Normal and Anomalous Diffusion of Gravel Tracer Particles in Rivers

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Abstract. One way to study the mechanism of gravel bedload transport is to seed the bed with marked gravel tracer particles within a chosen patch, and to follow the pattern of migration and dispersal of particles from this patch. In this study, we invoke the probabilistic Exner equation for sediment conservation of bed gravel, formulated in terms of the difference between the rate of entrainment of gravel into motion and the rate of deposition from motion. Assuming an active layer formulation, stochasticity in particle motion is introduced by considering the step length (distance traveled by a particle once entrained until it is deposited) as a random variable. For step lengths with a relatively thin (e.g., exponential) tail, the above formulation leads to the standard advection-diffusion equation for tracer dispersal. However, the complexity of rivers, characterized by a broad distribution of particle sizes, and extreme flood events transporting bed material, can give rise to a heavy tailed distribution of step lengths. This consideration leads to an anomalous advection-diffusion equation involving fractional derivatives. By identifying the probabilistic Exner equation as a forward Kolmogorov equation for the location of a randomly selected tracer particle, a stochastic model describing the temporal evolution of the relative concentrations is developed. The fractional advection-diffusion equation is revealed as its long-time asymptotic solution. Sample numerical results illustrate the large differences that can arise in predicted tracer concentrations under the normal and anomalous diffusion models. They highlight the need for intensive data collection efforts to aid the selection of the appropriate model in real rivers.

1. Introduction

The stones that make up the bed of gravel-bed rivers are transported as bedload during floods. During periods of overall transport, each particle undergoes alternating periods of movement and rest. Movement consists of rolling, sliding or saltation, which continues until a single step length of motion is completed. The particle is at rest when it is deposited, either on the bed or deeper within the deposit. One way to study the mechanism of bedload transport in gravel bed rivers is to seed the bed with marked tracer particles within some small area of the bed (patch), and to follow the pattern of migration and dispersal of particles from that patch [e.g., *Hassan and Church*, 1991; *Church and Hassan*, 1992; *Wilcock*, 1997; *Ferguson and Wathen*, 1998; *Ferguson and Hoey*, 2002; *Pyrce and Ashmore*, 2003]. Tracers provide a way of characterizing not only mean parameters pertaining to transport, but also the stochasticity of particle motion itself.

This stochasticity was first elaborated by *Einstein* [1937]. Einstein based his analysis on experimental observations of painted tracer particles. He noted that: "The results demonstrated clearly that even under the same experimental conditions stones having essentially identical characteristics were transported to widely varying distances $\cdots$ Consequently, it seemed reasonable to approach the subject of particle movement as a probability problem." Einstein considered a particle that moves in discrete steps punctuated by periods of inactivity. He quantified the problem in terms of the statistics of step length and resting period (waiting time). *Einstein* [1942] went on to explain how these quantities enter into the delineation of macroscopic relations of bedload transport (i.e., relations that represent averages over the stochasticity of sediment motion). More specifically, *Einstein*

[1942] showed that the bedload transport rate is proportional to the step length and inversely proportional to the resting period. Following the seminal work of *Einstein* [1942], many stochastic theories for sediment transport have been proposed which account for the aforementioned stochasticity (see for example, *Einstein and El-Sammi* [1949]; *Paintal* [1971]; *Nelson et al.* [1995]; *Cheng and Chiew* [1998]; *Lopez and Garcia* [2001]; *Kleinhans and van Rijn* [2002]; *Cheng* [2004]; *Cheng et al.* [2004]; *Charru et al.* [2004], *Ancey et al.* [2006, 2008]; *Ancey* [2009]; *Furbish et al.* [2009],*Ganti et al.* [2009] and references therein).

Two macroscopic quantities that can be captured by means of statistical analyses of tracer motion are the overall tendencies of ensembles of tracers to be advected downstream, and to disperse, or diffuse. (Various authors use the terms "dispersion" or "diffusion" of tracers to describe the same process: here we rather arbitrarily use the term "diffusion".) Both advection and diffusion are governed by a wide range of factors.

Bedload particles may roll, slide or saltate over the bed. In the case of grains of uniform size, mean saltation length may be on the order of ten diameters [e.g., *Niño and Garcia*, 1998]; whereas mean step length may be on the order of 100 grain diameters [*Einstein*, 1950; *Tsujimoto*, 1978; *Wong et al.*, 2007]. *Einstein* [1950] suggested that mean step length can be approximated as a constant multiple of grain diameters, whereas *Wong et al.* [2007] indicate a weak variation with Shields number, which is a proxy for flow strength. Step length is known, however, to vary stochastically [e.g., *Tsujimoto*, 1978]. As illustrated below, this stochasticity is one source of diffusion.

When a particle comes to rest, it may deposit so as to be exposed at the bed surface, or it may become buried at depth [e.g., *Hassan and Church*, 1994]. From a statistical point of view, deeper burial in general implies a longer resting time before exhumation and reentrainment. This effect can influence both diffusion and advection [*Ferguson and Hoey*, 2002]. Most natural gravels consist of a mixture of grain sizes, each of which undergoes steps and resting periods according to size-specific probabilities. For example, *Tsujimoto* [1978] has shown that larger grains in a mixture have longer step lengths, but also longer resting times. As these different sizes move downstream, their motion is affected by the presence of bedforms such as dunes [e.g., *Blom et al.*, 2006], bars and bends associated with channel meandering/braiding [e.g., *Pyrce and Ashmore*, 2003], and large-scale variations in channel width. In addition, the bed may be undergoing aggradation, which enhances the capture of bedload particles, or degradation, which causes the exhumation of grains that have undergone long-term storage [e.g., *Ferguson and Hoey*, 2002]. Grains can also enter floodplain storage for long periods of time, and then be exhumed as the channel migrates into the relevant deposit [e.g., *Bradley*, 1970; *Lauer and Parker*, 2008a, b]. Again, all these effects can influence the advection/diffusion of tracer particles.

The macroscopic transport of grains undergoing steps and rest periods governed by statistical laws can be most simply characterized in terms of the classical advection-diffusion model, according to which the particles spread downstream with a constant diffusivity. When step lengths and rest periods are governed by a multiplicity of mechanisms over a very wide range of spatial and temporal scales, however, the advection/diffusion of tracer particles may no longer be characterizable in terms of the classical model. It is widely known in the groundwater literature that a multiplicity of scales over which transport takes place can lead to "anomalous diffusion", for which the advection/diffusion equation can be characterized by fractional derivatives [e.g., *Benson*, 1998; *Berkowitz et al.*, 2002; *Cushman and Ginn*, 2000; *Schumer et al.*, 2003].

Nikora et al. [2002] have studied the diffusion of bedload particles using the measured motion of individual particles in a canal as the basis for ensemble averaging. They extracted from their data various moments characterizing particle location as a function of time. They delineated three ranges of temporal and spatial scales, each with different regimes of diffusion: ballistic diffusion (at the scale of saltation length), normal/anomalous diffusion (at a scale of step length) and sub-diffusion (at global scale). Their study thus represents a pioneering effort in the identification of anomalous diffusion of bedload particles.

We develop here a theoretical model for the study of anomalous diffusion of tracer particles moving as bedload. The present model is not intended to be comprehensive, in that it covers only a restricted set of phenomena that might lead to anomalous diffusion. It is our desire, however, that this first model should serve as an example illustrating the pathway to more general models of anomalous diffusion.

The paper is structured as follows. In Section 2, a straightforward formulation of the Exner equation for sediment conservation is presented which incorporates the probability density function (pdf) for step lengths, i.e., the distances traveled by particles once they are entrained to when they are deposited again on the river bed. In Section 3 we show that the assumption of step lengths having a distribution with thin tails (e.g., exponential, normal, log-normal distributions) leads to a classical advection-diffusion equation for tracer dispersal. However, in real rivers the complexity resulting from broad distributions of particle sizes and flood events can lead to a heavy tail in the pdf of step lengths (arising, for example, from the combination of an exponential distribution for step length conditional on a particle size and a gamma distribution of particle sizes). In Section 4, we show that this consideration leads to an anomalous advection-diffusion formulation which includes fractional derivatives. That model was introduced earlier in the context of other problems, such as groundwater dispersion. Section 5 shows how a heavy-tail step length distribution can arise from a thin-tailed (exponential) pdf of step length for particles of a given size, together with a thin-tailed grain size distribution. In Section 6, we build a stochastic model to describe the time evolution of the relative concentration of the tracers in the active layer, and show that the approximate solutions obtained in Sections 3 and 4 are long-time asymptotic solutions of the derived model. Finally, in section 7, numerical results are presented showing the difference between normal and anomalous advection-diffusion of gravel tracer particles.

2. Formulation

The starting point for our analysis is the entrainment-based one-dimensional Exner equation for sediment balance [*Tsujimoto*, 1978; *Parker et al.*, 2000; *Garcia*, 2008b];

$$(1-\lambda_p)\frac{\partial\eta(x,t)}{\partial t} = D_b\left(x,t\right) - E_b\left(x,t\right) \tag{1}$$

where η denotes local mean bed elevation, t denotes time, x denotes the downstream co-ordinate, D_b denotes the volume rate per unit area of deposition of bedload particles onto the bed, E_b denotes the volume rate per unit area of entrainment of bed particles into bedload, and λ_p is the porosity of bed sediment. We assume that, once entrained, a particle undergoes a step with length r before depositing. We further assume that this step length is probabilistic, with a probability density $f_s(r)$ (pdf of step length). The deposition rate of tracers $D_b(x,t)$ is then given as:

$$D_b\left(x,t\right) = \int_0^{\infty} E_b\left(x-r,t\right) f_s(r)dr \tag{2}$$

In the above formulation E_b is a macroscopically determined parameter, which can be shown to vary inversely with the mean resting time of a particle. The formulation thus includes the effect of stochasticity in step length, but not in resting time.

A model formulation for tracers that simplifies the above-mentioned model of entrainment and deposition is the active layer formulation. According to this formulation, grains in an active bed layer of thickness L_a below the local mean bed surface exchange directly with bedload grains. Grains below the active layer, i.e., grains in the substrate, exchange with the active layer only by means of bed aggradation (when active layer grains are transferred to the substrate) and degradation (when substrate grains are transferred to the active layer). In such a model, substrate grains do not directly exchange with the bedload grains.

Let $f_a(x,t)$ denote the fraction of tracer particles in the active layer at any location x and time t. In addition, let $f_I(x,t)$ denote the fraction of tracer particles in the sediment that is exchanged across the interface between the active layer and the substrate as the bed aggrades or degrades. The equation of mass conservation of tracers can then be written as:

$$(1-\lambda_p)\left(f_I(x,t)\frac{\partial \eta(x,t)}{\partial t}+L_a\frac{\partial f_a(x,t)}{\partial t}\right)=D_{bT}(x,t)-E_{bT}(x,t) \quad (3)$$

where E_{bT} denotes the volume entrainment rate of tracers and D_{bT} denotes the corresponding deposition rate, which are given as [*Parker et al.*, 2000]:

$$E_{bT}(x,t)=E_b(x,t)f_a(x,t) \quad (4)$$

and

$$D_{bT}(x,t)=\int_0^{\infty} E_b\left(x-r,t\right)f_a\left(x-r,t\right)f_s(r)dr \quad (5)$$

Here we exclude the complication induced by bedforms such as dunes [e.g., *Blom et al.*, 2006] by considering conditions of lower regime plane-bed transport, such as those investigated by *Wong et al.* [2007].

The fraction f_I of tracers exchanged at the interface as the mean bed elevation fluctuates can be expected to differ depending upon whether or not the bed is aggrading or degrading. *Hoey and Ferguson* [1994] and *Toro-Escobar et al.* [1996] have suggested forms for interfacial exchange fractions which can be adapted to the problem of tracers. Here we restrict consideration to the case for which the bed elevation is at equilibrium, so that L_a, E_b, η and the pdf $f_s(r)$ are all constant in x and t. Under this condition, equations (3), (4) and (5) reduce to:

$$(1-\lambda_p)\,\frac{L_a}{E_b}\frac{\partial f_a(x,t)}{\partial t} = \int_0^\infty f_a\,(x-r,t)\,f_s(r)dr - f_a(x,t) \tag{6}$$

The nature of the pattern of tracer diffusion predicted by equation (6) depends on the nature of the pdf $f_s(r)$ of step lengths. As shown in the next sections, a thin-tailed pdf, i.e., one for which all moments of $f_s(r)$ exist, leads to a classical Fickian advection-diffusion equation, while a heavy-tailed pdf, i.e., one for which moments larger than a given order do not exist, can lead to an anomalous advection-diffusion equation.

3. Tracer transport with thin-tailed step length distribution

In this section, we show that a thin-tailed pdf for the step length distribution, $f_s(r)$, in equation (6) leads to a classical Fickian (normal) advection-diffusion equation. For simplicity, we assume the porosity to be zero, i.e., $\lambda_p = 0$. The simplest way to solve the integral equation (6) is to use Fourier transforms, since the convolution becomes a product in Fourier space. The Fourier transform of a function $f_a(x,t)$ is given by:

$$\hat{f}_a(k,t) = \int_{-\infty}^{\infty} e^{-ikx} f_a(x,t)\,\mathrm{d}x \tag{7}$$

Taking the Fourier transforms in equation (6) and manipulating yields:

$$\frac{L_a}{E_b}\frac{\partial \hat{f}_a(k,t)}{\partial t} = \left(\hat{f}_s(k) - 1\right)\hat{f}_a(k,t) \tag{8}$$

Expanding the Fourier transform of $f_s(r)$ as Taylor series gives:

$$\hat{f}_s(k) = 1 - ik\mu_1 + \frac{1}{2}(ik)^2\mu_2 + \cdots \tag{9}$$

where $\mu_n = \int r^n f_s(r)\mathrm{d}r$ denotes the n^{th} order moment of the step length distribution. The above expansion is valid provided that the moments μ_n exist and are finite, and the series converges uniformly in a neighborhood of $k = 0$ [*Papoulis and Pillai*, 2002]. Substituting equation (9) into (8) we obtain:

$$\frac{L_a}{E_b}\frac{\partial \hat{f}_a(k,t)}{\partial t} = \left(-ik\mu_1 + \frac{1}{2}(ik)^2\mu_2 + \cdots\right)\hat{f}_a(k,t) \tag{10}$$

Recall that $(ik)\hat{f}_a(k,t)$ is the Fourier transform of $\partial f_a(x,t)/\partial x$. By making the approximation that higher order terms can be neglected (which will be shown equivalent, in Section 6, to considering a long-time asymptotic solution), and by setting $v = \mu_1$ and $2D_d = \mu_2$, it follows by an inverse Fourier transform that the function $f_a(x,t)$ is the approximate solution to the advection-diffusion equation:

$$\frac{L_a}{E_b}\frac{\partial f_a}{\partial t} = -v\frac{\partial f_a}{\partial x} + D_d\frac{\partial^2 f_a}{\partial x^2} \tag{11}$$

This is the standard form of advection-diffusion equation for tracer dispersal, and applies under equilibrium bedload conditions where v and D_d can be considered constant. The associated Green's function, i.e., the solution to the above equation with a pulse as the initial condition at $t = 0$, is the Gaussian distribution, which describes the tracer concentration at any given time $t > 0$. If the source is distributed in space and/or time, the solution to equation (11) is the convolution of the Green's function with the source.

4. Tracer transport with heavy-tailed step length distribution

As detailed in the next section, a heavy-tailed, power-law distribution for step lengths in gravel bed rivers can result from a thin-tailed pdf of step length for particles of a given size, together with a thin-tailed pdf of grain sizes. In this section, we develop a formalism that incorporates heavy tails for the step length distribution in the probabilistic Exner equation. In equation (6), consider $f_s(r)$ to be a step length distribution with power-law decaying tail, i.e., $f_s(r) \approx C\alpha r^{-\alpha-1}$ for $r > 0$ sufficiently large, some constant $C > 0$, and some power law index $1 < \alpha < 2$. In this case, the Fourier transform expansion (9) in terms of statistical moments of $f_s(r)$ is not valid, as the integrals $\mu_n = \int r^n f_s(r) \mathrm{d}r$ do not converge for $n > 1$ [e.g., *Lamperti*, 1962]. Instead, we may use a fractional Taylor expansion to write [*Odibat and Shawagfeh*, 2007; *Wheatcraft and Meerschaert*, 2008]:

$$\hat{f}_s(k) = 1 - ik\mu_1 + c_\alpha \left(ik\right)^\alpha + \cdots \tag{12}$$

where c_α is a constant that depends only on C and α. Substituting back in equation (8) we obtain:

$$\frac{L_a}{E_b} \frac{\partial \hat{f}_a(k,t)}{\partial t} = \left(-ik\mu_1 + c_\alpha \left(ik\right)^\alpha + \cdots\right) \hat{f}_a(k,t) \tag{13}$$

This equation (13) can be understood in terms of fractional derivatives. Fractional derivatives are close cousins of their integer order counterparts. The fractional derivative $\partial^\alpha f_a(x,t)/\partial x^\alpha$ can be defined simply as the function whose Fourier transform is $\left(ik\right)^\alpha \hat{f}_a(k,t)$. As in the normal advection-diffusion case, we make an approximation by including the first two terms in the expansion and neglecting the higher order terms (shown equivalent in Appendix A to a long-time asymptotic solution). Then by setting $v = \mu_1$ and $D_d = c_\alpha$, it follows from (13) that the function $f_a(x,t)$ is approximately the solution of the fractional advection-diffusion equation:

$$\frac{L_a}{E_b}\frac{\partial f_a}{\partial t} = -v\frac{\partial f_a}{\partial x} + D_d\frac{\partial^\alpha f_a}{\partial x^\alpha} \tag{14}$$

Fractional advection-diffusion has been extensively used in modeling the dispersal of tracers or pollutants in porous media which exhibit multiple scales of variability, as in subsurface transport [e.g., *Benson et al.*, 2000a, b; *Berkowitz et al.*, 2002] and pollutant transport in rivers [e.g., *Deng et al.*, 2005, 2006]. However, to the best of our knowledge, its application has not yet been explored in the context of river transport, apart from a recent study which uses fractional advection for transporting sediment in buffered bedrock rivers [*Stark et al.*, 2009].

In most natural rivers, the distribution of step lengths holds in the near field, but eventually transport steps become limited by river features (e.g., bars) that change the intermediate and far field distributions. The application of the governing equations (11) and (14) depends on the natural truncation of the step length distributions. If the truncation occurs at a very small threshold, then the Central Limit Theorem applies and a standard advection-diffusion equation will be the governing equation for the fraction of tracers in the active layer. However, if the truncation occurs at a large threshold, then the distribution can still be approximated by a power-law in the intermediate field and the governing equation for the fraction of tracers in the active layer is the fractional advection-diffusion equation. It is worth noting that equation (14) is the governing equation on scales where the power-law approximation of the step length distribution is accurate. In the next section, we explain how a power-law distribution for step lengths could emerge by combining a thin-tailed pdf of step length for particles of a given size with a thin-tailed pdf of grain sizes. Then in Section 6 we describe the stochastic model underlying the probabilistic Exner equation (6), and we show how equations (11) and (14) represent long-time asymptotic solutions.

5. Transport of sediment mixtures

A generalization of equation (6) for a range of grain sizes D can be expressed as follows. Let $f_{ad}\left(x,t,D\right)$ denote the fraction of tracers in the active layer with grain size D, so that,

$$f_a(x,t) = \int_0^\infty f_{ad}(x,t,D)dD \tag{15}$$

In addition, let $E_{bu}(D)$ denote the entrainment rate per unit bed content of size D. The generalization of equation (6) is then [e.g., *Parker*, 2008],

$$\left(1-\lambda_p\right) L_a \frac{\partial f_a\left(x,t,D\right)}{\partial t} = E_{bu}(D)\left(\int_0^\infty f_{ad}\left(x-r,t,D\right) f_s(r|D)dr - f_{ad}\left(x,t,D\right)\right) \tag{16}$$

In the above formulation, the conditional pdf of step length f_s is specified as a function of grain size, but the thickness of the active layer L_a is taken to be a constant for all grain sizes. The form corresponding to equation (6) is obtained by integrating over all grain sizes,

$$\left(1-\lambda_p\right) L_a \frac{\partial f_a\left(x,t\right)}{\partial t} = \int_0^\infty E_{bu}(D)\left(\int_0^\infty f_{ad}\left(x-r,t,D\right) f_s(r|D)dr - f_{ad}\left(x,t,D\right)\right) dD \tag{17}$$

In general, E_{bu}, f_{ad} and f_s can be expected to vary significantly in D. Closure of equation (17) requires specification of forms for E_{bu}, f_s and f_{ad} as functions of, among other parameters, grain size D. Such forms are available in the literature [e.g., *Tsujimoto*, 1978].

The goal of the present analysis is, however, to study the role of heavy-tailed pdfs for step lengths in driving diffusion of tracer particles. With this in mind, the problem is simplified for the purposes of illustration to one in which f_s varies in D but E_{bu} and f_{ad} do not. More specifically, we express $f_s(r, D) = f_s(r|D)f(D)$ as a product of the conditional step length pdf for a grain size, and the grain size pdf $f(D)$, to show that a heavy tail pdf for step lenghts in a mixture of particles can emerge, under certain conditions, from two thin-tailed pdfs.

5.1. Power laws emerging from thin tails

A typical finding in sediment transport is that step lengths r are exponentially distributed for a given grain size D [e.g., *Nakagawa and Tsujimoto*, 1976, 1980], i. e.,

$$\mathbb{P}(R > r \,|\, D) = e^{-r/\mu_r(D)} \tag{18}$$

where $\mu_r(D)$ is the mean step length as a function of grain size D. If we let f denote the pdf of grain sizes, then the unconditional distribution of step length can be derived from:

$$\mathbb{P}(R > r) = \int_0^\infty e^{-r/\mu_r(D)}\, f(D)\, dD. \tag{19}$$

The resulting pdf for step length, relating to a mixture of particle sizes, depends on both the mean step length $\mu_r(D)$ for grains of size D, and the pdf of grain sizes.

In this study we explore two distinct cases, one in which $\mu_r(D)$ increases with grain size, and another for which $\mu_r(D)$ decreases with grain size. The true dependence of mean step length on grain size in sediment mixtures remains somewhat ambiguous. In the case of uniform sediment, *Niño and Garcia* [1998] found that grain saltation length decreases with increasing grain size. One step length, however, typically consists of around 10 saltation lengths. *Hassan and Church* [1992] have studied the travel distance of size mixtures of stones in gravel-bed rivers, and have found a marked tendency for travel distance to decrease with increasing grain size. This result must be qualified in light of the fact that the distance traveled by a grain during a flood can be expected to be associated with multiple step lengths. This qualification notwithstanding, the data suggest a range of conditions under which the dependence between grain size and mean travel distance can be approximated by the simplified model:

$$\mu_r(D) = \kappa/D \tag{20}$$

where κ is a constant. A lognormal pdf of grain sizes

$$f(D) = \frac{1}{D\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\ln D - \mu)^2}{\sigma^2}} \tag{21}$$

was invoked by *Wilcock and Southard* [1989]; *Garcia* [2008b]; *Lanzoni and Tubino* [1999]; *Parker* [2008], where μ, σ are the mean and standard deviation of the sedimentological scale $\psi = \ln D$. The overall (unconditional) step length distribution can then be obtained, in principle, by substituting equations (20) and (21) into equation (19) and computing the integral. However, this integral is difficult to compute analytically with a log-normal form for $f(D)$. Figure 1 shows the grain size data from *Wilcock and Southard* [1989] along with a lognormal fit, as well as an alternative gamma distribution fit to the same data. The gamma pdf

$$f(D) = \frac{\nu^\nu}{\Gamma(\nu) {D_\mathrm{m}}^\nu} D^{\nu-1} \exp\left(-\nu \frac{D}{D_\mathrm{m}}\right) \tag{22}$$

with mean D_m and shape parameter ν provides a convenient alternative to the lognormal distribution that makes it possible to analytically evaluate the integral (19). Following the argument of *Stark et al.* [2009], we substitute equations (20) and (22) into equation (19) and evaluate the integral to obtain the unconditional probability distribution of step length:

$$\mathbb{P}(R > r) = \left(1 + \left(\frac{D_{\mathrm{m}}}{\nu\kappa}\right) r\right)^{-\nu} \tag{23}$$

The above equation (23) represents a heavy-tailed power-law pdf for the step length distribution arising from a thin-tailed pdf of step length combined with a thin-tailed pdf of grain sizes. The distribution in equation (23) is known as the Generalized Pareto, and its variance exists only when the shape parameter $\nu > 2$ [*Feller*, 1971]. The Generalized Pareto distribution also arises from exceedances over a fixed high threshold, and has consequently been used in modeling extreme floods and other hydrological phenomena [e.g., *Hosking and Wallace*, 1987].

The relationship (20) between mean step length and grain size may not be applicable in all situations. Depending upon the grain size distribution and the flow conditions, large particles may roll over holes that trap smaller particles, so that step length increases with grain size. Such a tendency has been reported in the experiments of *Tsujimoto* [1978]. Also *Wong et al.* [2007] observed that, in the case of uniform sediment subject to the same bed shear stress, step length increases with grain size. Such an increase in step length does not directly translate into a higher bedload transport rate for coarser grains, because the entrainment rate $E_{bu}(D)$ in equation (17) may decline with increasing grain sizes. In the present simplified analysis, where E_{bu} is assumed to be independent of grain size, the tendency for step length to increase with grain size can be captured in terms of the following simple form:

$$\mu_r(D) = \kappa D \tag{24}$$

where κ is a constant.

If D has an inverse gamma pdf with mean D_{m} and shape parameter ν, also similar in shape to the lognormal,

$$f(D) = \frac{(\nu-1)^{\nu} {D_{\mathrm{m}}}^{\nu}}{\Gamma(\nu)} D^{-\nu-1} \exp\left(-\frac{(\nu-1)D_{\mathrm{m}}}{D}\right) \tag{25}$$

then a change of variables $y = 1/D$ in (19) leads to another generalized Pareto:

$$\mathbb{P}(R > r) = \left(1 + \left(\frac{1}{(\nu-1)D_{\mathrm{m}}\kappa}\right) r\right)^{-\nu} \tag{26}$$

as shown in *Hill et al.* [2009], so that again the step length distribution averaged over all particle sizes has a heavy tail.

Note that in both cases considered above, whether mean step length increases or decreases with grain size, a heavy-tailed distribution for step lengths can emerge from a combination of two thin-tailed distributions. The gamma and inverse gamma distributions are used for particle sizes, as opposed to the more typical log-normal distribution, in order to derive analytically the heavy-tail pdf of the resulting step length distribution for a mixture of grain sizes. The alternative pdf assumption should be considered reasonable if the reader accepts that the fitted log-normal and gamma distributions for the grain size data from *Wilcock and Southard* [1989] in Figure 1 are practically indistinguishable. We hasten to emphasize, however, that the finding of a possible heavy-tailed pdf for step length is by no means universal. Many different choices of the grain size pdf $f(D)$ would certainly lead to a thin-tailed pdf of step length. Our point is simply that both thin-tail and heavy-tail models are reasonable, and hence it becomes very important to investigate the grain size distributions more exhaustively, to determine which type of overall step length pdf applies in a given situation.

6. Stochastic model for gravel transport in rivers

In this section, we develop a stochastic model to describe the time evolution of the relative concentration of gravel tracer particles in rivers. We derive an exact solution for $f_a(x,t)$ and show that, in the long-time asymptotic limit, a thin tail for the step length distribution leads to classical advection-diffusion, whereas heavy tails for the step length distribution leads to anomalous advection-diffusion. We start by rewriting (6) in the equivalent form:

$$\frac{\partial f_a(x,t)}{\partial t} = -\lambda f_a(x,t) + \lambda \int_0^\infty f_a(x-r,t) f_s(r)\,\mathrm{d}r \tag{27}$$

where $\lambda = E_b/L_a$ is the rate at which particles are entrained. The Fourier transform of the above equation is given by:

$$\frac{\partial \hat{f}_a(k,t)}{\partial t} = -\lambda \hat{f}_a(k,t)\left(1 - \hat{f}_s(k)\right) \tag{28}$$

Equation (27) describes the time evolution of the pdf $f_a(x,t)$ and can be regarded as a Kolmogorov forward equation for some Markov process $X(t)$, where $X(t)$ represents the location of a randomly selected gravel particle at time $t > 0$ [see *Feller*, 1971]. In this context, $f_a(x,t)$ is the pdf of the random variable $X(t)$. In this Markov process, the waiting time between entrainments has an exponential distribution with a rate parameter λ, and the number of entrainment events, $N(t)$, by any time $t > 0$ has a Poisson distribution with mean λt [*Feller*, 1971], i.e.,

$$P[N(t) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{29}$$

Let Y_n denote the travel distance during the n^{th} entrainment period. Since there are $N(t)$ entrainment periods by time $t > 0$, the particle location at some time $t > 0$ is given by the random sum:

$$X(t) = Y_1 + \cdots + Y_{N(t)} = \sum_{i=1}^{N(t)} Y_i \tag{30}$$

This random sum is a compound Poisson process [e.g., *Feller*, 1971]. Its pdf can be derived directly from equation (28) whose point source solution is:

$$\hat{f}_a(k,t) = \exp\left(-\lambda t\left(1-\hat{f}_s(k)\right)\right) \tag{31}$$

As a result, the fraction of tracers in the active layer, $f_a(x,t)$, can be obtained by taking the inverse Fourier transform of (31) and is given by:

$$f_a(x,t) = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda n)^n}{n!}f_s^{n^*}(x) \tag{32}$$

where $f_s^{n^*}(x)$ is the n-fold convolution of the density function $f_s(x)$ (recall that $f_s^{n^*}(x)$ is the inverse Fourier transform of $\hat{f}_s(k)^n$), which is also the pdf of $Y_1+\cdots+Y_n$. One way to understand this formula for $f_a(x,t)$ is that it randomizes the density of the sum of the particle movements according to the pdf of the number of jumps $N(t)$. The random sum, equation (30), is a special case of a continuous time random walk (CTRW) [*Montroll and Weiss*, 1965; *Scher and Lax*, 1973; *Meerschaert and Scheffler*, 2004]. It is important to note that the connection of the probabilistic Exner equation with CTRWs allows one to obtain the exact solution of equation (27) via simulation of the tracer particle motion. For example, a forward Kolmogorov equation of a Markov process can be solved by simulating a CTRW with an exponential waiting time distribution and step length distribution $f_s(r)$ [e.g., *Scalas et al.*, 2004; *Fulger et al.*, 2008]. Even if the complete shape of the pdf of step lengths is not known, the behavior of the stochastic process $X(t)$ is well defined in the long-time limit as shown below.

Consider the standardized particle location:

$$Z(t) = \frac{X(t)-\lambda\mu_1 t}{\sqrt{\lambda\mu_2 t}} \tag{33}$$

This random process has a mean 0 and variance 1 at every time $t > 0$. An easy calculation shows that the pdf of $Z(t)$ has Fourier transform:

$$\hat{f}_a\left(\frac{k}{\sqrt{\lambda\mu_2 t}}, t\right)\exp\left(\frac{ik\lambda\mu_1 t}{\sqrt{\lambda\mu_2 t}}\right) \tag{34}$$

Combining this equation with:

$$\hat{f}_a(k,t) = \exp\left(-\lambda t\left(ik\mu_1 - \frac{1}{2}(ik)^2\mu_2 + \cdots\right)\right) \tag{35}$$

which is obtained by substituting equation (9) into equation (31) results in the Fourier transform of the pdf of $Z(t)$ taking the form:

$$\exp\left(-\lambda t\left(-\frac{1}{2}\frac{(ik)^2}{\lambda\mu_2 t}\mu_2 + \frac{1}{3!}\frac{(ik)^3}{(\lambda\mu_3 t)^{\frac{3}{2}}}\mu_3 + \cdots\right)\right) \tag{36}$$

As $t \to \infty$, (36) tends to $\exp\left(-\frac{1}{2}k^2\right)$ which is the Fourier transform of a standard normal density. This shows that $Z(t)$ tends to a standard normal deviate, Z, for all large $t > 0$. Substituting into equation (33) and solving, we see that the long-time asymptotic solution for the particle location is:

$$X(t) \approx \lambda\mu_1 t + \sqrt{\lambda\mu_2 t}Z \tag{37}$$

By taking the Fourier transforms of the corresponding pdfs we obtain:

$$\hat{f}_a(k,t) = \exp\left(-\lambda\mu_1 t(ik) + \frac{1}{2}\lambda\mu_2 t(ik)^2\right) \tag{38}$$

which is the point source solution to the differential equation:

$$\frac{\partial\hat{f}_a(x,t)}{\partial t} \approx \left(-\lambda\mu_1(ik) + \frac{1}{2}\lambda\mu_2(ik)^2\right)\hat{f}_a(k,t) \tag{39}$$

Inverting this Fourier transform yields the advection-diffusion equation (11) with $v = \lambda\mu_1$ and $2D_d = \lambda\mu_2$, as in Section 3. In summary, equation (11) governs the asymptotic particle density in the long-time limit.

Now consider the case of a particle jump length density with a heavy tail. A similar argument shows that equation (14) governs the asymptotic particle density in the long-time limit, when the particle jump length density $f_s(r)$ has a heavy tail with a power-law decay, i.e., $f_s(r) \approx C\alpha r^{-\alpha-1}$ for $r > 0$ sufficiently large, some constant $C > 0$, and some power law index $1 < \alpha < 2$ (see Appendix A for a detailed proof). In this case, we note that the governing equation in the long-time asymptotic limit for $\hat{f}_a(k,t)$ is given by:

$$\frac{\partial \hat{f}_a(k,t)}{\partial t} \approx \left(-\lambda\mu_1(ik) + \lambda c_\alpha (ik)^\alpha\right) \hat{f}_a(k,t) \tag{40}$$

Inverting the Fourier transform yields the fractional advection-diffusion equation (14) with $v = \lambda\mu_1$ and $D_d = \lambda c_\alpha$, as in Section 4. We remark that, while the derivation in this section is new in the context of stone tracer dispersion, a similar approach was taken to derive the fractional advection-diffusion equation for tracers in ground water, under a different set of assumptions [*Schumer et al.*, 2001]. The next section provides a numerical demonstration to illustrate how a source of tracers will disperse over time under normal or anomalous diffusion.

7. Tracer dispersal under normal and anomalous diffusion

Consider a patch of tracers entrained instantaneously in the flow at a location x_0 and initial time t_0. This patch will advect and diffuse on the gravel bed over time. It is useful to track the time evolution of the fraction of tracers $f_a(x,t)$ in the active layer at any location x and time t. As was shown in the previous sections, the probabilistic Exner equation can be approximated at late time by a normal or anomalous diffusion, equations (11) and (14) respectively, depending on the pdf of step length. In this section we illustrate the time evolution of a patch of tracers under normal and anomalous advection-diffusion. We know from theory that the Green's function solution to the normal advection-diffusion equation is the Gaussian distribution, and the Green's function solution to the fractional advection-diffusion is the α-stable distribution [*Benson et al.*, 2000b]. The α-stable distributions are also known as Lévy distributions. Specifically, in our case, the Green's function solution to the fractional advection-diffusion equation is an α-stable distribution with a skewness parameter $\beta = 1$, owing to the fact that step lengths are positive, so that the stable pdf has a heavy lead leading tail (see Appendix B for a description of stable distributions). Figure 2(a) shows the evolution of $f_a(x, t)$ under normal advection-diffusion from a pulse at $t = 0$ and $x = 0$, i.e., $f_a(0, 0) = 1$. Figure 2(b) shows the evolution of $f_a(x, t)$ under anomalous advection-diffusion with $\alpha = 1.5$ from a pulse at $x = 0$. The α-stable densities in Figures 2(a) and 2(b) were simulated using the method of *Nolan* [1997]. In this hypothetical experiment, we chose the parameter values of the normal and anomalous diffusion equations to be unity, i.e., $v = 1$ m/day and $D_d = 1$ m$^\alpha$/day. Note that the units of the diffusion coefficient, D_d, is $[L^\alpha/T]$. As can be seen by comparing Figures 2(a) and 2(b), anomalous advection-diffusion predicts a faster spreading of tracers downstream (heavy leading tails). For example, the leading tails of the fraction of tracers at $t = 100$ reaches a near-zero value at ~ 50 m downstream of its mean in normal advection-diffusion, whereas it reaches this value at ~ 200 m downstream of its mean in fractional advection-diffusion with $\alpha = 1.5$. Note that the mean of $f_a(x, t)$ in both cases is the same. Both the Gaussian pdf, and the skewed stable pdf, assign some extremely small but mathematically nonzero probability to the interval $x < 0$, while the probabilistic Exner equation assigns zero probability to $x < 0$. This illustrates the fact that both the Gaussian and skewed stable are only approximations. However, in practice, the probability assigned to $x < 0$

is exceedingly small, since both the Gaussian and skewed stable pdf fall off at a super-exponential rate on the left tail, and the approximation is perfectly reasonable.

As seen in the previous section, under equilibrium bedload transport conditions, the long-time asymptotic solutions of the probabilistic Exner equation converge to the normal and anomalous advection-diffusion equation depending on the pdf of the step length. Therefore, long-time asymptotic solutions of the probabilistic Exner equation are the Gaussian and α-stable distributions in the respective cases of thin or heavy tailed pdfs for step length. In Figure 3 we compare the long-time asymptotic solutions for several values of α, starting from $\alpha = 2$ (Gaussian corresponding to the solution of normal advection-diffusion equation) to $\alpha = 1.1$. One can easily see the marked difference in the dispersal of tracers downstream in normal and anomalous advection-diffusion. For example, after 500 days, only $\sim 5\%$ of the tracers have been recovered at ~ 550 m in standard advection-diffusion, whereas $\sim 8\%$ and $\sim 18\%$ of tracers are recovered at the same distance in fractional advection-diffusion for $\alpha = 1.5$ and $\alpha = 1.1$, respectively. Note that in the case of $\alpha = 1.1$ the gravel tracer particles are transported very long distances downstream when compared with the normal advection-diffusion case ($\alpha = 2$). Note that the parameter α of the fractional advection-diffusion relates to the heaviness of the tail of the pdf of particle step lengths, in effect determining how far downstream the tracers disperse from the source. In practice, the parameter α will have to be estimated from observations which typically will not be in the form of step lengths but in the form of "breakthrough curves" or pdfs of particle concentration at a given location downstream of the source. Tracer experiments in a large experimental flume are currently under development to document the possibility of faster-than-normal diffusion of tracers and the estimation of the parameter α.

8. Conclusions

In this work, a mathematical framework for the continuum treatment of tracer particle dispersal in rivers has been proposed, based on the probabilistic Exner equation. We have shown that when the step length distribution is thin-tailed, the governing equation for the tracer dispersal in the long-time limit is given by the standard advection-diffusion equation. However, the step length distributions can be heavy-tailed with power-law decay arising from heterogeneity in grain sizes and other complexities in real gravel bed rivers. It was shown that these heavy tails can be modeled using fractional derivatives, akin to contaminant transport in subsurface hydrology [e.g., *Benson*, 1998; *Benson et al.*, 2000a, b; *Berkowitz et al.*, 2002]. For a simplified active layer formulation, the probabilistic Exner equation was shown to be governed by a Markov process that describes the tracer dispersal problem. Further, it was shown that the classical (normal) advection-diffusion and fractional (anomalous) advection-diffusion equations arise as long-time asymptotic solutions of that stochastic model. A numerical example was then provided to illustrate the profound effect of fractional diffusion on the leading edge of the particle distribution.

The material presented here is intended to serve as an introduction to the problem of anomalous diffusion in the context of transport in gravel-bed rivers. The full power of the techniques introduced here remains to be realized through future research. For example, the innate variability of rivers is such that the entrainment rate E_b and bed elevation η are unlikely to be constant in x and t. This variability can lead to long-term sequestration, and subsequent long-delayed exhumation of tracers. *Parker et al.* [2000] and

Blom et al. [2006] have shown how the fractional Exner equation (1) can be generalized to a formulation that assigns a probabilistic structure not only to step length, but also to the probabilities of entrainment and deposition as continuously varying functions of vertical position within the bed deposit. These complications can lead to anomalous sub-diffusion, if particle resting times have a heavy, power-law tail. A model that can explain the deposition and exhumation of particles at arbitrary depth, including variability in entrainment rate and bed elevation as well as grain size, has the potential to explain at least part of the tendency for a decrease in advection velocity over time described by *Ferguson and Hoey* [2002]. The anomalous advection-diffusion model proposed herein, as well as further extensions to accommodate additional stochastic elements of transport as discussed above, will require extensive experiments and data collection to directly verify the nature of the distribution of step lenghts, waiting times and entrainment rates of particles in order to select the most appropriate model for transport.

Appendix A: Long-time asymptotics for heavy-tailed distributions

The standardized particle location cannot be expressed using equation (33) when the step length distribution has a heavy tail, because the second moment μ_2 of the distribution $f_s(r)$ does not exist, i.e., the population variance is infinite while the sample variance diverges unstably as the number of samples increases [*Lamperti*, 1962]. Instead, we consider the normalized process:

$$S(t) = \frac{X(t) - \lambda\mu_1 t}{(\lambda c_\alpha t)^{\frac{1}{\alpha}}} \tag{A1}$$

The pdf of $S(t)$ has the Fourier transform:

$$\hat{f}_a\left(\frac{k}{(\lambda c_\alpha t)^{\frac{1}{\alpha}}}, t\right) \exp\left(\frac{ik\lambda\mu_1 t}{(\lambda c_\alpha t)^{\frac{1}{\alpha}}}\right) \tag{A2}$$

Substitution of equation (12) into equation (31) results in:

$$\hat{f}_a(k,t) = \exp\left(-\lambda t\left(ik\mu_1 - c_\alpha(ik)^\alpha - d_\alpha(ik)^{2\alpha} + \cdots\right)\right) \tag{A3}$$

which combined with (A2) gives the left-hand side of the equation (A4) for the Fourier transform of the PDF of $S(t)$. In the long-time limit, i.e., as $t \to \infty$ this tends to the limit in the right-hand side below, i.e.,

$$\exp\left(\lambda t\left(c_\alpha \frac{(ik)^\alpha}{\lambda c_\alpha t} + d_\alpha \frac{(ik)^{2\alpha}}{(\lambda c_\alpha t)^2} + \cdots\right)\right) \to \exp\left((ik)^\alpha\right) \tag{A4}$$

since the higher order terms tend to zero as $t \to \infty$. This limit is the Fourier transform of a standard stable density, and the limit argument is closely related to the convergence criterion for compound Poisson random variables (see Chapter 3 in *Meerschaert and Scheffler* [2001] for more details and extensions). Hence, $S(t) \approx S$ is standard stable for all large $t > 0$. Substituting into equation (A1) and solving, we see that the long-time asymptotic approximation for the particle location is:

$$X(t) \approx \lambda\mu_1 t + (\lambda c_\alpha t)^{\frac{1}{\alpha}} S \tag{A5}$$

Taking the Fourier transforms of the corresponding pdfs, we obtain:

$$\hat{f}_a(k,t) \approx \exp\left(-\lambda\mu_1 t(ik) + \lambda c_\alpha t(ik)^\alpha\right) \tag{A6}$$

This is the Fourier transform of $f_a(x,t)$ with the higher order terms removed, as well as the point source solution to the differential equation:

$$\frac{\partial f_a(\hat{k},t)}{\partial t} \approx \left(-\lambda\mu_1(ik) + \lambda c_\alpha(ik)^\alpha\right)\hat{f}_a(k,t) \tag{A7}$$

Inverting this Fourier transform results in the fractional advection-diffusion equation (14).

Appendix B: Stable distributions

If $X, X_1, X_2, \ldots$ are mutually independent random variables with a common distribution F_s, then the distribution F_s is said to be stable if for each $n \in \mathbb{Z}$, there exists constants C_n and r_n such that [e.g., *Lamperti*, 1962; *Feller*, 1971]:

$$S_n \stackrel{d}{=} C_n X + r_n \tag{B1}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ and $\stackrel{d}{=}$ means identical in distribution. In other words, stable distributions are aggregation invariant up to a scale parameter, C_n, and location parameter, r_n. The norming constant C_n is of the form $n^{\frac{1}{\alpha}}$ for $0 < \alpha \leq 2$, where α is called the characteristic exponent of the distribution F_s. The distribution F_s is said to be strictly stable when $r_n = 0$. Closed-form expressions of the density functions of stable distributions exist for values of α equal to $2, 1$ and 0.5. In general, the stable pdf is defined by its Fourier transform [see *Stuart and Ord*, 1987]:

$$\hat{\rho}(k) = \{-i\delta k - |\gamma k|^{\alpha}\left(1 + i\beta \mathrm{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)\right)\} \tag{B2}$$

for $0 < \alpha \leq 2$ and $\alpha \neq 1$. In the above equation $\mathrm{sgn}(\cdot)$ denotes the signum function. The remaining parameters of the distribution are the location parameter $(-\infty < \delta < \infty)$, scale parameter $(\gamma > 0)$ and the skewness parameter $(-1 \leq \beta \leq 1)$. The distribution is symmetric for $\beta = 0$ and is said to be completely skewed for $\beta = -1$ and $\beta = 1$. For $\alpha = 2$, $\hat{\rho}(k)$ gives the Fourier transform of a Gaussian density with mean δ and variance $2\gamma^2$. For the special case $\alpha = 1$, the Fourier transform is expressed in a slightly different way. When $\alpha = 1$ and $\beta = 0$, the stable distribution is also called a Cauchy distribution.

If a random variable X has an α-stable distribution, then its statistical moments exist only up to order α. The mean of the distribution exists when $1 < \alpha \leq 2$, but when $0 < \alpha < 1$ both the mean and variance of the distribution do not exist. Thus, the

Gaussian distribution is the only stable distribution with finite mean and variance. Stable distributions provide good approximations for spatial rainfall fluctuations in convective storms [e.g., *Kumar and Foufoula-Georgiou*, 1993], daily discharges in river flows [e.g., *Dodov and Foufoula-Georgiou*, 2004], spatial snapshots of tracer plumes in underground aquifers [e.g., *Benson et al.*, 2000a] and river flows [e.g., *Deng et al.*, 2004].

Notation

x	streamwise co-ordinate.
t	time.
η	local mean bed elevation.
t	time.
η	local mean bed elevation.
λ_p	porosity.
D_b	volume rate per unit area of deposition of bedload particles.
E_b	volume rate per unit area of entrainment of bedload particles.
$f_s(r)$	pdf of step lengths.
$f_s(r\|D)$	pdf of step lengths conditioned on grain size.
$f_a\left(x,t\right)$	fraction of tracer particles in the active layer.

$f_I(x,t)$	fraction of the tracer particles in the sediment that is exchanged across the interface between active layer and substrate.
L_a	thickness of the active layer.
E_{bT}	volume entrainment rate of tracers.
D_{bT}	volume deposition rate of tracers.
v	advection velocity of tracers.
D_d	diffusivity of tracers.
D	grain-size.
D_m	arithmetic-mean of the grain-size distribution.
D_g	geometric mean of the grain-size distribution.
α	tail index of the stable distribution and the order of fractional differentiation.
$\mu_r(D)$	mean step length for grain-size D.
$f_{ad}(x,t,D)$	fraction of tracers in the active layer with grain-size D.
$E_{bu}(D)$	entrainment rate per unit bed content of grain-size D.

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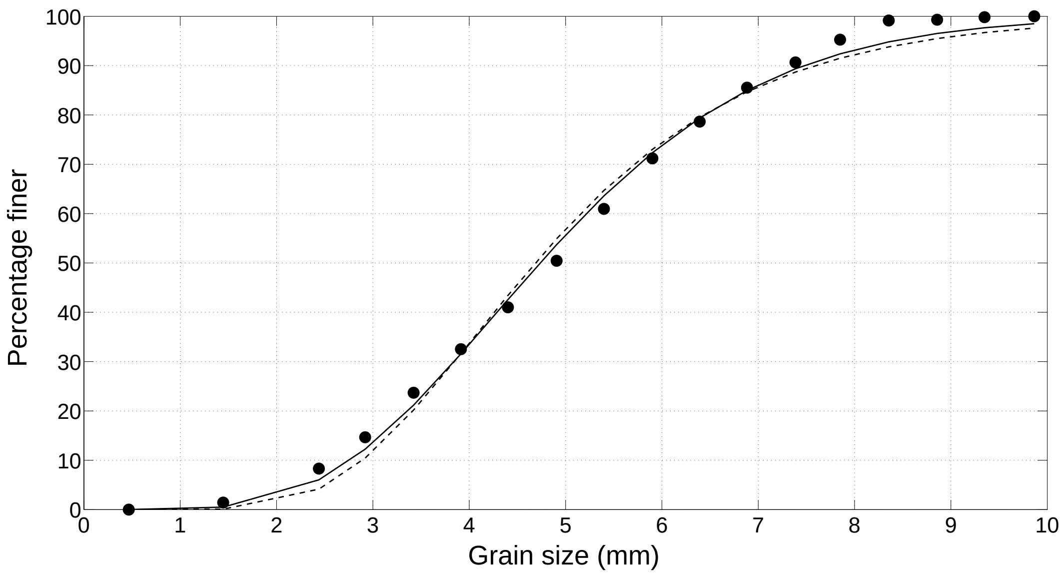

Figure 1. Plot showing fitted log-normal (dashed line) and gamma (solid line) distributions, to a grain-size distribution (solid points) reproduced from *Wilcock and Southard* [1989].

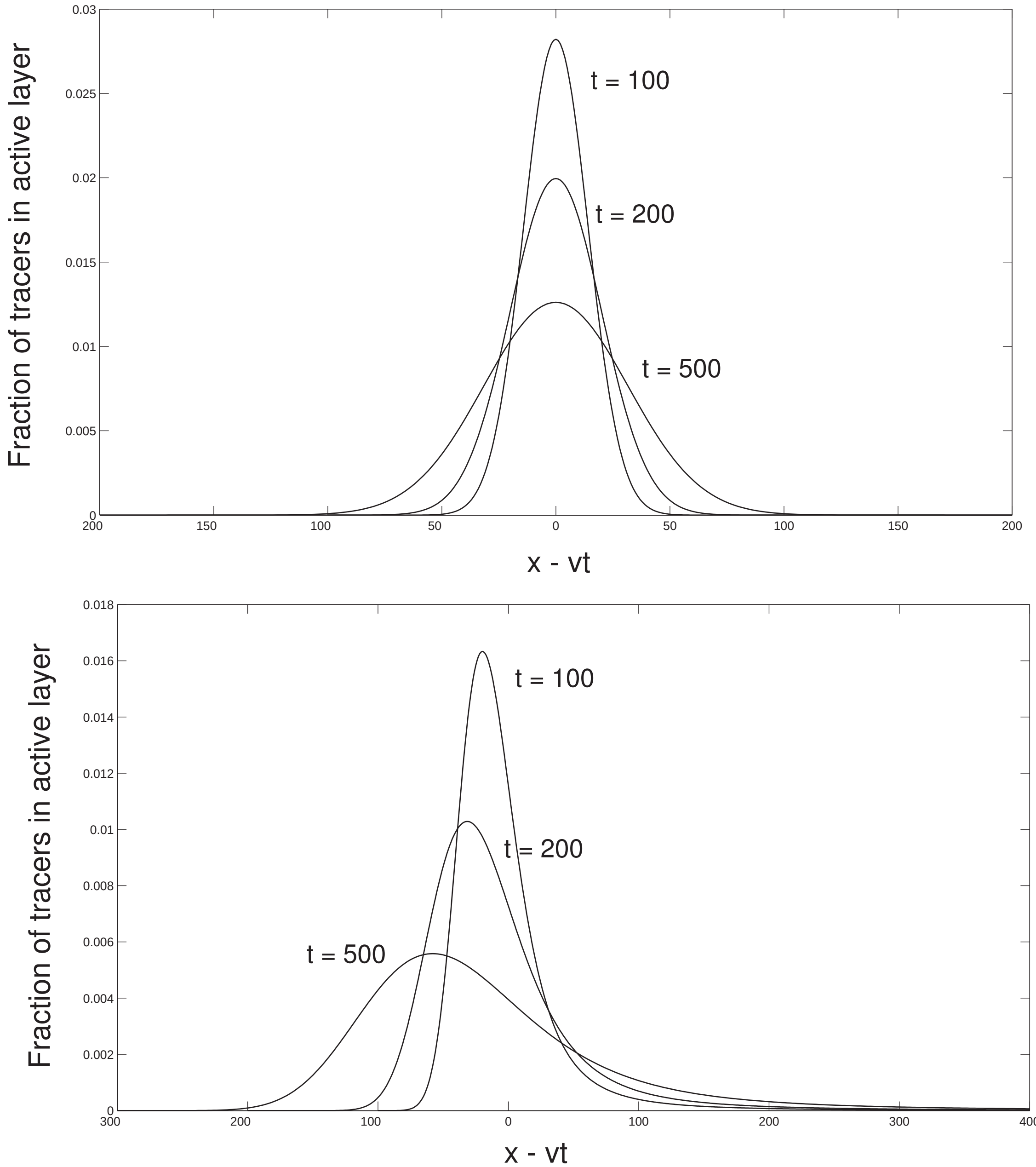

Figure 2. Time evolution of the fraction of tracers in the active layer, $f_a(x,t)$, by (a) normal advection-diffusion ($\alpha = 2$), and (b) anomalous advection-diffusion with $\alpha = 1.5$. Note that the advection term has been removed to facilitate comparison of the dispersion of the tracers at different times. The initial condition is a pulse at $x = 0$. The solutions are obtained with parameters $v = 1$ m/day and $D_d = 1$ m$^\alpha$/day. The times (in days) at which the solutions are obtained are labeled in the figure.

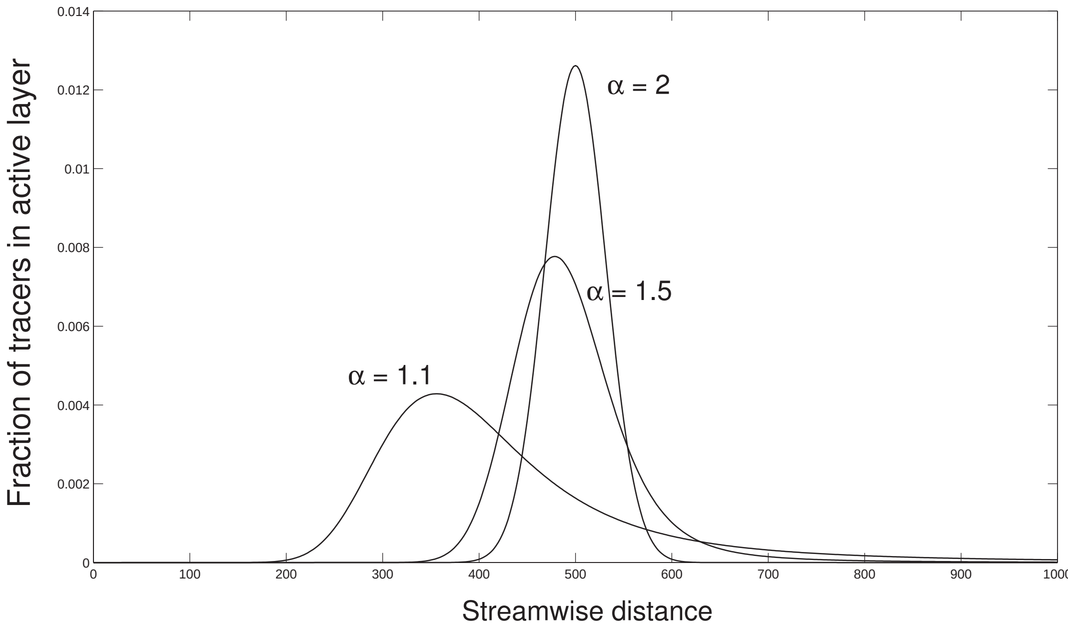

Figure 3. Long-time asymptotic solutions of the anomalous advection-diffusion equation for three different values of α. The solutions shown above are for 500 days after a patch of tracers is entrained into the flow. Normal advection-diffusion corresponds to $\alpha = 2$.