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TAUBERIAN THEOREMS FOR MATRIX REGULAR VARIATION

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Abstract. Karamata's Tauberian theorem relates the asymptotics of a nondecreasing right-continuous function to that of its Laplace-Stieltjes transform, using regular variation. This paper establishes the analogous Tauberian theorem for matrix-valued functions. Some applications to time series analysis are indicated.

1. Introduction

Regular variation is an asymptotic property of functions that captures power behavior. In essence, a regularly varying function grows like a power, times another factor that varies more slowly than any power. The book of Bingham, Goldie, and Teugels [5] describes numerous applications to number theory, analysis, and probability. Karamata's Tauberian theorem proves that a nondecreasing right-continuous function is regularly varying if and only if its Laplace-Stieltjes transform is regularly varying, and establishes an asymptotic equivalence between these two functions. This paper establishes the corresponding Tauberian theorem for matrix-valued functions, along with some related results on power series with matrix coefficients. This work was originally motivated by a problem in time series analysis; see Section 5 for a discussion.

2. Matrix regular variation

We say that a Borel measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is regularly varying at infinity with index ρ, and we write $f \in \mathrm{RV}_\infty(\rho)$, if

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The functions x^ρ and $x^\rho \log x$ are both in $\mathrm{RV}_\infty(\rho)$. We say that a function $g(x)$ is regularly varying at zero with index $-\rho$, and we write $g \in \mathrm{RV}_0(-\rho)$, if the function $g(x) = f(1/x)$ is in $\mathrm{RV}_\infty(\rho)$. If $\rho = 0$, we also say that $f(x)$ is slowly varying at infinity. It is easy to check that any $f \in \mathrm{RV}_\infty(\rho)$ can be written in the form $f(x) = x^\rho L(x)$, where L is slowly varying at infinity. Then for any $\delta > 0$, there exists an $x_0 > 0$ such that $x^{-\delta} < L(x) < x^\delta$ for all $x \geq x_0$; see for example Feller [8, Lemma 2, VIII.8]. It follows that

$$x^{\rho-\delta} < f(x) < x^{\rho+\delta} \quad \text{for all } x \geq x_0,$$

so that $f(x)$ grows like a power.

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Let $\mathrm{GL}(\mathbb{R}^m)$ denote the space of invertible $m \times m$ matrices with real entries. We say that a Borel measurable function $f : \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^m)$ is regularly varying at infinity with index E, and we write $f \in \mathrm{RV}_\infty(E)$, if

$$\lim_{x\to\infty} f(\lambda x)f(x)^{-1} = \lambda^E \quad \text{for all } \lambda > 0. \tag{2.1}$$

Here the matrix power $\lambda^E = \exp(E \log \lambda)$, where $\exp(A) = I + A + A^2/2! + \cdots$ is the usual matrix exponential. If $f \in \mathrm{RV}_\infty(E)$, then we also say that the function $g(x) = f(1/x)$ is regularly varying at zero with index $-E$, and we write $g \in \mathrm{RV}_0(-E)$. Matrix regular variation was first considered by Balkema [1] and Meerschaert [10]. They proved that, if (2.1) holds, then we have *uniform convergence* in (2.1) on compact sets $\lambda \in [a, b]$ for $0 < a < b < \infty$ (e.g., see [11, Theorem 4.2.1]).

A sequence of matrices (C_n) is regularly varying at infinity with index E if the function $f(x) = C_{[x]}$ is in $\mathrm{RV}_\infty(E)$. This is equivalent to $C_{[\lambda n]}C_n^{-1} \to \lambda^E$ for all $\lambda > 0$; see [11, Theorem 4.2.9]. For matrix regular variation, a spectral decomposition reveals the power behavior. Factor the minimal polynomial of E into $f_1(x) \cdots f_p(x)$, where all roots of f_i have real part a_i, and $a_i < a_j$ for $i < j$. Define $V_i = \mathrm{Ker}(f_i(E))$. Then we can write $\mathbb{R}^m = V_1 \oplus \cdots \oplus V_p$, a direct sum decomposition of $\mathbb{R}^m$ into E-invariant subspaces, called the *spectral decomposition* of $\mathbb{R}^m$ with respect to E. The spectral decomposition of E is $E = E_1 \oplus \cdots \oplus E_p$, where $E_i : V_i \to V_i$, and every eigenvalue of E_i has real part a_i. The matrix for E in an appropriate basis is then block-diagonal with p blocks, the ith block corresponding to the matrix for E_i. This is a special case of the primary decomposition theorem of linear algebra (see, e.g., Curtis [7]).

Write $C_n \sim D_n$ for matrices $C_n, D_n \in \mathrm{GL}(\mathbb{R}^m)$ if $C_n D_n^{-1} \to I$, the identity matrix. The *spectral decomposition theorem* [11, Theorem 4.3.10] states that (C_n) varies regularly with index E if and only if $C_n \sim D_n T$ for some invertible matrix T and some (D_n) regularly varying with index E such that each V_i in the spectral decomposition of $\mathbb{R}^m$ with respect to E is D_n-invariant for all n, and $D_n = D_{1n} \oplus \cdots \oplus D_{pn}$, where each $D_{in} : V_i \to V_i$ is regularly varying with index E_i. We say that (D_n) is *spectrally compatible* with E. The role of T is clear, since $D_{[\lambda n]}T(D_n T)^{-1} = D_{[\lambda n]}D_n^{-1}$ for any T. Then for any nonzero $x \in V_i$, for any $\varepsilon > 0$, for some n_0 we have

$$n^{a_i-\varepsilon} < \|D_n x\| < n^{a_i+\varepsilon} \quad \text{for all } n \geq n_0;$$

see [11, Theorem 4.3.1]. Then $\|C_n x\|$ grows like a power, with an exponent depending on x.

3. Matrix Tauberian theorem

Let $u(x)$ be a nondecreasing right-continuous function defined on $x \geq 0$, and suppose that its Laplace-Stieltjes transform

$$\tilde{u}(s) := \int_0^\infty e^{-sx} u(dx) \tag{3.1}$$

exists for some $s > 0$. Karamata's Tauberian theorem (e.g., see [8, Theorem 1, XIII.5]) states that

$$u(x) \sim \frac{x^\rho \ell(x)}{\Gamma(1+\rho)} \quad \text{as } x \to \infty \quad \iff \quad \tilde{u}(s) \sim s^{-\rho}\ell(1/s) \quad \text{as } s \to 0, \tag{3.2}$$

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where $\rho \geq 0$ and $\ell(x)$ is slowly varying at infinity. In order to extend this result to matrix-valued Laplace transforms, we require the matrix gamma function, defined by

$$\Gamma(P) := \int_0^\infty y^{P-I} e^{-y} dy \tag{3.3}$$

for any matrix P whose eigenvalues $a + ib$ all satisfy $a > 0$.

Proposition 3.1. *$\Gamma(P)$ exists, is invertible, and $\Gamma(P + I) = P\Gamma(P) = \Gamma(P)P$.*

Proof. Let $0 < b_1 < \cdots < b_p$ denote the real parts of the eigenvalues of P. Then, by Theorem 2.2.4 of [11], for any $\delta > 0$ there exists a constant $K > 0$ such that $\|y^{P-I}\| \leq K y^{b_1-1-\delta}$ for $0 < y \leq 1$ and $\|y^{P-I}\| \leq K y^{b_p-1+\delta}$ for $y > 1$. Hence $\Gamma(P)$ is well defined. Since $\frac{d}{dt}(t^P) = Pt^{P-I}$, integration by parts yields $\Gamma(P+I) = P\Gamma(P)$. That $\Gamma(P)$ and P commute follows directly from (3.3). Finally, it follows from [9] that $\Gamma(P)$ is invertible. □

Given a sequence of matrices $(C_j) \in \mathrm{RV}_\infty(E)$, let $a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of E, and suppose that $a_1 > -1$. Define

$$U(x) := \sum_{j=0}^{[x]} C_j \tag{3.4}$$

for $x > 0$. The function $U(x)$ has the matrix-valued Laplace transform

$$\tilde{U}(s) := \int_0^\infty e^{-sx}\, U(dx) = \sum_{j=0}^\infty e^{-sj} C_j. \tag{3.5}$$

It follows from [11, Theorem 4.2.4] that for any $\delta > 0$, there exists a constant $K > 0$ such that $\|C_j\| \leq K j^{a_p+\delta}$ for all $j > 0$, and hence $\tilde{U}(s)$ exists for all $s > 0$. Our next goal is to show that regular variation of (C_j) implies regular variation of the function $U(x)$ at infinity, as well as regular variation of its Laplace transform $\tilde{U}(s)$ at zero. We begin by establishing two convergence results, which we will later prove are equivalent to regular variation.

Theorem 3.2. *Let $(C_n) \in \mathrm{RV}_\infty(E)$ and assume that every eigenvalue $a + ib$ of E has real part $a > -1$. Define $B_n = nC_n$. Then*

$$U(nx)B_n^{-1} \to \Phi(x) \tag{3.6}$$

uniformly on compact subsets of $\{x > 0\}$, where

$$\Phi(x) := \int_0^x s^E ds = P^{-1} x^P \tag{3.7}$$

is invertible for all $x > 0$, where $P := I + E$.

Before we give a proof of Theorem 3.2, we establish the existence of the limit in (3.7).

Lemma 3.3. *The function $\Phi(x)$ in (3.7) exists for all $x > 0$, and $\Phi(x) \to 0$ as $x \to 0$.*

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Proof. Choose $\delta > 0$ such that $a_1 - \delta > -1$. Then, by Theorem 2.2.4 of [11], there exists a constant $K > 0$ such that $\|s^E\| \le K s^{a_1-\delta}$ for any $0 < s \le 1$. Hence the integral in (3.7) exists. Moreover we have

$$\|\Phi(x)\| \le \int_0^x \|s^E\|\, ds \le K x^{1+a_1-\delta} \to 0$$

as $x \to 0$. It remains to evaluate that integral. It is well known that the function $x = \exp(tA)$ solves the linear system of differential equations $x' = Ax$. This and the chain rule imply that $f(t) = t^A = \exp(A \log t)$ has derivative $f'(t) = At^A(\log t)' = At^{A-I}$, and then the fundamental theorem of calculus yields (3.7). $\square$

The following two lemmas are essential for the proof of Theorem 3.2.

Lemma 3.4. *Given $\delta > 0$ such that $a_1 - \delta > -1$, there exists a constant $K > 0$ and a natural number k_0 such that*

$$\left\|C_k C_n^{-1}\right\| \le K \left(\frac{k}{n}\right)^{a_1-\delta} \tag{3.8}$$

for all $k_0 \le k \le n$.

Proof. Let E^t denote the transpose of the matrix E with respect to the usual Euclidean inner product. Since both E and E^t have the same eigenvalues, the real parts of the eigenvalues of $-E^t$ are $-a_p < \cdots < -a_1 < 1$. By [11, Theorem 2.2.4] there exists a $\lambda_0 > 1$ such that $\|\lambda_0^{-E^t}\| \le \lambda_0^{-a_1+\delta/2}$. Choose $\varepsilon_1 > 0$ such that $\lambda_0^{-a_1+\delta/2} + \varepsilon_1 \le \lambda_0^{-a_1+\delta}$. Since (C_n) is in $\mathrm{RV}_\infty(E)$, it follows easily that $(C_n^t)^{-1} = (C_n^{-1})^t$ is in $\mathrm{RV}_\infty(-E^t)$. Then for any $0 < a < b < \infty$ we have

$$(C_{[\lambda n]}^{-1})^t C_n^t \to \lambda^{-E^t} \quad \text{uniformly on } \lambda \in [a,b] \text{ as } n \to \infty \tag{3.9}$$

by the uniform convergence theorem [11, Theorem 4.2.1] for regularly varying matrices. Hence, there exists a $k_0 \ge 1$ such that

$$\left\|(C_{[\lambda k]}^{-1})^t C_k^t - \lambda^{-E^t}\right\| < \varepsilon_1$$

for all $1 \le \lambda \le \lambda_0$ and all $k \ge k_0$. Then, using the general fact that $\|A^t\| = \|A\|$, we get

$$\left\|C_k C_{[\lambda_0 k]}^{-1}\right\| \le \varepsilon_1 + \left\|\lambda_0^{-E^t}\right\| \le \lambda_0^{-a_1+\delta}$$

for all $k \ge k_0$. Given $k_0 \le k \le n$, write

$$\frac{n}{k} = \lambda_0^{m(n,k)} \mu_{n,k}$$

for some integer $m(n,k) \ge 0$ and $1 \le \mu_{n,k} < \lambda_0$. Using (3.9) again, there exists a constant $K > 0$ such that

$$\left\|C_{[\lambda_0^{m(n,k)}k]} C_{[\mu_{n,k}\lambda_0^{m(n,k)}k]}^{-1}\right\| \le K$$

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for all $k_0 \le k \le n$. Moreover

$$\begin{aligned}\left\|C_k C_n^{-1}\right\| &= \left\|C_k C_{[(n/k)k]}^{-1}\right\| \\ &= \left\|C_k C_{[\mu_{n,k}\lambda_0^{m(n,k)}k]}^{-1}\right\| \\ &\le \left\|C_k C_{[\lambda_0 k]}^{-1}\right\| \cdots \left\|C_{[\lambda_0^{m(n,k)-1}k]} C_{[\lambda_0^{m(n,k)}k]}^{-1}\right\| \left\|C_{[\lambda_0^{m(n,k)}k]} C_{[\mu_{n,k}\lambda_0^{m(n,k)}k]}^{-1}\right\| \\ &\le K\left(\lambda_0^{-a_1+\delta}\right)^{m(n,k)} \\ &\le K\left(\frac{n}{k}\right)^{-a_1+\delta}\end{aligned}$$

for all $k_0 \le k \le n$ and the proof is complete. □

Lemma 3.5. *For any $\delta > 0$ such that $a_1 - \delta > -1$, there exists a constant $K' > 0$ and a natural number n_0 such that*

$$\left\|\frac{1}{n}\sum_{k=0}^{[n\varepsilon]} C_k C_n^{-1}\right\| \le K'\varepsilon^{1+a_1-\delta} \tag{3.10}$$

for all $n \ge n_0$ and all $\varepsilon > 0$.

Proof. By Lemma 3.4, there exists a $k_0 \ge 1$ such that (3.8) holds for $k_0 \le k \le n$. Write

$$\begin{aligned}J_n^\varepsilon &:= \frac{1}{n}\sum_{k=0}^{[n\varepsilon]} C_k C_n^{-1} \\ &= \frac{1}{n}\sum_{k=0}^{k_0-1} C_k C_n^{-1} + \frac{1}{n}\sum_{k=k_0}^{[n\varepsilon]} C_k C_n^{-1} \\ &=: A_n + D_n^\varepsilon.\end{aligned}$$

Since $B_n := nC_n$ is in $\mathrm{RV}_\infty(E+I)$, and the real parts of the eigenvalues of $E+I$ are positive, we get using [11, Corollary 4.2.6] that

$$\|A_n\| \le \left\|\sum_{k=0}^{k_0-1} C_k^{-1}\right\| \, \|B_n^{-1}\| \to 0 \quad \text{as } n \to \infty.$$

Furthermore, by (3.8) we get

$$\begin{aligned}\|D_n^\varepsilon\| &\le \frac{1}{n}\sum_{k=k_0}^{[n\varepsilon]} \|C_k C_n^{-1}\| \\ &\le K\frac{1}{n}\sum_{k=k_0}^{[n\varepsilon]} \left(\frac{k}{n}\right)^{a_1-\delta} \\ &\le Kn^{-1-a_1+\delta}\int_{k_0-1}^{n\varepsilon} y^{a_1-\delta}\,dy \\ &\le K'\varepsilon^{1+a_1-\delta}\end{aligned}$$

for all large n, where $K' > 0$ is a constant independent of n. □

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Proof of Theorem 3.2. For $n \geq 1$ and $s \geq 0$ let

$$\psi_n(s) := \frac{1}{n} \sum_{k=0}^{\infty} (k+1) \mathbf{1}_{[k,k+1)}(s)$$

and observe that $\psi_n(ns) \to s$ as $n \to \infty$. Fix any $x > 0$. Then for any $0 < \varepsilon < x$ we can write

$$\begin{aligned} U(nx) B_n^{-1} &= \frac{1}{n} \sum_{k=0}^{[nx]} C_k C_n^{-1} \\ &= \int_0^{[nx]} C_{[t]} C_n^{-1} \, d\psi_n(t) \\ &= \int_0^{x_n} C_{[ns]} C_n^{-1} \, d\psi_n(ns) \\ &= \int_\varepsilon^{x_n} C_{[ns]} C_n^{-1} \, d\psi_n(ns) + \int_0^\varepsilon C_{[ns]} C_n^{-1} \, d\psi_n(ns) \\ &=: I_n^{x,\varepsilon} + J_n^\varepsilon, \end{aligned}$$

where $x_n := [nx]/n \to x$ as $n \to \infty$. By uniform convergence on compact subsets in (2.1) we get

$$C_{[ns]} C_n^{-1} \to s^E \quad \text{uniformly in } s \in [\varepsilon, x]$$

as $n \to \infty$. Then it follows by standard arguments that

$$I_n^{x,\varepsilon} \to \int_\varepsilon^x s^E \, ds \quad \text{as } n \to \infty.$$

By Lemma 3.3 we know that $\Phi(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then it follows from Lemma 3.5 that (3.6) holds.

To show uniform convergence, we have to show that whenever $x_n \to x > 0$ we have $U(nx_n) B_n^{-1} \to \Phi(x)$ as $n \to \infty$. Assume first that $x_n \downarrow x$. Then we have

$$U(nx_n) B_n^{-1} = U(nx) B_n^{-1} + \frac{1}{n} \sum_{k=[nx]+1}^{[nx_n]} C_k C_n^{-1}.$$

The argument for Lemma 3.5 yields

$$\left\| \frac{1}{n} \sum_{k=[nx]+1}^{[nx_n]} C_k C_n^{-1} \right\| \leq K \big(x_n^{1+a_1-\delta} - x^{1+a_1-\delta} \big) \to 0$$

as $n \to \infty$. The proof of the case $x_n \uparrow x$ is similar. □

Remark 3.6. In the scalar case $m = 1$, there is a partial converse to Theorem 3.2, called the monotone density theorem (e.g., see Feller [8, Theorem 4, XIII]): If the sequence (c_n) is eventually monotone, then regular variation of $u(x) = c_1 + \cdots + c_{[x]}$ with index ρ implies regular variation of a sequence c_n with index $\rho - 1$. If every element $[C_n]_{ij}$ forms a monotone sequence, then this result can be extended to sequences and series of matrices, with a very similar proof. However, this assumption is rather strong. For example, if every C_n is a diagonal matrix, and if the diagonal entries are all eventually monotone, then $U(x) \in \mathrm{RV}_\infty(P)$ implies that (C_n) is in $\mathrm{RV}_\infty(P - I)$. However, since each diagonal entry forms a regularly

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varying sequence of real numbers, everything reduces to the scalar case. A more general monotone density theorem for matrix-valued functions seems difficult.

Next we state and prove the analogue of Theorem 3.2 for the Laplace-Stieltjes transform. Note that $\Phi(dx) = s^E dx$ in light of (3.7).

Theorem 3.7. *Let $(C_n) \in \mathrm{RV}_\infty(E)$ and assume that every eigenvalue $a + ib$ of E has real part $a > -1$. Define $B_n := nC_n$. Then*

$$\tilde{U}(n^{-1}s)B_n^{-1} \to \tilde{\Phi}(s) \tag{3.11}$$

uniformly on compact subsets of $\{s > 0\}$, where

$$\tilde{\Phi}(s) := \int_0^\infty e^{-sx}\Phi(dx) = s^{-P}\Gamma(P) \tag{3.12}$$

for all $s > 0$, with $P := I + E$.

Before we prove Theorem 3.7, we need some preliminary results.

Lemma 3.8. *The integral $\tilde{\Phi}(s)$ in* (3.12) *exists, and*

$$\tilde{\Phi}(\lambda s) = \lambda^{-P}\tilde{\Phi}(s) \tag{3.13}$$

for all $s > 0$ and all $\lambda > 0$.

Proof. As in the proof of Lemma 3.3 we have

$$\left\| \int_0^1 e^{-sx} x^E \, dx \right\| \le \int_0^1 \|x^E\| \, dx < \infty.$$

By Theorem 2.2.4 of [11], for any $\delta > 0$ there exists a $K > 0$ such that $\|x^E\| \le Kx^{a_p+\delta}$ for all $x \ge 1$. Then we have

$$\left\| \int_1^\infty e^{-sx} x^E \, dx \right\| \le K \int_1^\infty e^{-sx} x^{a_p+\delta} \, dy < \infty,$$

so $\tilde{\Phi}(s)$ is well defined. Equation (3.12) follows from the definition (3.3) of the matrix gamma function, by a simple change of variable. Then (3.13) follows from (3.12). □

Lemma 3.9. *Given $\delta > 0$, there exists a constant $K > 0$ and an integer $n_0 \ge 1$ such that*

$$\left\|C_n C_k^{-1}\right\| \le K\left(\frac{k}{n}\right)^{a_p+\delta} \tag{3.14}$$

for all $n_0 \le n \le k$.

Proof. By Theorem 2.2.4 of [11], there exists a $\lambda_0 > 1$ such that $\|\lambda_0^E\| \le \lambda_0^{a_p+\delta/2}$. Choose $\varepsilon_1 > 0$ such that $\lambda_0^{a_p+\delta/2} + \varepsilon_1 \le \lambda_0^{a_p+\delta}$. By uniform convergence in (2.1), there exists an $n_0 \ge 1$ such that

$$\left\|C_{[\lambda n]}C_n^{-1} - \lambda^E\right\| < \varepsilon_1$$

for all $n \ge n_0$ and $1 \le \lambda \le \lambda_0$. Especially, for all $n \ge n_0$,

$$\left\|C_{[\lambda_0 n]}C_n^{-1}\right\| \le \varepsilon_1 + \|\lambda_0^E\| \le \lambda_0^{a_p+\delta}.$$

Now for $n_0 \le n \le k$ write

$$\frac{k}{n} = \lambda_0^{m(n,k)}\mu_{n,k}$$

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for some integer $m(n,k) \geq 0$ and some $1 \leq \mu_{n,k} < \lambda_0$. Then we have for $n_0 \leq n \leq k$,

$$\begin{aligned}
\|C_k C_n^{-1}\| &= \left\| C_{[\mu_{n,k}\lambda_0^{m(n,k)} n]} C_n^{-1} \right\| \\
&\leq \left\| C_{[\mu_{n,k}\lambda_0^{m(n,k)} n]} C^{-1}_{[\lambda_0^{m(n,k)} n]} \right\| \left\| C_{[\lambda_0^{m(n,k)} n]} C^{-1}_{[\lambda_0^{m(n,k)-1} n]} \right\| \cdots \left\| C_{[\lambda_0 n]} C_n^{-1} \right\| \\
&\leq K \left(\lambda_0^{a_p+\delta}\right)^{m(n,k)} \\
&\leq K \left(\frac{k}{n}\right)^{a_p+\delta}
\end{aligned}$$

using uniform convergence again. This concludes the proof. □

Proof of Theorem 3.7. Fix any $s > 0$. Given $0 < \varepsilon < M$, use the notation from the proof of Theorem 3.2 to write

$$\begin{aligned}
\tilde{U}(s/n) B_n^{-1} &= \frac{1}{n} \sum_{k=0}^{\infty} e^{-(s/n)k} C_k C_n^{-1} \\
&= \int_0^{\infty} e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny) \\
&= \int_0^{\varepsilon} e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny) + \int_{\varepsilon}^{M} e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny) \\
&\quad + \int_M^{\infty} e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny).
\end{aligned}$$

By uniform convergence on compact subsets in (2.1) we get by a standard argument that

$$\int_{\varepsilon}^{M} e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny) \to \int_{\varepsilon}^{M} e^{-sy} y^E\, dy$$

as $n \to \infty$.

Furthermore, as in the proof of Lemma 3.3 we have

$$\left\| \int_0^{\varepsilon} e^{-sy} y^E\, dy \right\| \leq \int_0^{\varepsilon} \|y^E\|\, dy \to 0 \quad \text{as } \varepsilon \to 0.$$

As in the proof of Lemma 3.8 we have

$$\left\| \int_M^{\infty} e^{-sy} y^E\, dy \right\| \leq K \int_M^{\infty} e^{-sy} y^{a_p+\delta}\, dy \to 0 \quad \text{as } M \to \infty.$$

Now

$$\left\| \int_0^{\varepsilon} e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny) \right\| = \left\| \frac{1}{n} \sum_{k=0}^{[n\varepsilon]} e^{-(s/n)k} C_k C_n^{-1} \right\| \leq \frac{1}{n} \sum_{k=0}^{[n\varepsilon]} \|C_k C_n^{-1}\|,$$

which can be made arbitrarily small, uniformly for all large n, if $\varepsilon > 0$ is chosen small enough, using Lemma 3.5.

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Finally, for any $\delta > 0$, by Lemma 3.9, there exists an $n_0 \geq 1$ such that (3.14) holds for all $n_0 \leq n \leq k$. Then, for $M \geq 1$ and $n \geq n_0$ we get that

$$\begin{aligned}\Big\|\int_M^\infty e^{-sy} C_{[ny]} C_n^{-1}\, d\psi_n(ny)\Big\| &= \Big\|\frac{1}{n}\sum_{k=[nM]}^{\infty} e^{-s(k/n)} C_k C_n^{-1}\Big\| \\ &\leq \frac{1}{n}\sum_{k=[nM]}^{\infty} e^{-s(k/n)} \|C_k C_n^{-1}\| \\ &\leq K\frac{1}{n}\sum_{k=[nM]}^{\infty} e^{-s(k/n)} \Big(\frac{k}{n}\Big)^{a_p+\delta} \\ &\leq K' n^{-1-a_p-\delta}\int_{[nM]-1}^{\infty} e^{-s(y/n)} y^{a_p+\delta}\, dy \\ &\leq K'\int_{M-1}^{\infty} e^{-su} u^{a_p+\delta}\, du,\end{aligned}$$

which can be made arbitrarily small if $M \geq 1$ is chosen large enough.

To prove uniform convergence let $s_n \downarrow s > 0$ and write

$$\begin{aligned}\tilde{U}(s/n)B_n^{-1} - \tilde{U}(s_n/n)B_n^{-1} &= \frac{1}{n}\sum_{k=0}^{n-1} e^{-s(k/n)}\Big(1 - e^{-(k/n)(s_n-s)}\Big) C_k C_n^{-1} \\ &\quad + \frac{1}{n}\sum_{k=n}^{\infty} e^{-s(k/n)}\Big(1 - e^{-(k/n)(s_n-s)}\Big) C_k C_n^{-1} \\ &=: E_n + F_n.\end{aligned}$$

Now, using $|1 - e^{-x}| \leq x$ for $x > 0$ we get

$$\|E_n\| \leq |s_n - s|\frac{1}{n}\sum_{k=0}^{n}\big\|C_k C_n^{-1}\big\|.$$

Using Lemma 3.5 with $\varepsilon = 1$, it follows that $E_n \to 0$ as $n \to \infty$. Moreover, using Lemma 3.9 again, there exists an $n_0 \geq 1$ such that (3.14) holds for all $n_0 \leq n \leq k$. Then we get

$$\begin{aligned}\|F_n\| &\leq |s_n - s|\,\frac{1}{n}\sum_{k=n}^{\infty} e^{-s(k/n)}\left(\frac{k}{n}\right)\big\|C_k C_n^{-1}\big\| \\ &\leq K|s_n - s|\,\frac{1}{n}\sum_{k=n}^{\infty} e^{-s(k/n)}\left(\frac{k}{n}\right)^{1+a_p+\delta} \\ &\leq K'|s_n - s|\end{aligned}$$

for some constant $K' > 0$. The proof of the case $s_n \uparrow s$ is similar. □

Equation (3.6) is a sequential version of the definition for regular variation. Our next goal is to show that this sequential definition is equivalent to the standard definition (2.1).

Theorem 3.10. *A Borel measurable function $U : \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^m)$ is $\mathrm{RV}_\infty(P)$ if and only if there exists a sequence (B_n) in $\mathrm{RV}_\infty(P)$ such that (3.6) holds uniformly on compact subsets of $x > 0$, for some $\Phi(x) \in \mathrm{GL}(\mathbb{R}^m)$. Then $\Phi(x) = x^P Q$ for some $Q \in \mathrm{GL}(\mathbb{R}^m)$.*

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The proof requires a simple lemma.

Lemma 3.11. *The function $U(x)$ in* (3.4) *is invertible for all $x > 0$ sufficiently large.*

Proof. Theorem 3.2 implies that $U(nx)B_n^{-1} \to \Phi(x)$ in the topological vector space $\mathrm{L}(\mathbb{R}^m)$ of $m \times m$ matrices with real entries, where $\Phi(x)$ is invertible. The set of invertible matrices is an open subset of $\mathrm{L}(\mathbb{R}^m)$, since it is the inverse image of the open set $\{y \in \mathbb{R} : y \neq 0\}$ under the determinant function, which is continuous. Then $U(nx)B_n^{-1}$ is invertible for all large n, and since B_n is also invertible, it follows that $U(x)$ is invertible for all large x. $\square$

Proof of Theorem 3.10. Suppose that (3.6) holds uniformly on compact subsets, with $\Phi(x) \in \mathrm{GL}(\mathbb{R}^m)$. Then for any $\lambda > 0$, using the uniform convergence, we have

$$U(\lambda x)U(x)^{-1} = \left(U(\lambda x)B_{[x]}^{-1}\right)\left(U(x)B_{[x]}^{-1}\right)^{-1} \to \Phi(\lambda)\Phi(1)^{-1} =: \Psi(\lambda)$$

as $x \to \infty$. Then [11, Theorem 4.1.2] implies that $\Psi(\lambda) = \lambda^R$ for some matrix R, and $U(x)$ is $\mathrm{RV}_\infty(R)$. Write

$$U(n\lambda x)B_n^{-1} = U(n\lambda x)B_{[\lambda n]}^{-1}\, B_{[\lambda n]}B_n^{-1}$$

and take limits to see that $\Psi(\lambda x) = \Psi(x)\lambda^P$, showing that $R = P$. Then $\Phi(x) = x^P Q$, where $Q = \Phi(1)$. Conversely, if $U(x)$ is $\mathrm{RV}_\infty(P)$ and (2.1) holds, then this convergence is also uniform on compact subsets of $x > 0$ by [11, Theorem 4.2.1], and so we have $U(nx)U(n)^{-1} \to x^P$ as $n \to \infty$, uniform on compact subsets. Then (3.6) holds uniformly on compact subsets with $B_n = U(n)$ and $\Phi(x) = x^P$. $\square$

Corollary 3.12. *A Borel measurable function $\tilde{U} : \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^m)$ is $\mathrm{RV}_0(-P)$ if and only if there exists a sequence (B_n) in $\mathrm{RV}_\infty(P)$ such that* (3.11) *holds uniformly on compact subsets of $s > 0$, for some $\tilde{\Phi}(s) \in \mathrm{GL}(\mathbb{R}^m)$. Then $\tilde{\Phi}(s) = s^{-P}Q$ for some $Q \in \mathrm{GL}(\mathbb{R}^m)$.*

Proof. Apply Theorem 3.10 to the function $f(x) = \tilde{U}(1/x)$. $\square$

Now we will state and prove the analogue of Karamata's Tauberian theorem for matrix-valued functions. The scalar result (3.2) implies that $u(x)$ varies regularly at infinity with index $\rho \geq 0$ if and only if $\tilde{u}(s)$ varies regularly at zero with index $-\rho$, and in either case

$$\tilde{u}(1/x) \sim \Gamma(\rho + 1)u(x) \quad \text{as } x \to \infty.$$

In the matrix version of this result, the index ρ becomes a matrix P, whose eigenvalues $a+ib$ all satisfy $a > 0$. Suppose that $U : \mathbb{R}^+ \to \mathrm{GL}(\mathbb{R}^m)$ is Borel measurable and that every component $[U(x)]_{ij}$ is of bounded variation. Assume that the Laplace transform

$$\tilde{U}(s) := \int_0^\infty e^{-sx} U(dx)$$

exists for any $s > 0$. Here the integral is defined as a Stieltjes integral componentwise.

Theorem 3.13 (Karamata theorem for matrices)**.** *Suppose that every eigenvalue $a + ib$ of P has $a > 0$. Then $U(x)$ is $\mathrm{RV}_\infty(P)$ if and only if $\tilde{U}(s)$ is $\mathrm{RV}_0(-P)$, and in either case*

$$\tilde{U}(1/x) \sim \Gamma(P + I)U(x) \quad \text{as } x \to \infty. \tag{3.15}$$

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Proof. Suppose $U(x)$ is $\mathrm{RV}_\infty(P)$ and that every component $[U(x)]_{ij}$ is of bounded variation. Then Theorem 3.10 implies that (3.6) holds uniformly on compact subsets of $x > 0$ for some sequence (B_n) in $\mathrm{RV}_\infty(P)$, with $\Phi(x) = x^P Q$ for some $Q \in \mathrm{GL}(\mathbb{R}^m)$. Let $b_1 < \cdots < b_p$ be the real parts of the eigenvalues of P. It follows from [11, Theorem 4.2.4] that for any $\delta > 0$, there exists a constant $K > 0$ such that $\|U(x)\| \le K x^{b_p+\delta}$ for all $x > 0$, and hence $\tilde{U}(s)$ exists for all $s > 0$. Applying the continuity theorem for Laplace transforms (e.g., see Feller [8, Theorem 2a, p. 433]) component-wise, we obtain by a simple change of variables

$$\begin{aligned}\tilde{U}(s/n)B_n^{-1} &= \int_0^\infty e^{-(s/n)x}\, U(dx)B_n^{-1} \\ &= \int_0^\infty e^{-sy}\, U(ndy)B_n^{-1} \to \int_0^\infty e^{-sy}\Phi(dy) = \tilde{\Phi}(s)\end{aligned}$$

for any $s > 0$, where

$$\tilde{\Phi}(s) = \int_0^\infty e^{-sy} P y^{P-I} Q dy = \int_0^\infty e^{-x} x^{P-I} s^{I-P} Q s^{-1} dx = s^{-P}\Gamma(P+I)Q.$$

To prove uniform convergence, given $s_n \to s > 0$, let $x_n = (s_n/s)x \to x$ and substitute $y_n = x_n/n$ to get

$$\begin{aligned}\tilde{U}(s_n/n)B_n^{-1} &= \int_0^\infty e^{-(s/n)x_n}\, U(dx)B_n^{-1} \\ &= \int_0^\infty e^{-sy_n}\, U(ndy_n)B_n^{-1} \to \int_0^\infty e^{-sy}\Phi(dy) = \tilde{\Phi}(s)\end{aligned}$$

since $U(ny_n)B_n^{-1} \to \Phi(y)$. Then Corollary 3.12 implies that $\tilde{U}(s)$ is $\mathrm{RV}_0(-P)$.

Conversely, suppose that $\tilde{U}(s)$ is $\mathrm{RV}_0(-P)$. Then Corollary 3.12 and Proposition 3.1 imply that (3.11) holds uniformly on compact subsets of $x > 0$ for some sequence (B_n) in $\mathrm{RV}_\infty(P)$, with $\tilde{\Phi}(s) = s^{-P}\Gamma(P+I)Q$ for some $Q \in \mathrm{GL}(\mathbb{R}^m)$. A simple change of variables yields

$$\tilde{U}(s/n)B_n^{-1} = \int_0^\infty e^{-sx} U(dx)B_n^{-1} = \int_0^\infty e^{-sy} U(ndy)B_n^{-1} \to \tilde{\Phi}(s) = \int_0^\infty e^{-sy}\Phi(dy)$$

as $n \to \infty$. Applying the continuity theorem for Laplace transforms component-wise, we obtain $U(ndy)B_n^{-1} \to \Phi(dy)$. Integrating over the set $[0,x]$ yields $U(nx)B_n^{-1} \to \Phi(x)$, uniformly on compact subsets of $x > 0$. Then Theorem 3.10 implies that $U(x)$ is $\mathrm{RV}_\infty(P)$.

In either case, we have from (3.6) with $\Phi(x) = x^P Q$ that

$$U(x)B_{[x]}^{-1} \to Q \quad \text{as } x \to \infty.$$

We also have from (3.11) with $\tilde{\Phi}(s) = s^{-P}\Gamma(P+I)Q$ and $s = 1/x$ that

$$\tilde{U}(1/x)B_{[x]}^{-1} \to \tilde{\Phi}(1) = \Gamma(P+I)Q \quad \text{as } x \to \infty,$$

and hence, in view of Lemma 3.11 it follows that

$$\tilde{U}(1/x)U(x)^{-1} = \tilde{U}(1/x)B_{[x]}^{-1}\,(U(x)B_{[x]}^{-1})^{-1} \to \Gamma(P+I)QQ^{-1} \quad \text{as } x \to \infty.$$

Then using Proposition 3.1 we get

$$\tilde{U}(1/x)U(x)^{-1}\Gamma(P+I)^{-1} \to I \quad \text{as } x \to \infty,$$

which is the same as (3.15). $\square$

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4. Sharp growth bounds

In this section, we prove sharp bounds on the growth behavior of the function $U(x)$ in (3.4) and its Laplace transform $\tilde{U}(s)$ in (3.5), assuming that the underlying sequence of matrices (C_n) is in $\mathrm{RV}_\infty(E)$, where every eigenvalue $a+ib$ of E has real part $a>-1$. Recall the spectral decomposition $\mathbb{R}^m = V_1 \oplus \cdots \oplus V_p$ and $E = E_1 \oplus \cdots \oplus E_p$, where $E_i : V_i \to V_i$ and every eigenvalue of E_i has real part a_i. Apply [11, Corollary 4.3.12] to obtain a matrix T_0 and a regularly varying sequence (G_n) such that $T_0 C_n \sim G_n$, where (G_n) is $\mathrm{RV}_\infty(E_0)$ with $E_0 = T_0 E T_0^{-1}$ and:

(a) The subspaces $W_i = T_0(V_i)$ in the spectral decomposition of $\mathbb{R}^m$ with respect to E_0 are mutually orthogonal.
(b) These subspaces are also G_n invariant.
(c) If we write $E_0 = E_{10} \oplus \cdots \oplus E_{p0}$ and $G_n = G_{1n} \oplus \cdots \oplus G_{pn}$ with respect to this direct sum decomposition, then each $G_{in} : W_i \to W_i$ is regularly varying with index $E_{i0} = T_0 E_i T_0^{-1}$.
(d) Every eigenvalue of E_{i0} has real part a_i.
(e) If $x_n \to x$ in $W_1 \oplus \cdots \oplus W_i$, then $n^{-\rho}\|G_n x_n\| \to 0$ for all $\rho > a_i$.
(f) If $x_n \to x \neq 0$ in $\mathbb{R}^m \setminus (W_1 \oplus \cdots \oplus W_i)$, then $n^{-\rho}\|G_n x_n\| \to \infty$ for all $\rho < a_{i+1}$.

Define

$$\alpha(\theta) := \max\{a_j : \theta_j \neq 0\},$$

for $\theta \neq 0$, where we write $\theta = \theta_1 + \cdots + \theta_p$ with respect to the spectral decomposition $W_1 \oplus \cdots \oplus W_p$. The following result gives a sharp bound on the growth in different directions for the regularly varying matrix-valued function $U(x)$.

Theorem 4.1. *For any direction $\theta \neq 0$ and any $\delta > 0$ small, there exist $m, M > 0$ and $x_0 > 0$ such that*

$$m x^{1+\alpha(\theta)-\delta} \le \|U(x)\theta\| \le M x^{1+\alpha(\theta)+\delta} \tag{4.1}$$

for all $x \ge x_0$.

Proof. Since all norms are equivalent in $\mathbb{R}^m$, it suffices to consider the Euclidean norm. Write

$$U(x)\theta = \left(U(x)B_{[x]}^{-1}\right)\, B_{[x]}\theta$$

so that

$$\begin{aligned}\|U(x)\theta\| &= \left\|\left(U(x)B_{[x]}^{-1}\right)\, B_{[x]}\theta\right\| \\ &\le \left\|U(x)B_{[x]}^{-1}\right\|\, \left\|B_{[x]}\theta\right\|.\end{aligned}$$

In view of Theorem 3.2, there exists a constant $K > 0$ such that

$$\|U(x)B_{[x]}^{-1}\| \le K$$

for all $x > 0$. Then it remains to obtain an upper bound on $\|B_{[x]}\theta\|$. By definition, $\alpha(\theta) = a_i$ if and only if both $\theta \in W_1 \oplus \cdots \oplus W_i$ and $\theta \in \mathbb{R}^m \setminus (W_1 \oplus \cdots \oplus W_{i-1})$. Recall that $B_n = nC_n \sim nT_0^{-1}G_n$. By property (e), there exists $K > 0$ such that

$$\|G_n\theta\| \le K n^{a_i+\delta}$$

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for all large n. Since $C_n \sim D_n$ for matrices implies that $\|C_n x\| \sim \|D_n x\|$ for all $x \in \mathbb{R}^m$, it follows that

$$\left\|B_{[x]}\theta\right\| \sim [x] \ \left\|T_0^{-1}G_{[x]}\theta\right\| \leq K\|T_0^{-1}\|x^{1+a_i+\delta}$$

for all large x. This proves the upper bound in (4.1).

For the proof of the lower bound, use the general fact that $\|Ax\| \geq \|x\|/\|A^{-1}\|$ for all $x \in \mathbb{R}^m$ to write

$$\|U(x)\theta\| = \left\|U(x)B_{[x]}^{-1} \ B_{[x]}\theta\right\| \geq \frac{\left\|B_{[x]}\theta\right\|}{\left\|(U(x)B_{[x]}^{-1})^{-1}\right\|} \geq M\left\|B_{[x]}\theta\right\|$$

for some constant $M > 0$, since $(U(x)B_{[x]}^{-1})^{-1} \to \Phi(1)^{-1}$ as $x \to \infty$ by Theorem 3.2. By property (f), there exists $K > 0$ such that

$$\left\|G_n\theta\right\| \geq Kn^{a_i-\delta}$$

for all large n. Then

$$\left\|B_{[x]}\theta\right\| \sim [x] \ \left\|T_0^{-1}G_{[x]}\theta\right\| \geq K\|T_0^{-1}\|x^{1+a_i-\delta}$$

for all large x. This proves the lower bound in (4.1), and hence the proof is complete. $\square$

Next we prove sharp growth bounds on the behavior of the matrix-valued Laplace transform $\tilde{U}(s)$ near zero.

Theorem 4.2. *For any direction $\theta \neq 0$ and any $\delta > 0$ small, there exist constants $m, M > 0$ and $s_0 > 0$ such that*

$$ms^{-1-\alpha(\theta)+\delta} \leq \|\tilde{U}(s)\theta\| \leq Ms^{-1-\alpha(\theta)-\delta} \tag{4.2}$$

for all $0 < s < s_0$.

Proof. For $x > 0$ write

$$\tilde{U}(1/x)\theta = \left(\tilde{U}(1/x)B_{[x]}^{-1}\right) \ B_{[x]}\theta$$

and observe that by Theorem 3.7 we have $\tilde{U}(1/x)B_{[x]}^{-1} \to \Gamma(I+E)$ as $x \to \infty$. Then, it follows as in Theorem 4.1 that, for some constants $m, M > 0$ and $x_0 > 0$, we have

$$mx^{1+\alpha(\theta)-\delta} \leq \|\tilde{U}(1/x)\theta\| \leq Mx^{1+\alpha(\theta)+\delta}$$

for all $x \geq x_0$. Setting $x = 1/s$, the result follows. $\square$

5. Applications

This paper was motivated by a problem in time series analysis. Some recent papers of Barbe and McCormack [3, 4] apply regular variation to model linear processes

$$X_t := \sum_{j=0}^{\infty} c_j Z_{t-j},$$

where (Z_j) is a sequence of iid random variables. In time series analysis, it is common to represent the linear process $X_t = p(B)Z_t$, where $p(z) := \sum_j c_j z^j$, using

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the backward shift operator $BZ_t = Z_{t-1}$. For example, in the FARIMA$(0,d,0)$ process we take

$$c_j = w_j^{(d)} := \binom{-d}{j}(-1)^j \sim \frac{d}{\Gamma(1+d)} j^{\,d-1} \quad \text{as } j \to \infty$$

so that $p(B) = (1-B)^{-d}$, X_t is the fractional integral of the noise sequence Z_t, and we take $0 < d < 1/2$ for long range dependence [6]. The more general approach in [3, 4] uses a regularly varying sequence c_j with the same index $d-1$. The analysis of these long range dependent times series relies on a Tauberian theorem, Theorem 5.1.1 in [4] (see also Corollary 1.7.3 in [5]), that relates the function $u(x) = c_0 + \cdots + c_{[x]}$ to its Laplace transform $\tilde{u}(\lambda) = \sum_j e^{-\lambda j} c_j$ using regular variation. The connection to Laplace transforms comes from $p(z) = \tilde{u}(-\ln z)$.

For vector time series, it is natural to consider the linear process

$$X_t := \sum_{j=0}^{\infty} C_j Z_{t-j}, \tag{5.1}$$

where the Z_j are iid random vectors, and the C_j are matrices. Then we can write $X_t = p(B)Z_t$, where $p(z) = \sum_j C_j z^j$. For example, a vector FARIMA time series with a different order of fractional integration in each coordinate can be defined using

$$C_j = \operatorname{diag}[w_j^{(d_1)}, \ldots, w_j^{(d_p)}] = \begin{pmatrix} w_j^{(d_1)} & & \\ & \ddots & \\ & & w_j^{(d_p)} \end{pmatrix}.$$

Then C_j varies regularly with index $E = \operatorname{diag}[d_1 - 1, \ldots, d_p - 1]$ and the eigenvalues of E are $(d_j - 1) \in (-1, -1/2)$. In this example, we have

$$p(s) = \operatorname{diag}[(1-s)^{-d_1}, \ldots, (1-s)^{-d_p}]$$

and the ith coordinate of X_t is a FARIMA$(0, d_i, 0)$ time series, where we emphasize that d_i varies with the coordinate. The Laplace transform

$$\tilde{U}(s) = \sum_{j=0}^{\infty} e^{-sj} C_j$$

exists for $s > 0$, and clearly we have

$$p(1-s) = \tilde{U}(-\log(1-s)) \sim \tilde{U}(s) \quad \text{as } s \to 0. \tag{5.2}$$

Since $p(1-s) = \operatorname{diag}[s^{-d_1}, \ldots, s^{-d_p}]$ varies regularly at zero with index $-P = -I - E$, so does $\tilde{U}(s)$, consistent with Corollary 3.12.

The vector time series (5.1), where $(C_j) \in \mathrm{RV}_\infty(E)$, and every eigenvalue $a+ib$ of E has real part $a \in (-1, -1/2)$, provides a flexible model for long range dependence. The strength of the long range dependence varies with the coordinate, and the coordinate system is completely arbitrary. The convergence and other properties of the moving average (5.1) depend on analysis of the matrix-valued power series $p(1-s) = \sum_j C_j (1-s)^j$ as $s \to 0$; see Barbe and McCormack [3, 4] for the scalar case. It follows from (5.2) and the matrix Tauberian theorem, Theorem 3.13, that

$$p(1-s) \sim U(1/s)\Gamma(P+I) \quad \text{as } s \to 0,$$

where $P = E + I$, so that $p(1-s)$ is regularly varying at zero with index $-P$. This indicates one possible application of the results in this paper. Since vector regular

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variation has proven useful in many areas (e.g., see Balkema and Embrechts [2] for a regular variation approach to extreme value theory in $\mathbb{R}^m$), it is possible that the results of this paper will also find applications in other contexts.

Remark 5.1. For modeling purposes, we are free to choose the sequence (C_n) in (5.1). Theorems 4.1 and 4.2 illustrate the advantage of choosing the regularly varying sequence to be spectrally compatible with its index E. Then the sharp growth bounds in those results are governed by the spectral decomposition of E. Specifically, one can take $W_i = V_i$ in the definition of the index function $\alpha(\theta)$, and $G_n = C_n$ in the proofs.

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