becker-Kern et al. (2004) once we verify that, for every $\mathbf{x} \in (0, \infty)^m$, $\mu_c(\mathbf{x}^{(c)}) \Rightarrow \nu(\mathbf{x})$ for every sequence $\{\mathbf{x}^{(c)}\} \subset (0, \infty)^m$ that satisfies $\mathbf{x}^{(c)} \to \mathbf{x}$ as $c \to \infty$.

The last statement is equivalent to

$$c^{-H} \left(S(cx_1^{(c)}), \ldots, S(cx_m^{(c)})\right) \Rightarrow K_1 \left(L_{\alpha,H}(x_1), \ldots, L_{\alpha,H}(x_m)\right) \tag{3.4}$$

whenever $\mathbf{x}^{(c)} \to \mathbf{x}$ as $c \to \infty$. This is stronger than (3.3), where the fixed time-instants $0 < x_1 < x_2 < \cdots < x_m$ on the left-hand side are replaced now by $x_1^{(c)}, \ldots, x_m^{(c)}$. Our proof of (3.4) is a modification of the proof of Theorem 5.2 in Kasahara and Maejima (1988).

To this end, we define the step function $r \mapsto A_c(r)$ on $\mathbb{R}$ by

$$A_c(r) = \begin{cases} c^{-1/\alpha} \sum_{j=1}^{[cr]} Z_j & \text{if } r > 0, \\ c^{-1/\alpha} \sum_{j=[cr]}^{0} Z_j & \text{if } r \le 0. \end{cases} \tag{3.5}$$

In the above, we use the convention $\sum_{j=1}^{0} Z_j = 0$. Then it is known that, as $c \to \infty$, $A_c(r) \Rightarrow A(r)$ in the J_1 topology on $D(\mathbb{R}, \mathbb{R})$. This follows, for example, from Theorem 4.5.3 in Whitt (2002). For any function g on $\mathbb{R}$, as in Kasahara and Maejima (1988, p. 88), we define

$$\int_{-\infty}^{\infty} g(r)\, \mathrm{d}A_c(r) := \frac{1}{c^{1/\alpha}} \sum_{j=-\infty}^{\infty} g\left(\frac{j}{c}\right) Z_j. \tag{3.6}$$

By using (3.2), (3.5) and (3.6) we can rewrite $c^{-H} S(cx)$ ($x > 0$ and $c > 0$) as

$$\begin{aligned} c^{-H} S(cx) &= \frac{1}{c^{1/\alpha}} \sum_{j=-\infty}^{\infty} \frac{1}{c^{H-1/\alpha}} \left(\sum_{k=1-j}^{[cx]-j} \widetilde{c}_k \right) Z_j \\ &= \int_{\mathbb{R}} g_c(x, r)\, \mathrm{d}A_c(r), \end{aligned} \tag{3.7}$$

where the integrand $g_c(x, r)$ ($x > 0$, $r \in \mathbb{R}$) is given by

$$\begin{aligned} g_c(x, r) &= \frac{1}{c^{H-1/\alpha}} \sum_{k=1-[cr]}^{[cx]-[cr]} \widetilde{c}_k \\ &= \frac{1}{c^{H-1/\alpha}} \left(\sum_{k=-[cr]+1}^{\infty} \widetilde{c}_k - \sum_{k=[cx]-[cr]+1}^{\infty} \widetilde{c}_k \right) \\ &:= \widetilde{g}_c(0, r) - \widetilde{g}_c(x, r). \end{aligned} \tag{3.8}$$

In the above, we have used the fact that $\sum_{j=-\infty}^{\infty} |\widetilde{c}_j| < \infty$ to derive the second equality.

It follows from (3.7) that (3.4) can be rewritten as

$$\left\{ \int_{\mathbb{R}} g_c(x_i^{(c)}, r)\, \mathrm{d}A_c(r) \right\}_{i=1}^{m} \Rightarrow \left\{ K_1 \int_{\mathbb{R}} g(x_i, r)\, \mathrm{d}A(r) \right\}_{i=1}^{m}, \tag{3.9}$$

where $g(x, r) = (x - r)_+^{H-1/\alpha} - (-r)_+^{H-1/\alpha}$ is the function in (2.2).

Now let us fix $\mathbf{x} = (x_1, \ldots, x_m) \in (0, \infty)^m$ and an arbitrary sequence $\{\mathbf{x}^{(c)}\} \subset (0, \infty)^m$ that satisfies $\mathbf{x}^{(c)} \to \mathbf{x}$ as $c \to \infty$. Without loss of generality, we assume $|x_i - x_i^{(c)}|$ is sufficiently small. By Corollary 3.3 in Kasahara and Maejima (1988) (with $f_n^i(\cdot)$ being taken as $g_c(x_i^{(c)}, \cdot)$), the convergence in (3.9) will follow once we verify that for every $1 \le i \le m$ the following conditions are satisfied:

(A1)′ for dr-almost every $r \in \mathbb{R}$,

$$g_c(x_i^{(c)}, r_c) \longrightarrow K_1\, g(x_i, r) \tag{3.10}$$

whenever $r_c \to r$ as $c \to \infty$.

(A2)′ for every $T > 0$, there exists a constant $\gamma > \alpha$ such that

$$\sup_{c \ge 1} \int_{|r| \le T} \left| g_c(x_i^{(c)}, r) \right|^{\gamma} \mathrm{d}\lambda_c(r) < \infty, \tag{3.11}$$

where $\lambda_c(r) = [cr]/c$, and

(A3)′ there exists a constant $\varepsilon > 0$ such that

$$\lim_{T \to \infty} \limsup_{c \to \infty} \int_{|r| > T} \left\{ \left| g_c(x_i^{(c)}, r) \right|^{\alpha - \varepsilon} + \left| g_c(x_i^{(c)}, r) \right|^{\alpha + \varepsilon} \right\} \mathrm{d}\lambda_c(r) = 0. \tag{3.12}$$

For simplicity of notation, we will from now on omit the subscript i. To verify Condition (A1)′, note that by the property of $\{c_k\}$, we have

$$\lim_{c\to\infty} \frac{1}{c^{H-1/\alpha}} \sum_{k=[cr]+1}^{\infty} \widetilde{c}_k = \begin{cases} -K_1 r^{H-1/\alpha} & \text{if } r > 0, \\ 0 & \text{if } r \le 0, \end{cases} \tag{3.13}$$

and the convergence is uniform in r on every compact set in $\mathbb{R}\setminus\{0\}$. For any $x \in \mathbb{R}_+$ and $r \in \mathbb{R}$, we may distinguish three cases $r < 0$, $0 < r < x$ and $r > x$. By applying (3.13) to (3.8) we derive that, as $c \to \infty$, $g_c(x, r) \to g(x, r)$ uniformly in (x, r) on every compact set in $\{(x, r) : x \in \mathbb{R}_+, r \in \mathbb{R}\setminus\{0, x\}\}$. This implies that $g_c(x^{(c)}, r_c) \to g(x, r)$ whenever $r \notin \{0, x\}$ and $r_c \to r$ as $c \to \infty$. Hence (A1)′ is satisfied.

To verify Condition (A2)′, we choose and fix a constant $\gamma > \alpha$ such that $\gamma(H - 1/\alpha) > -1$, say, $\alpha < \gamma < \min\{2, \alpha/(1 - H\alpha)\}$. For any $x \ge 0$, consider the integral

$$\begin{aligned} \int_{|r|\le T} |\widetilde{g}_c(x, r)|^\gamma \, d\lambda_c(r) &= \int_{|r|\le T} \left| \frac{1}{c^{H-1/\alpha}} \sum_{k=[cx]-[cr]+1}^{\infty} \widetilde{c}_k \right|^\gamma d\lambda_c(r) \\ &= \sum_{|j|\le cT} \frac{1}{c^{\gamma(H-1/\alpha)+1}} \left| \sum_{k=[cx]-j+1}^{\infty} \widetilde{c}_k \right|^\gamma . \end{aligned} \tag{3.14}$$

Let $N > 1$ be a constant such that $|c_k| \le 2c_0 k^{H-1/\alpha-1}$ for all $k \ge N$. We split the summation on the right-hand side of (3.14) according to whether $[cx] - j \le N$ or $[cx] - j > N$. Thanks to the fact that $\sum_k \widetilde{c}_k = 0$ we have

$$\sum_{|j|\le cT, [cx]-j\le N} \frac{1}{c^{\gamma(H-1/\alpha)+1}} \left| \sum_{k=[cx]-j+1}^{\infty} \widetilde{c}_k \right|^\gamma \le \frac{K_3}{c^{\gamma(H-1/\alpha)+1}} \tag{3.15}$$

for some finite constant $K_3 > 0$ which is independent of x and c. In the above we have also used the fact that there are at most $N + 1$ non-zero terms in the summation in j.

On the other hand, we have

$$\begin{aligned} \sum_{|j|\le cT, [cx]-j> N} \frac{1}{c^{\gamma(H-1/\alpha)+1}} \left| \sum_{k=[cx]-j+1}^{\infty} \widetilde{c}_k \right|^\gamma &\le K_4 \sum_{|j|\le cT, [cx]-j> N} \frac{([x] - [j/c])^{\gamma(H-1/\alpha)}}{c} \\ &\le K_5 \int_{|r|\le T} |x - r|^{\gamma(H-1/\alpha)} \, dr \end{aligned} \tag{3.16}$$

for some finite constants $K_4, K_5 > 0$ which are independent of x and c. Note that the last integral is convergent because $\gamma(H - 1/\alpha) > -1$. Combining (3.15) and (3.16) yields that for all $x \ge 0$

$$\int_{|r|\le T} |\widetilde{g}_c(x, r)|^\gamma \, d\lambda_c(r) \le \frac{K_3}{c^{\gamma(H-1/\alpha)+1}} + K_5 \int_{|r|\le T} |x - r|^{\gamma(H-1/\alpha)} \, dr. \tag{3.17}$$

Thanks to (3.8) and the c_τ-inequality $[(a + b)^\tau \le \max(1, 2^{\tau-1})(a^\tau + b^\tau)]$, we have

$$|g_c(x, r)|^\gamma \le \max\{1, 2^{\gamma-1}\} \left(|\widetilde{g}_c(0, r)|^\gamma + |\widetilde{g}_c(x, r)|^\gamma \right). \tag{3.18}$$

It follows from (3.17) and (3.18) that for every constant $R > 0$, all $x \in [0, R]$ and all $c \ge 1$

$$\begin{aligned} \int_{|r|\le T} |g_c(x, r)|^\gamma \, d\lambda_c(r) &\le \frac{4K_3}{c^{\gamma(H-1/\alpha)+1}} + 4K_5 \int_{|r|\le T} |x - r|^{\gamma(H-1/\alpha)} \, dr \\ &\le 4K_3 + K_6, \end{aligned} \tag{3.19}$$

where we have use the fact that $\gamma < 2$ and where $K_6 > 0$ is a finite constant which depends only on H, α, γ, R and T. Hence (A2)′ follows from (3.19).

The verification of (A3)′ is similar to the above, but we will not consider $\widetilde{g}_c(0, r)$ and $\widetilde{g}_c(x, r)$ in (3.8) separately. We choose and fix a constant $\varepsilon > 0$ such that $(H - 1 - 1/\alpha)(\alpha - \varepsilon) < -1$. This is possible because $0 < H < 1$. Let $x > 0$ be fixed. Then for all T and c sufficiently large (say, $T > x$), we use (3.8) and our assumption on $\{c_j\}$ to derive that

$$\begin{aligned} \int_{|r|>T} |g_c(x, r)|^{\alpha\pm\varepsilon} \, d\lambda_c(r) &\le \sum_{|j|>cT} \frac{1}{c^{(\alpha\pm\varepsilon)(H-1/\alpha)+1}} \left| \sum_{k=1-j}^{[cx]-j} \widetilde{c}_k \right|^{\alpha\pm\varepsilon} \\ &\le \sum_{j<-cT} \frac{(2c_0)^{\alpha\pm\varepsilon}}{c^{(\alpha\pm\varepsilon)(H-1/\alpha)+1}} \left| \sum_{k=1-j}^{[cx]-j} k^{H-1-1/\alpha} \right|^{\alpha\pm\varepsilon} \\ &\le K_7 \int_{-\infty}^{-T} \left[(x - r)^{H-1/\alpha} - (-r)^{H-1/\alpha} \right]^{\alpha\pm\varepsilon} dr, \end{aligned} \tag{3.20}$$

where $K_7 > 0$ is a finite constant that is independent of x and c. In the above we have used the fact that $\widetilde{c}_k = 0$ for all $k < 0$. Thanks to our choice of $\varepsilon > 0$, we can verify directly that

$$\lim_{T\to\infty} \int_{-\infty}^{-T} \left[(x-r)^{H-1/\alpha} - (-r)^{H-1/\alpha}\right]^{\alpha\pm\varepsilon} \, \mathrm{d}r = 0. \tag{3.21}$$

Therefore, condition (A3)′ follows from (3.20) and (3.21). This finishes the proof of Theorem 2.3. □

Finally we prove Theorem 2.4.

Proof of Theorem 2.4. Recall that $\tilde{b}(c)^{-1}N_{ct} \Rightarrow E_t$ in the J_1 topology (Meerschaert and Scheffler, 2004, Corollary 3.4). Theorem 4.6.1 in Whitt (2002) shows that, as $n \to \infty$, $\sigma_n^{-1}S(nt) \Rightarrow W_H(t)$ in the J_1 topology on $D([0,\infty), \mathbb{R})$. This result is originally due to Davydov (1970), see also Giraitis et al. (2003, p. 276). Since the sequence $\{J_n\}$ is independent of $\{Y_n\}$, we have $\left(\sigma^{-1}_{[\tilde{b}(c)]}S(\tilde{b}(c)t), \tilde{b}(c)^{-1}N_{ct}\right) \Rightarrow (W_H(t), E_t)$ in the product space, and then continuous mapping along with Theorem 13.3.1 in Whitt (2002) yields (2.5) in the J_1 topology. □

4. Discussion

Self-similar processes arise naturally in limit theorems of random walks and other stochastic processes, and they have been applied to model various phenomena in a wide range of scientific areas including telecommunications, turbulence, image processing and finance (see, e.g., Embrechts and Maejima (2002)). The most prominent example is fractional Brownian motion (FBM). However, many real data sets are non-Gaussian, which motivates the development of alternative models. Many authors have constructed and investigated various classes of non-Gaussian self-similar processes. Samorodnitsky and Taqqu (1994) provide a systematic account on self-similar stable processes with stationary increments. Burdzy (1993, 1994) introduced *iterated Brownian motion* (IBM) which replaces the time parameter of a two-sided Brownian motion by an independent one-dimensional Brownian motion $B = \{B_t, t \geq 0\}$. In this paper we have shown that the limit processes of CTRWs with dependent jumps form a wide class of self-similar processes which are different from the existing ones.

When $0 < \beta \leq 1/2$, the inner process E_t in (2.1) or (2.5) is also the local time at zero L_t of a stable Lévy process, and the iterated process $\{W_H(L_t), t \geq 0\}$ is called a local time fractional Brownian motion (LTFBM) in Meerschaert et al. (2008), a self-similar process with index βH and continuous sample paths. Large deviation and modulus of continuity results for LTFBM are developed in a companion paper Meerschaert et al. (2008). Strassen-type law of the iterated logarithm has been proved by Csáki et al. (1997) for local time Brownian motion (LTBM, the case $H = 1/2$). It is interesting to note that our Theorem 2.4 shows that the "randomly-stopped stationary sequence" $\{(Y_n : n \leq N_t), t \geq 0\}$ belongs to the "domain of attraction" of $\{W_H(L_t), t \geq 0\}$ for all $H \in (0, 1)$. This theorem provides a physical interpretation of the process $\{W_H(L_t), t \geq 0\}$.

One interesting property of LTBM is that its increments are uncorrelated (this follows by a simple conditioning argument), but not independent. It has long been recognized that price returns are essentially uncorrelated, but not independent (Baillie et al., 1996; Mandelbrot, 1963). Hence LTBM, the scaling limit of a CTRW with (weakly) correlated price jumps, may be useful to model financial price returns. This approach could provide an interesting alternative to the subordinated variance-Gamma model of Madan and Seneta (1990), Carr et al. (2002) or the FATGBM model of Heyde (2002).

LTBM has a close connection to fractional partial differential equations. Meerschaert and Scheffler (2004) and Baeumer and Meerschaert (2001) showed that the probability density $u(x, t)$ of LTBM solves the fractional Cauchy problem

$$\partial_t^\beta u(t,x) = \partial_x^2 u(t,x). \tag{4.1}$$

Baeumer et al. (in press) further showed that the density of the iterated Brownian motion solves the same equation (4.1). As we mentioned at the end of Section 2, the connection between the limit processes in this paper and fractional partial differential equations remains to be investigated.

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