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INHOMOGENEOUS FRACTIONAL DIFFUSION EQUATIONS

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Abstract

Fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. They are useful to model anomalous diffusion, where a plume of particles spreads in a different manner than the classical diffusion equation predicts. An initial value problem involving a space-fractional diffusion equation is an abstract Cauchy problem, whose analytic solution can be written in terms of the semigroup whose generator gives the space-fractional derivative operator. The corresponding time-fractional initial value problem is called a fractional Cauchy problem. Recently, it was shown that the solution of a fractional Cauchy problem can be expressed as an integral transform of the solution to the corresponding Cauchy problem. In this paper, we extend that results to inhomogeneous fractional diffusion equations, in which a forcing function is included to model sources and sinks. Existence and uniqueness is established by considering an equivalent (non-local) integral equation. Finally, we illustrate the practical application of these results with an example from groundwater hydrology, to show the effect of the fractional time derivative on plume evolution, and the proper specification of a forcing function in a time-fractional evolution equation.

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1. Introduction

Fractional derivatives are almost as old as their more familiar integer-order counterparts [23, 29]. Fractional derivatives have recently been applied to many problems in physics [5, 10, 11, 13, 18, 20, 21, 22, 28, 30, 35], finance [16, 19, 26, 30, 27], and hydrology [3, 7, 8, 17, 32, 33]. Fractional derivatives are used to model anomalous diffusion, where a particle plume spreads at a rate inconsistent with the classical model, and the plume may be asymmetric. When a fractional derivative replaces the second derivative in the diffusion equation, it leads to enhanced diffusion (also called super-diffusion). A fractional time derivative leads to sub-diffusion, where a cloud of particles spreads slower than the classical $t^{1/2}$ rate. In applications to groundwater flow and transport, the diffusion term in the transport equation models mechanical dispersion, the spreading of contaminants due to velocity contrast as the fluid passes through a porous medium [6]. A fractional derivative in space models the anomalous super-diffusion caused by large velocity contrasts in heterogeneous porous media [7, 14]. A fractional time derivative models particle sticking and trapping, a sub-diffusive effect [3, 9, 31].

Fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. Since fractional derivatives are non-local operators, solution methods are significantly different than for integer-order partial differential equations. An initial value problem $\partial_t g = Lg$ involving a space-fractional derivative operator L is an abstract Cauchy problem [1]. Its analytic solution can be written in terms of the semigroup whose generator L defines the space-fractional derivative operator. The corresponding time-fractional initial value problem $\partial_t^\beta g = Lg$ is called a fractional Cauchy problem [2]. Here the time-fractional derivative is the Caputo derivative [12]. The solution of this fractional Cauchy problem can be expressed as an integral transform of the solution to the corresponding Cauchy problem [21]. In this paper, we develop an analytical formula for the solution of an inhomogeneous fractional diffusion equation

$$\partial_t^\beta g = Lg + r. \tag{1}$$

Then we prove existence and uniqueness of solutions by considering an equivalent integral equation of convolution type. The connection between fractional diffusion equations, and evolution equations with a convolution flux integral term, was pointed out in [15]. In practical applications, the forcing function $r(x,t)$ is used to model sources and sinks. At the end of this paper, we include an example from groundwater hydrology to illustrate the proper modeling of the forcing term in a time-fractional partial differential equation.

A FINAL NOTE: After this paper was accepted, we learned of a new paper by Umarov and Saydamatov [34], where equation (1) is solved using different methods. That paper also considers the case $\beta > 1$ which is not considered here.

2. Fractional diffusion equations

The Riemann-Liouville fractional integral of order $\beta > 0$ is defined by

$$J_t^\beta g(t) = \int_0^t \frac{(t-u)^{\beta-1}}{\Gamma(\beta)} g(u)du \tag{2}$$

and $J_t^0 g(t) = g(t)$. The Riemann-Liouville fractional derivative of order $\beta > 0$ is

$$D_t^\beta g(t) = \frac{d^m}{dt^m} J_t^{m-\beta} g(t) = \frac{d^m}{dt^m} \int_0^t \frac{(t-u)^{m-\beta-1}}{\Gamma(m-\beta)} g(u)du; \quad m = \lceil \beta \rceil, \tag{3}$$

where $\lceil \beta \rceil$ denotes the smallest integer $m \geq \beta$. The Caputo fractional derivative is

$$\partial_t^\beta g(t) = J_t^{m-\beta} \frac{d^m}{dt^m} g(t) = \int_0^t \frac{(t-u)^{m-\beta-1}}{\Gamma(m-\beta)} g^{(m)}(u)du; \quad m = \lceil \beta \rceil. \tag{4}$$

If β is a positive integer then $D_t^\beta = \partial_t^\beta$ is the usual derivative operator. In this paper we consider time-fractional derivatives of order $0 < \beta < 1$ for functions $g \in C^\infty([0,\infty); X)$ for some Banach space X, and the integrals in the definitions (2), (3), and (4) are Bochner integrals [1]. In that case, if $g(t)$ has Laplace transform $\hat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)dt$ then the Riemann-Liouville fractional integral $J_t^\beta g$ has Laplace transform $\lambda^{-\beta}\hat{g}(\lambda)$, since (2) is the convolution of $g(t)$ with a function $k_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ whose Laplace transform (as one can easily check) is $\lambda^{-\beta}$, and since the Laplace transform of a convolution is the product of the corresponding Laplace transforms (see, e.g., [1] Proposition 1.6.4). Furthermore, $J_t^\beta g \in C^\infty([0,\infty);X)$, and it is easy to check that $J_t^\beta g(0) = 0$. Using the fact that $D_t^1 g(t)$ has Laplace transform $\lambda\hat{g}(\lambda) - g(0)$, it follows that the Riemann-Liouville derivative $D_t^\beta g$ has Laplace transform $\lambda^\beta \hat{g}(\lambda)$ and the Caputo derivative $\partial_t^\beta g$ has Laplace transform $\lambda^\beta \hat{g}(\lambda) - \lambda^{\beta-1} g(0)$. Additional properties of time-fractional derivatives can be found in [4].

For fractional derivatives in space, some additional definitions are useful. The positive Liouville fractional derivative of order $\alpha > 0$ is

$$\mathbb{D}_x^\alpha h(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x-y)^{n-\alpha-1} h(y)dy; \quad n = \lceil \alpha \rceil. \tag{5}$$

The negative Liouville fractional derivative is

$$\mathbb{D}_{-x}^\alpha h(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x}^{\infty} (y-x)^{n-\alpha-1} h(y)dy; \quad n = \lceil \alpha \rceil. \tag{6}$$

If $h(x)$ has the Fourier transform $\hat{h}(k) = \int e^{-ikx} h(x)dx$ then $\mathbb{D}_x^\alpha h$ has Fourier transform $(ik)^\alpha \hat{f}(k)$ and $\mathbb{D}_{-x}^\alpha h$ has Fourier transform $(-ik)^\alpha \hat{h}(k)$. Zaslavsky [35] introduced the space-time fractional diffusion equation $\partial_t^\beta g = Lg$ with $L = \mathbb{D}_x^\alpha$ as a model for Hamiltonian chaos. Benson *et al.* [7, 8, 9] use $L = -vD_x + p\mathbb{D}_x^\alpha + (1-p)\mathbb{D}_{-x}^\alpha$ to model advection and dispersion of contaminants in underground water.

More generally, we assume in this paper that the spatial derivative operator L is the generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ on some Banach space X. For example, L could be the generator of a convolution semigroup on $\mathbb{R}^d$ defined by $T(t)h(x) = \int h(x-y)\mu_t(dy) = \int p(x-y,t)h(y)dy$ where $p(x,t)$ is the density function of some infinitely divisible probability measure μ^t on $\mathbb{R}^d$ [2]. As another example, L could be a uniformly elliptic operator on some bounded domain in $\mathbb{R}^d$. In any case the domain $D(L)$ of the generator

$$Lh = \lim_{t\to 0^+} \frac{T(t)h - h}{t} \tag{7}$$

is dense in X (see, e.g., Theorem 3.1.2 in [1]) and L is closed (see, e.g., Proposition 3.1.9 in [1]). Furthermore, the function $u : t \mapsto T(t)h$ solves the abstract Cauchy problem

$$\partial_t u(t) = Lu(t); \quad u(0) = h \tag{8}$$

for any $h \in D(L)$, see for example [1, 24]. In the case of L being the generator of a convolution semigroup, the function $p(x,t)$ is called the Green's function solution to (8) since formally it corresponds to the Dirac delta function initial condition $h(x) = \delta(x)$. For the classical diffusion equation with drift, take $L = -v\partial_x + D\partial_x^2$ in (8). The simplest space-fractional diffusion equation with drift is (8) with $L = -v\partial_x + D\partial_x^\alpha$ for $1 < \alpha < 2$. It models anomalous super-diffusion, where a particle cloud spreads out from its center of mass faster than the classical diffusion model predicts.

For $0 < \beta < 1$ and $h \in X$ the fractional Cauchy problem

$$D_t^\beta g(x,t) = Lg(x,t) + h(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{9}$$

has solution $g(x,t) = \int q(x-y,t)h(y)dy$, where

$$q(x,t) = \frac{t}{\beta}\int_0^\infty p(x,s)g_\beta(ts^{-1/\beta})s^{-1/\beta-1}ds, \tag{10}$$

and p is the Green's function solution to (8), see [2, 21]. Here g_β is a probability density function called the stable subordinator, whose Laplace transform is $\exp(-s^\beta)$, see for example [29]. We also call q the Green's function solution to (9) since formally it corresponds to the case $a(x) = \delta(x)$. Theorem 3.1 in Bajlekova [4] leads to a different formula:

$$q(x,t) = \int_0^\infty p(x,u)t^{-\beta}\Phi_\beta(ut^{-\beta})du; \quad \Phi_\beta(z) = \sum_0^\infty \frac{(-z)^n}{n!\Gamma(-n\beta+1-\beta)}\,, \tag{11}$$

but it seems difficult to equate these two forms (10) and (11) directly, and we note that (10) can be computed explicitly. In the special case $L = -v\partial_x + D\partial_x^2$, equation (9) is called the time-fractional diffusion equation. It models anomalous sub-diffusion, in which a particle cloud spreads slower than the classical diffusion model predicts. The simplest space-time fractional diffusion equation, (9) with $L = D\partial_x^\alpha$, was first used by Zaslavsky [35] as a model for Hamiltonian chaos. Here the sub-diffusive effect of the fractional time derivative mitigates the super-diffusive nature of the fractional space derivative, resulting in a model where the particle cloud spreads like $t^{\beta/\alpha}$, see [21].

3. Fractional diffusion equation with forcing term

In order to illuminate the nature of the last term in (9), and to facilitate the introduction of a forcing term, we note that the fractional Cauchy problem (9) can be written in several equivalent forms. The proof of the next result extends the main results in [2] and [21]. The arguments are similar. Recall that a strongly continuous semigroup $\{T(t) : t \geq 0\}$ on some

Banach space X with generator L has a resolvent $R(\lambda, L) = (\lambda - L)^{-1}$, a linear operator on X whose domain is all of X (see, e.g., [1] p. 41) and such that $\int_0^\infty e^{-\lambda t} T(t) h dt = (\lambda - L)^{-1} h$ for $\Re(\lambda) > \omega(T)$ and any $h \in X$ (see, e.g., Theorem 3.1.7 in [1]). The notation $g = (\lambda - L)^{-1} h$ means that $h = (\lambda - L) g$, and $\omega(T) = \inf\{\omega > 0 : \sup_{t \geq 0} \|e^{-\omega t} T(t)\| < \infty\}$ is the exponential growth bound for the semigroup (see, e.g., [1] Theorem 1.4.3).

Proposition 1. *Assume $0 < \beta < 1$. Let L be the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space X and $g \in C([0, \infty); X)$ be Laplace transformable. Then for all $h \in X$ the following are equivalent:*

(i) For all $t > 0$, the Riemann-Liouville derivative of g exists, $g(t) \in D(L)$, the Laplace transform of $D_t^\beta g(t)$ exists, and

$$D_t^\beta g(t) = L g(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} h. \tag{12}$$

(ii) For all $t > 0$, the Caputo derivative of g exists, $g(t) \in D(L)$, the Laplace transform of $\partial_t^\beta g(t)$ exists, and

$$\partial_t^\beta g(t) = L g(t); \quad g(0) = h. \tag{13}$$

(iii) For all $t > 0$ the function g is differentiable, $g(t) \in D(L)$, the Laplace transform of $\partial_t g(t)$ exists, and

$$\partial_t g(t) = D_t^{1-\beta} L g(t); \quad g(0) = h. \tag{14}$$

(iv) For all $t > 0$, $J_t^\beta g(t) \in D(L)$

$$g(t) = L J_t^\beta g(t) + h. \tag{15}$$

(v) The function $g(t)$ is analytic on $0 < t < \infty$, satisfies $\|g(t)\| \leq M e^{\omega t}$ on $0 < t < \infty$ for some $M, \omega \geq 0$ and

$$g(t) = \int_0^\infty \frac{t}{\beta s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) T(s) h \, ds, \tag{16}$$

where g_β is the stable subordinator; i.e. $\int_0^\infty e^{-\lambda t} g_\beta(t)\, dt = \exp(-\lambda^\beta)$.

(vi) The function g has Laplace transform defined in terms of the resolvent

$$\hat{g}(\lambda) = \lambda^{\beta-1}\left(\lambda^{\beta} - L\right)^{-1} h \tag{17}$$

for all λ with $\Re(\lambda) > \omega(T)$, the exponential growth bound of this semigroup.

P r o o f. We first show that (i) through (iv) each imply (vi). Starting with (i), and recalling that the Riemann-Liouville fractional derivative $D_t^{\beta} g(t)$ of a function $g(t)$ with Laplace transform $\hat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)dt$ equals $\lambda^{\beta}\hat{g}(\lambda)$, take Laplace transforms in (12) to obtain

$$\lambda^{\beta}\hat{g}(\lambda) = \widehat{Lg}(\lambda) + \lambda^{\beta-1}h$$

using the fact, which one can easily check, that $\lambda^{\beta-1}$ is the Laplace transform of $t^{-\beta}/\Gamma(1-\beta)$ for any $\beta < 1$. Fix $B > 0$ and let $x = \int_0^B e^{-\lambda t} g(t)dt$ the Bochner integral [1] and let $y = \int_0^B e^{-\lambda t} Lg(t)dt$. Take x_n to be an approximation to the integral x obtained by replacing $g(t)$ by a step function and let $y_n = Lx_n$. Using the fact that L is closed, it follows that $y = Lx$ exists and $y_n = Lx_n \to y$. This shows that $\int_0^B e^{-\lambda t} Lg(t)dt = L\int_0^B e^{-\lambda t} g(t)dt$. Then another similar argument shows that $\widehat{Lg}(\lambda) = L\hat{g}(\lambda)$. Note that for a function f taking values in X to be Bochner integrable, it is necessary and sufficient that f is measurable and $\|f\|$ is integrable (see, e.g., Theorem 1.1.4 in [1]). Hence the Laplace transform of g exists as long as $\int_0^\infty e^{-\lambda t}\|g(t)\|dt < \infty$. Now it follows that

$$(\lambda^{\beta} - L)\hat{g}(\lambda) = \lambda^{\beta-1}h$$

and hence we have in terms of the resolvent $R(\lambda, L) = (\lambda - L)^{-1}$ that (17) holds for all λ with $\Re(\lambda) > \omega(T)$, the exponential growth bound of this semigroup.

If (ii) holds, then taking Laplace transforms in (13), and using the fact that the Caputo fractional derivative $\partial_t^{\beta} g(t)$ has Laplace transform $\lambda^{\beta}\hat{g}(\lambda) - \lambda^{\beta-1}g(0)$, we obtain $\lambda^{\beta}\hat{g}(\lambda) - \lambda^{\beta-1}h = \widehat{Lg}(\lambda)$. Then using $\widehat{Lg}(\lambda) = L\hat{g}(\lambda)$ again we arrive at (17) as before.

If (iv) holds, then taking Laplace transforms in (15) leads to $\hat{g}(\lambda) = L\lambda^{-\beta}\hat{g}(\lambda) + \lambda^{-1}h$, using the fact that L is closed to obtain that $\widehat{LJ_t^{\beta}g}(\lambda) = L\widehat{J_t^{\beta}g}(\lambda)$, and recalling that $\lambda^{-\beta}\hat{g}(\lambda)$ is the Laplace transform of $J_t^{\beta}g$, and that $\lambda^{-1}C$ is the Laplace transform of a constant C. Then (17) follows.

If (iii) holds, then taking the Laplace transform on both sides of (14) yields

$$\lambda\hat{g}(\lambda) - h = \int_0^\infty e^{-\lambda t} D_t^{1-\beta} Lg(t)dt = \int_0^\infty e^{-\lambda t} D_t^1 J_t^\beta Lg(t)dt.$$

Using the fact that L is closed, and the definition (2) of the Riemann-Liouville fractional integral, it is not hard to show that $J_t^\beta Lg(t) = LJ_t^\beta g(t)$. A similar argument was used previously to show that $\widehat{Lg}(\lambda) = L\hat{g}(\lambda)$. Approximating $D_t^1 f(t)$ by a difference quotient and using the fact that L is closed, it follows that $D_t^1 LJ_t^\beta g(t) = LD_t^1 J_t^\beta g(t)$. Then the same argument as before shows that

$$\lambda\hat{g}(\lambda) - h = \int_0^\infty e^{-\lambda t} D_t^1 J_t^\beta Lg(t)dt = L\int_0^\infty e^{-\lambda t} D_t^1 J_t^\beta g(t)dt.$$

Integrate by parts to get

$$\lambda\hat{g}(\lambda) - h = \lambda L\int_0^\infty e^{-\lambda t} J_t^\beta g(t)\,dt - J_t^\beta Lg(0) = \lambda^{1-\beta} L\hat{g}(\lambda) - J_t^\beta Lg(0),$$

then rearrange and apply the resolvent on both sides to obtain

$$\hat{g}(\lambda) = \lambda^{\beta-1}\left(\lambda^\beta - L\right)^{-1}(h - J_t^\beta Lg(0))$$

for all λ with $\Re(\lambda) > \omega(T)$. We now show that $J_t^\beta Lg(0) = 0$. For Laplace transformable continuous functions g we have that $\lim_{\lambda\to\infty}\lambda\hat{g}(\lambda) = g(0)$ (see, e.g., [1] Proposition 4.1.3). Also, Proposition 3.1.9 in [1] shows that $\lambda(\lambda - L)^{-1}g \to g$ as $\lambda \to \infty$ for all $g \in X$. Hence on one hand $\lambda\hat{g}(\lambda) = \lambda^\beta\left(\lambda^\beta - L\right)^{-1}(h - J_t^\beta Lg(0)) \to h - J_t^\beta Lg(0)$ as $\lambda \to \infty$, and on the other $\lambda\hat{g}(\lambda) \to g(0) = h$. Hence $J_t^\beta Lg(0) = 0$ and again (17) follows. Hence we have shown that any of (i) through (iv) implies (vi).

Next we show that (v) implies (vi). Equation (16) along with the fact that $g_\beta(t)$ has Laplace transform $\hat{g}_\beta(\lambda) = e^{-\lambda^\beta}$ implies that

$$\begin{aligned}\int_0^\infty e^{-\lambda t} g(t)\,dt &= \int_0^\infty e^{-\lambda t}\int_0^\infty \frac{t}{\beta s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) T(s)h\,ds\,dt\\ &= \int_0^\infty\int_0^\infty e^{-\lambda t}\frac{t}{\beta s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) dt\,T(s)h\,ds \qquad (18)\\ &= \int_0^\infty\int_0^\infty e^{-\lambda u s^{1/\beta}}\frac{u s^{1/\beta}}{\beta s} g_\beta(u)\,du\,T(s)h\,ds\end{aligned}$$

$$= \int_0^\infty -\frac{d}{d\lambda}\int_0^\infty e^{-\lambda u s^{1/\beta}} \frac{1}{\beta s} g_\beta(u)\, du\, T(s)h\, ds = \int_0^\infty -\frac{d}{d\lambda}\left(e^{-\lambda^\beta s}\frac{1}{\beta s}\right) T(s)h\, ds$$

$$= \int_0^\infty \lambda^{\beta-1} e^{-\lambda^\beta s}\, T(s)h\, ds = \lambda^{\beta-1}\left(\lambda^\beta - L\right)^{-1} h,$$

using Fubini and the dominated convergence theorem. Hence (vi) holds.

Now we show that (vi) implies (v). Since L is the generator of a semigroup, for some $\omega > \omega(T)$ the exponential bound of this semigroup, there exists $M \geq 0$ such that $\lambda \mapsto (\lambda - L)^{-1}h$ is analytic in the half plane $\{\lambda : \Re(\lambda) > \omega\}$ (see, e.g., [1] p. 122) and $\|(\lambda-\omega)(\lambda-L)^{-1}\| \leq M$ for all $\lambda > \omega$. (This is a special case of the Hille-Yosida Theorem, see, e.g., [1] Theorem 3.3.4). Now if $\hat{g}$ satisfies (17), then $\hat{g}$ has an analytic extension to the region $R = \{\lambda \in \mathbb{C} \setminus (-\infty, 0] : \Re(\lambda^\beta) > \omega\}$ in the complex plane with $\|\lambda\hat{g}(\lambda)\| \leq M$ for all λ in that region. Since $0 < \beta < 1$, the argument of the curve $(\omega + i\mathbb{R})^{1/\beta}$ approaches $\pm\pi/2\beta$. Hence for any $0 < \alpha < \min\{\pi, \frac{\pi}{2\beta}\}$ there exists $\omega' > 0$ such that the region R contains the sectorial region $\omega' + \{re^{-\theta} : r > 0, |\theta| < \alpha\}$. Now an application of Theorem 2.6.1 in [1] shows that $g(t)$ is analytic for $t > 0$ and also satisfies $\|e^{-\omega t}g(t)\| \leq M$ for all $t > 0$. By the uniqueness of the Laplace transform, it follows from (18) that (vi) implies (v). Hence (v) and (vi) are equivalent.

Finally we wish to show that (vi) implies each of (i) through (iv). If (vi) holds, then (v) also holds, and from (17) it follows that $(\lambda^\beta - L)\hat{g}(\lambda) = \lambda^{\beta-1}h$. Then $\hat{g}(\lambda) = \lambda^{-\beta}L\hat{g}(\lambda) + \lambda^{-1}h$. Since $\hat{g}(\lambda)$ is analytic in a sectorial region, so is $\lambda^{-\beta}\hat{g}(\lambda)$, and hence this product represents the Laplace transform of some analytic function. Since $J_t^\beta g(t)$ is the convolution of $g(t)$ with the function $k_\beta(t) = t^{\beta-1}/\Gamma(\beta)$, whose Laplace transform is $\lambda^{-\beta}$, Proposition 1.6.4 in [1] shows that $\lambda^{-\beta}\hat{g}(\lambda)$ is the Laplace transform of that convolution. Since L is closed, the convolution $k_\beta * (Lg) = L(k_\beta * g)$, and the Laplace transform of $J_t^\beta Lg(t)$ is $\lambda^{-\beta}L\hat{g}(\lambda)$. Recalling that $\lambda^{-1}C$ is the Laplace transform of a constant C and inverting the Laplace transform yields (17), and so (v) implies (iv). Now (i) through (iii) follow in the same manner, since the Laplace transforms of the corresponding equations are all equivalent to (17). ■

Define a family of bounded, strongly continuous (even strongly analytic, see Theorem 3.1 in [2]) linear operators $\{S(t)\}_{t\geq 0}$ on X via

$$S(t)h := \int_0^\infty \frac{t}{\beta s^{1+1/\beta}} g_\beta\left(\frac{t}{s^{1/\beta}}\right) T(s)h\, ds. \tag{19}$$

In view of Proposition 1, item (v), the function $g(t) = S(t)h$ defines a solution to the fractional Cauchy problem given by any of the equivalent forms in (i) through (iv) for any initial condition $h \in X$, and this solution depends continuously on the initial condition h. The purpose of this paper is to solve an inhomogeneous fractional diffusion equation which, in view of Proposition item (iii), can be written in the form

$$\partial_t g(t) = D_t^{1-\beta} Lg(t) + f(t); \quad g(0) = h. \tag{20}$$

In practical applications, the Banach space X is specified as a suitable function space of real-valued functions on some domain in $\mathbb{R}^d$, and the forcing function $f(t) \in X$ can be written with some abuse of notation as $f(x,t)$, denoting a source/sink term at location x at time t, which has the same units as $\partial_t g(x,t)$. We can also rewrite this equation in the form

$$\partial_t^\beta g(t) = Lg(t) + r(t); \quad g(0) = h, \tag{21}$$

or equivalently

$$D_t^\beta g(t) = Lg(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} h + r(t), \tag{22}$$

or in Volterra integral form as

$$g(t) = LJ_t^\beta g(t) + h + \int_0^t f(s)\, ds. \tag{23}$$

The equations are equivalent if g is differentiable and if we take $f = \partial_t^{1-\beta} r$ with $r(0) = 0$ (and then $f = D_t^{1-\beta} r$ as well), but note that the "fractional forcing function" $r(t)$ here does not have the interpretation of a source or sink, or even the same units.

Remark. Formally, one can also write the homogeneous version of (20) by taking $g(x,0) = 0$ and $f(x,t) = h(x)\delta(t)$. Then the "fractional forcing function"

$$r(x,t) = J_t^{1-\beta} f(x,t) = h(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}$$

which motivates the equivalence between the Caputo form (13) and Riemann-Liouville form (9) of the fractional diffusion equation.

We now show that the fractional diffusion equation (20) can be solved by an extension of the usual methods, involving superposition and Duhamel's principle, for integer-order partial differential equations with a forcing term. A similar idea was used in Bajlekova [4], p.55, to study existence and uniqueness of solutions to the equation $D_t^\beta g = Ag + f; J_t^{1-\beta} g(x,0) = h(x)$ or in the context of Volterra integral equations by J. Prüss [25].

THEOREM 1. *Assume that* $0 < \beta < 1$, $f(t) \in L^1([0,T);X)$ *for some* $T > 0$, *and* $h \in X$ *where* L *is the generator of some strongly continuous semigroup on a Banach space* X. *Then the unique solution of (23) is given by*

$$g(t) = S(t)h + \int_0^t S(t-s)f(s)ds, \tag{24}$$

where $S(t)$ *is the solution family to the fractional Cauchy problem (9) given by* (19).

P r o o f. If g_1 and g_2 are solutions to (23), then $g_1 - g_2$ solves (15) with zero initial condition ($h = 0$). Hence (16) implies that $g_1 - g_2 = 0$.

The function g defined in (24) is well defined and continuous by Prop. 1.3.4 in [1] along with Theorem 3.1 in [2]. Since L is closed, Proposition 1 implies that

$$\begin{aligned} g(t) =& S(t)h + \int_0^t S(t-s)f(s)\,ds \\ =& LJ_t^{1-\beta}S(t)h + h + \int_0^t LJ_{t-s}^{1-\beta}S(t-s)f(s) + f(s)\,ds \\ =& LJ_t^{1-\beta}S(t)h + h + L\int_0^t\int_0^{t-s}\frac{(t-s-u)^{-\beta}}{\Gamma(1-\beta)}S(u)f(s)duds + \int_0^t f(s)ds \\ =& LJ_t^{1-\beta}S(t)h + h + L\int_0^t\int_s^t\frac{(t-v)^{-\beta}}{\Gamma(1-\beta)}S(v-s)f(s)\,dv\,ds + \int_0^t f(s)\,ds \\ =& LJ_t^{1-\beta}S(t)h + h + L\int_0^t\frac{(t-v)^{-\beta}}{\Gamma(1-\beta)}\int_0^v S(v-s)f(s)\,ds\,dv + \int_0^t f(s)\,ds \\ =& LJ_t^{1-\beta}S(t)h + h + LJ_t^{1-\beta}\int_0^t S(t-s)f(s)\,ds + \int_0^t f(s)\,ds \\ =& LJ_t^{1-\beta}g(t) + h + \int_0^t f(s)\,ds, \end{aligned} \tag{25}$$

which agrees with (23). ∎

In case that L generates a convolution semigroup, we have the following representation.

COROLLARY 1. *Assume that* $0 < \beta < 1$, *the forcing function* $f(x,t) \in L^1(\mathbb{R}^d \times (0,T))$ *for some* $T > 0$, *the initial condition* $h \in L^1(\mathbb{R}^d)$ *and assume that* L *is the generator of some strongly continuous convolution semigroup* *on $\mathbb{R}^d$ associated with a family of infinitely divisible probability distributions μ^t with density $p(x,t)$. Then the solution of (23) is given by*

$$g(x,t) = \int q(x-y,t)h(y)dy + \int_0^t \int q(x-y,t-u)f(y,u)dydu, \tag{26}$$

where q is the Green's function solution to the fractional Cauchy problem (9) given by (10).

4. Application

We consider a laboratory experiment where a stable left-to-right flow is established through a long thin homogeneous sandbox, and then a tracer is injected along with the flow at a constant concentration c starting at time $t=0$ and ending at time $t=t_1$, after which the flow continues without any additional tracer. The tracer enters at the left boundary $x=0$ and concentration $g(x,t)$ is measured inside the sandbox at location $x>0$ and time $t>0$. We use the standard advection-dispersion operator and a fractional time derivative of order $0<\beta<1$ in equation (21) to model sticking and/or trapping of tracer particles. Hence our equation of transport is

$$\frac{\partial^\beta g(x,t)}{\partial^\beta t} = -v\frac{\partial g(x,t)}{\partial x} + D\frac{\partial^2 g(x,t)}{\partial^2 x} + r(x,t) \tag{27}$$

with initial condition $g(x,0)=0$ at all x. Note that the parameters v and D have a different interpretation in this equation than the classical velocity and dispersion, since the time derivative is fractional. (Even the units of v and D are different.) The proper forcing function is most easily described by adopting the equivalent form (20)

$$\frac{\partial g(x,t)}{\partial t} = D_t^{1-\beta}\left(-v\frac{\partial g(x,t)}{\partial x} + D\frac{\partial^2 g(x,t)}{\partial^2 x}\right) + f(x,t) \tag{28}$$

where

$$f(x,t) = c\delta(x)\left[H(t) - H(t-t_1)\right] \tag{29}$$

and $H(t)$ is the Heaviside function (which equals zero for $t<0$ and 1 otherwise). In eq. (21) the corresponding "fractional forcing function" is

$$r(x,t) = J_t^{1-\beta} f(x,t) = \begin{cases} c\delta(x)\dfrac{t^{1-\beta}}{\Gamma(2-\beta)} & \text{for } t \le t_1, \\ c\delta(x)\dfrac{t^{1-\beta}-(t-t_1)^{1-\beta}}{\Gamma(2-\beta)} & \text{for } t > t_1. \end{cases} \tag{30}$$

In general, the appropriate forcing term $r(x,t)$ in (27) is simply the fractional integral of the usual source/sink term $f(x,t)$ in (28).

The solution of the model equation (27) was obtained by numerically evaluating the integrals in equation (26) from Corollary . We assume that tracer was injected for $t_1 = 1$ hour at concentration $c = 1$. We use a fractional derivative of order $\beta = 0.9$ to model light retardation, and we assume parameter values of $v = 1$ and $D = 0.1$. In an actual laboratory experiment, the values β, v, D can be determined experimentally. The first integral in (26) disappears since $h(x) \equiv 0$. The well-known Green's function solution of (8) for the operator $L = -v\partial_x + D\partial_x^2$ is

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\Big(-\frac{(x-vt)^2}{4Dt}\Big), \tag{31}$$

and then (10) gives

$$q(x,t) = \frac{t}{\beta}\int_0^\infty g_\beta(u^{-1/\beta}t)u^{-1/\beta-1}\frac{1}{\sqrt{4\pi Du}}\exp\Big(-\frac{(x-vu)^2}{4Du}\Big)du. \tag{32}$$

Inserting (32) along with (29) into equation (26), we evaluate the second integral in (26) using existing numerical routines for computing the stable density $g_\beta(t)$ (see, e.g., [29]). The resulting solution curves $g(x,t)$ are plotted in Figure 1 for several values of $t > 0$ to show the plume evolution. We also include the corresponding solution at one time $t = 10$ for the integer-order time derivative case $\beta = 1$, to show the effect of the fractional time derivative that models particle sticking and trapping. Note that the $\beta = 0.9$ curves are skewed to the left while the $\beta = 1$ curve is symmetrical.

5. Conclusion

A closed analytical solution is obtained for the inhomogeneous fractional diffusion equation. Existence and uniqueness is proven by considering an equivalent integral equation. The proper specification of a forcing function for a time-fractional evolution equation is explained. An example is included to demonstrate the practical application of these results to a problem in groundwater hydrology.

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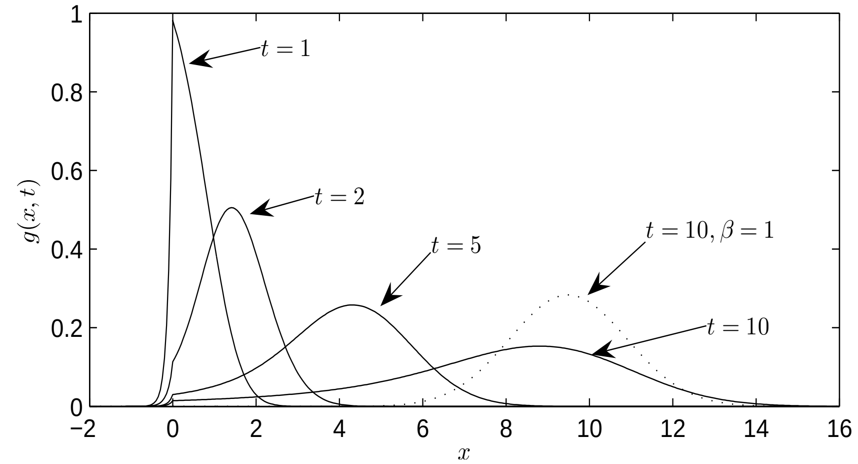

Figure 1: Numerical solution of the time-fractional advection-dispersion equation (27) with $v = 1$, $D = 0.1$ and forcing function (30) with $c = 1$ and $t_1 = 1$.

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