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PERIODIC MOVING AVERAGES OF RANDOM VARIABLES WITH REGULARLY VARYING TAILS

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In this paper we establish the basic asymptotic theory for periodic moving averages of i.i.d. random variables with regularly varying tails. The moving average coefficients are allowed to vary according to the season. A simple reformulation yields the corresponding results for moving averages of random vectors. Our main result is that when the underlying random variables have finite variance but infinite fourth moment, the sample autocorrelations are asymptotically stable. It is well known in this case that sample autocorrelations in the classical stationary moving average model are asymptotically normal.

Introduction. Regular variation is used to characterize those i.i.d. sequences of random variables for which a version of the central limit theorem holds. When these random variables have infinite variance, the sum is asymptotically stable instead of asymptotically normal. Stable random variables have found many practical applications beginning with the work of Holtsmark (1919) on gravitation. Elegant scaling properties of these distributions and the fact that the sample paths of the associated stochastic processes are random fractals form the basis for an impressive array of physical applications found in Mandelbrot (1963). Infinite variance noise processes are important in electrical engineering; see, for example, Stuck and Kleiner (1974) and Rybin (1978). Mandelbrot (1963) and Fama (1965) argued that variations in stock market prices should be modeled as stable random variables. Taylor (1986) recounted the controversy among economists over the use of stable laws in modeling economic time series. If a random variable X has regularly varying tails with index $-\alpha$, then $P[|X| > t] \to 0$ about as fast as $t^{-\alpha}$. We say that X has heavy tails, since in this case $E|X|^{\beta} = \infty$ for all $\beta > \alpha$. Jansen and de Vries (1991) invoke a heavy tail model to explain the stock market crash of 1987. They calculate that for many stock price returns the parameter α is between 2 and 4, which makes large fluctuations in price more likely than standard models or intuition would suggest. Loretan and Phillips (1994) demonstrate that fluctuations in aggregate stock market returns and currency exchange rates also exhibit heavy tails with α between 2 and 4. The recent book of Mittnik and Rachev (1995) provides details on heavy tail models in finance, including recent developments in the theory of option pricing. Resnick and Stărică (1995) show that the duration of quiet pe-

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riods between communications for a networked terminal has heavy tails with an infinite mean. Janicki and Weron (1994a) survey applications of stable laws and stable processes in economics, physics and geology. A modern reference on stable laws and processes is Samorodnitsky and Taqqu (1994). Janicki and Weron (1994b) discuss practical methods for simulating stable stochastic processes.

Davis and Resnick (1985a, b, 1986) compute the asymptotic distribution of the sample autocovariance and sample autocorrelation for moving averages of random variables with regularly varying tails. They employ methods which elucidate the Poisson nature of the underlying point process. A summary of these results along with some practical applications can be found in Brockwell and Davis [(1991), Section 13.3]. Brockwell and Davis advised that "any time series which exhibits sharp spikes or occasional bursts of outlying observations suggests the possible use" of these methods. These results also form the basis for the analysis of ARMA models with infinite variance innovations in Kokoszka and Taqqu (1994) and Mikosch, Gadrich, Klüppenberg and Adler (1995). Kokoszka (1996) and Kokoszka and Taqqu (1996) discuss prediction and parameter estimation for infinite variance fractional ARIMA models. Bhansali (1993) gives a general method for parameter estimation for linear infinite variance processes. Asymptotic results for the sample autocovariances and sample autocorrelations of periodic ARMA processes have been derived by Tjostheim and Paulsen (1982) and Anderson and Vecchia (1993), but only in the case where the noise sequence has finite fourth moment. Adams and Goodwin (1995) discuss parameter estimation for the periodic ARMA model with finite fourth moments. Forecasting for this model including the multivariate case is considered in Ula (1993). Gardner and Spooner (1994) include an extensive review of results on periodic time series models with finite fourth moments and their applications in signal processing. Tiao and Grupe (1980) demonstrate the pitfalls of ignoring seasonal behavior in time series modeling. Seasonal variations in the mean of time series data can easily be removed by a variety of methods, but when the variance (or dispersion in the infinite variance case) as well as the mean varies with the season, then the use of periodic time series models is indicated. If the data also indicate heavy tails, then the methods of this paper are relevant.

In this paper we develop the basic asymptotic theory for periodic moving averages of random variables with regularly varying tails. The moving average coefficients are allowed to vary according to the season. A simple reformulation yields the corresponding results for moving averages of random vectors with heavy tails. Our main focus is on the case where the underlying distribution possesses a finite variance but an infinite fourth moment (the case $2 < \alpha < 4$). This case is of considerable practical importance in economics. Our main result in this case is that for periodic moving averages the sample autocorrelations are asymptotically stable with index $\alpha/2$. It is well known [see, for example, Brockwell and Davis (1991), Proposition 7.3.8] that in this case the sample autocorrelations for the classical moving average model are asymptotically normal. This paradoxical result occurs because of a cancellation in the formula for the classical case which does not occur in the periodic case; see the remark following the Proof of Theorem 3.1.

In Section 1, we compute the asymptotic distribution of periodic moving averages of random variables with regularly varying tails. We show that the asymptotics of moving averages are essentially the same as for the underlying i.i.d. sequence. In Section 2, we compute the asymptotic distribution of the sample autocovariance function of a periodic moving average. In Section 3, we apply the results of Section 2 to obtain the asymptotic distribution of the sample autocorrelation function. In Section 4, we reformulate our results in terms of vector moving averages. These results should provide a useful guide to further research. Section 5 discusses the representation of a periodic ARMA model as a periodic moving average and the application of our results to these models.

1. Periodic moving averages. In this section we discuss the asymptotic behavior of moving averages of an i.i.d. sequence with regularly varying tails, assuming that the moving average coefficients vary according to the season. We will call X_t a periodic moving average if

$$X_t = \sum_{j=-\infty}^{\infty} \psi_t(j)\varepsilon_{t-j}, \tag{1.1}$$

where $\psi_t(j)$ is periodic in t with the same period $\nu \geq 1$ for all j and ε_t is an i.i.d. sequence of random variables. We will say that the i.i.d. sequence ε_t is $\mathrm{RV}(\alpha)$ if $P[|\varepsilon_t| > x]$ varies regularly with index $-\alpha$ and $P[\varepsilon_t > x]/P[|\varepsilon_t| > x] \to p$ for some $p \in [0, 1]$. The periodic moving average (1.1) of an $\mathrm{RV}(\alpha)$ sequence converges almost surely provided that

$$\sum_{j=-\infty}^{\infty} |\psi_t(j)|^{\delta} < \infty \tag{1.2}$$

for all t and for some $\delta < \alpha$ with $\delta \leq 1$; compare Brockwell and Davis [(1991), Proposition 13.3.1]. In this paper we will always assume that (1.2) holds, so that (1.1) is well defined.

If ε_t is an $\mathrm{RV}(\alpha)$ sequence, then $E|\varepsilon_t|^{\beta}$ exists for $0 < \beta < \alpha$ and is infinite for $\beta > \alpha$. For $0 < \alpha < 2$ the sequence ε_t belongs to the domain of attraction of a stable law with index α; see Feller (1971). If $\alpha \geq 2$, then ε_t belongs to the domain of attraction of a normal law. The following result shows that periodic moving averages have essentially the same asymptotic behavior as the underlying $\mathrm{RV}(\alpha)$ sequence. When $E\varepsilon_t^2 < \infty$ this follows directly from the central limit theorem, and so we only consider the case where the underlying variables ε_t have an infinite variance. When $\nu = 1$ our result reduces to the classical case considered by Davis and Resnick (1985a, 1986), but our proof does not require their point process machinery. While these results are of some independent interest, they will also be used to compute the asymptotic distribution of the sample autocovariance in the next section. The proofs of Theorem 1.1 and Corollary 1.2 are straightforward but technical; see the Appendix. We use $\Rightarrow$ to denote convergence in distribution.

THEOREM 1.1. *Suppose that* $Y_t = \sum_j c_t(j) Z_{t-j}$ *is a periodic moving average of some* $RV(\beta)$ *sequence* Z_t *with* $EZ_t^2 = \infty$. *Then for some* $d_N \to \infty$ *and some i.i.d. random variables* $S_0, \dots, S_{\nu-1}$ *we have*

$$(1.3) \qquad Nd_N^{-1}\left(N^{-1}\sum_{t=0}^{N-1} Y_{t\nu+i} - \sum_{j=-\infty}^{\infty} c_i(j) b_N\right) \Rightarrow \sum_{r=0}^{\nu-1} C_{i,r} S_r,$$

where $b_N = EZ_t I(|Z_t| \le d_N)$ *and* $C_{i,r} = \sum_j c_i(j\nu + i - r)$. *If* $\beta < 2$, *we can choose* d_N *to satisfy* $NP[|Z_t| > d_N] \to 1$ *and* $S_0, \dots, S_{\nu-1}$ *i.i.d. stable with index* β. *If* $\beta = 2$, *we can choose* d_N *to satisfy* $Nd_N^{-2} EZ_t^2 I(|Z_t| \le d_N) \to 1$ *and* $S_0, \dots, S_{\nu-1}$ *i.i.d. normal.*

If $1 < \beta \le 2$, we can also take $b_N = EZ_t$. Then if $\beta = 2$, the limit in (1.3) is normal with mean zero and variance $\sum_r C_{i,r}^2$, and if $1 < \beta < 2$, the limit in (1.3) is stable with index β, mean zero and dispersion $\sum_r |C_{i,r}|^\beta$. If $0 < \beta < 1$, we can also take $b_N = 0$ and then the limit in (1.3) is centered stable with index β and dispersion $\sum_r |C_{i,r}|^\beta$. The skewness of the stable limit in (1.3) is the same as for the stable limit of the underlying RV(α) sequence.

COROLLARY 1.2. *Suppose that* $Y_t(k) = \sum_j c_t(j,k) Z_{t-j}$ *for* $k = 0, \dots, h-1$ *are period* ν *moving averages of some* $RV(\beta)$ *sequence* Z_t *with* $EZ_t^2 = \infty$. *Then*

$$(1.4) \qquad Nd_N^{-1}\left(N^{-1}\sum_{t=0}^{N-1} Y_{t\nu+i}(k) - \sum_{j=-\infty}^{\infty} c_i(j,k) b_N\right) \Rightarrow \sum_{r=0}^{\nu-1} C_{i,r}(k) S_r$$

jointly over all seasons $i = 0, \dots, \nu-1$ *and all* $k = 0, \dots, h-1$, *where* b_N, d_N *and* S_r *are as in Theorem* 1.1 *and* $C_{i,r}(k) = \sum_j c_i(j\nu + i - r, k)$.

2. The sample autocovariance. In this section we compute the asymptotic distribution of the sample autocovariance of the periodic moving average X_t defined in (1.1), where ε_t is RV(α) and $E\varepsilon_t^4 = \infty$. When $E\varepsilon_t^4 < \infty$, the results of Anderson and Vecchia (1993) apply. Recall that if ε_t is RV(α), then $E|\varepsilon_t|^\beta$ is finite for $0 < \beta < \alpha$ and infinite for $\beta > \alpha$, so that we need only consider the case $0 < \alpha \le 4$. We define the sample autocovariance at season i and lag ℓ by

$$(2.1) \qquad \hat{\gamma}_i(\ell) = N^{-1} \sum_{n=0}^{N-1} X_{n\nu+i} X_{n\nu+i+\ell}.$$

By substituting (1.1) into (2.1) we obtain an expression for the sample autocovariance in terms of the errors ε_t. Our first result shows that in this formula the ε_t^2 terms dominate. The proof is straightforward, but technical; see the Appendix.

LEMMA 2.1. *Suppose that ε_t is RV(α) with $E\varepsilon_t^4 = \infty$ and that $E\varepsilon_t = 0$ if $\alpha \geq 2$. Then for some $a_N \to \infty$ we have*

$$(2.2) \qquad Na_N^{-2}\left(\hat{\gamma}_i(\ell) - N^{-1}\sum_{t=0}^{N-1}\sum_{j=-\infty}^{\infty}\psi_i(j)\psi_{i+\ell}(j+\ell)\varepsilon_{t\nu+i-j}^2\right) \to_P 0.$$

If $\alpha < 4$, we can choose a_N to satisfy $NP[|\varepsilon_t| > a_N] \to 1$ and if $\alpha = 4$, we can choose a_N to satisfy $Na_N^{-4}E\varepsilon_t^4 I(|\varepsilon_t| \leq a_N) \to 1$.

Next we present the main result of this section, which gives the asymptotic distribution of the sample autocovariance in the case $E\varepsilon_t^4 = \infty$. The analogous results for the classical moving average of an RV(α) sequence can be found in Davis and Resnick (1985a, 1986). For $0 < \alpha < 2$, we have $E\varepsilon_t^2 = \infty$ and the autocovariance $\gamma_t(\ell) = \text{cov}(X_t, X_{t+\ell})$ is undefined. For $\alpha > 2$, we have $\sigma^2 = E\varepsilon_t^2 < \infty$ and the autocovariance can be written in the form

$$(2.3) \qquad \gamma_t(\ell) = \sigma^2\sum_j \psi_t(j)\psi_{t+\ell}(j+\ell).$$

THEOREM 2.2. *Suppose that X_t is a periodic moving average of an RV(α) sequence ε_t with $E\varepsilon_t^4 = \infty$ and that $E\varepsilon_t = 0$ if $\alpha \geq 2$. Then for some $a_N \to \infty$ we have*

$$(2.4) \qquad Na_N^{-2}\left(\hat{\gamma}_i(\ell) - \sigma_N^2\sum_{j=-\infty}^{\infty}\psi_i(j)\psi_{i+\ell}(j+\ell)\right) \Rightarrow \sum_{r=0}^{\nu-1} C_r(i,\ell)S_r,$$

where $\sigma_N^2 = E\varepsilon_t^2 I(|\varepsilon_t| \leq a_N)$ and $C_r(i,\ell) = \sum_j \psi_i(j\nu + i - r)\psi_{i+\ell}(j\nu + i + \ell - r)$. If $\alpha < 4$, then we can choose a_N to satisfy $NP[|\varepsilon_t| > a_N] \to 1$ and $S_0, S_1, \ldots, S_{\nu-1}$ i.i.d. stable with index $\alpha/2$. If $\alpha = 4$, then we can choose a_N to satisfy $Na_N^{-4}E\varepsilon_t^4 I(|\varepsilon_t| \leq a_N) \to 1$ and $S_0, S_1, \ldots, S_{\nu-1}$ i.i.d. normal.

PROOF. If ε_t is RV(α), then $Z_t = \varepsilon_t^2$ is RV($\alpha/2$). Apply Theorem 1.1 with $d_N = a_N^2$, $c_t(j) = \psi_t(j)\psi_{t+\ell}(j+\ell)$ and $\beta = \alpha/2$ to obtain

$$(2.5) \qquad Na_N^{-2}\left(N^{-1}\sum_{t=0}^{N-1}\sum_{j=-\infty}^{\infty} c_i(j)\varepsilon_{t\nu+i-j}^2 - \sigma_N^2\sum_{j=-\infty}^{\infty} c_i(j)\right) \Rightarrow \sum_{r=0}^{\nu-1} C_{i,r}S_r$$

and then apply Lemma 2.1 to see that (2.4) holds. □

REMARKS. Note that the norming sequence in (2.4) varies regularly with index $1 - 2/\alpha$. If $2 < \alpha \leq 4$, we can also substitute $\sigma_N^2 = \sigma^2 = E\varepsilon_t^2$ in (2.4). Then the left-hand side of (2.4) becomes $Na_N^{-2}(\hat{\gamma}_i(\ell) - \gamma_i(\ell))$ and the limit on the right-hand side is normal with mean zero and variance $\sum_r C_r(i,\ell)^2$ if $\alpha = 4$ and stable with index $\alpha/2$, mean zero, skewness 1 and dispersion $\sum_r |C_r(i,\ell)|^{\alpha/2}$ if $2 < \alpha < 4$. Since $Na_N^2 \to \infty$ when $\alpha > 2$, this also shows that $\hat{\gamma}_i(\ell) \to \gamma_i(\ell)$ in probability in this case. If $0 < \alpha < 2$, then we can substitute $\sigma_N^2 = 0$ in (2.4) and then the limit in (2.4) is centered stable with index $\alpha/2$, skewness 1 and dispersion $\sum_r |C_r(i,\ell)|^{\alpha/2}$. Since $Na_N^{-2} \to 0$ when $\alpha < 2$, this also shows that $\hat{\gamma}_i(\ell)$ is not bounded in probability in this case. Note also that the limit is almost surely positive in this case, since the left-hand side of (2.5) is nonnegative and the right-hand side, being stable, has a density with respect to Lebesgue measure.

COROLLARY 2.3. *The convergence* (2.4) *holds jointly for all seasons* $i = 0, 1, \dots, \nu - 1$ *and lags* $\ell = 0, \dots, h-1$.

To prove Corollary 2.3, apply Corollary 1.2 with $c_t(j, \ell) = \psi_t(j)\psi_{t+\ell}(j+\ell)$ in place of Theorem 1.1 in the proof of Theorem 2.2.

COROLLARY 2.4. *The convergence* (2.4) *still holds if we define*

$$\hat{\gamma}_i(\ell) = N^{-1} \sum_{t=0}^{N-1} (X_{t\nu+i} - \bar{X}_i)(X_{t\nu+i+\ell} - \bar{X}_{i+\ell}), \tag{2.6}$$

where $\bar{X}_i = N^{-1} \sum_{t=0}^{N-1} X_{t\nu+i}$. *Therefore, we need not assume* $E\varepsilon_t = 0$ *if* $\alpha \geq 2$.

PROOF. The difference between (2.1) and (2.6) is

$$\bar{X}_{i+\ell} N^{-1} \sum_{t=0}^{N-1} X_{t\nu+i} + \bar{X}_i N^{-1} \sum_{t=0}^{N-1} X_{t\nu+i+\ell} - \bar{X}_i \bar{X}_{i+\ell} = \bar{X}_i \bar{X}_{i+\ell}$$

and so it suffices to show that $N a_N^{-2} \bar{X}_i \bar{X}_{i+\ell} \to 0$ in probability. The proof is a straightforward application of regular variation theory; see the Appendix. Then if $\alpha \geq 2$, we can assume without loss of generality that $E\varepsilon_t = 0$ since the mean always exists, and in formula (2.6) a nonzero mean cancels.

3. The sample autocorrelation. In this section we compute the asymptotic distribution of the sample autocorrelation of the periodic moving average X_t defined in (1.1) where ε_t is RV(α) with $E\varepsilon_t^4 = \infty$. When $E\varepsilon_t^4 < \infty$ the results of Anderson and Vecchia (1993) apply. We define the sample autocorrelation at season i and lag ℓ by

$$\hat{\rho}_i(\ell) = \frac{\hat{\gamma}_i(\ell)}{\sqrt{\hat{\gamma}_i(0)\hat{\gamma}_{i+\ell}(0)}}, \tag{3.1}$$

where $\hat{\gamma}_i(\ell)$ is given by (2.1). For $\alpha > 2$ the autocorrelation $\rho_t(\ell) = \operatorname{corr}(X_t, X_{t+\ell}) = \gamma_t(\ell)/\sqrt{\gamma_t(0)\gamma_{t+\ell}(0)}$ and in view of (2.3) this reduces to

$$\rho_t(\ell) = \frac{\sum_j \psi_t(j)\psi_{t+\ell}(j+\ell)}{\sqrt{\sum_j \psi_t(j)^2 \sum_j \psi_{t+\ell}(j)^2}}. \tag{3.2}$$

For $0 < \alpha < 2$ the autocorrelation is undefined, but we will persist in using (3.2) for ease of notation.

In the following theorem, it is interesting to note that for $0 < \alpha < 2$ the sample autocorrelation of the periodic moving average (1.1) converges in distribution to a limit which can be expressed as a function of stable laws. The limit is similar to the formula of Logan, Mallows, Rice and Shepp (1973) for self-normalized sums. For the classical moving average model, Theorem 4.2 of Davis and Resnick (1985a) shows that the sample autocorrelation converges in probability. For $2 < \alpha < 4$ the sample autocorrelation of the periodic moving average model is typically asymptotically stable, while the sample autocorrelation of the classical moving average is always asymptotically normal. This is especially curious since the periodic moving average model reduces to the classical model when $\nu = 1$. See the remarks following the proof for a simple explanation.

THEOREM 3.1. *Suppose that X_t is a periodic moving average of the $RV(\alpha)$ sequence ε_t with $E\varepsilon_t^4 = \infty$. If $0 < \alpha < 2$, then*

$$\hat{\rho}_i(\ell) \Rightarrow \frac{\sum_{r=0}^{\nu-1} C_r(i,\ell) S_r}{\sqrt{\sum_{r=0}^{\nu-1} C_r(i,0) S_r \sum_{r=0}^{\nu-1} C_r(i+\ell,0) S_r}}, \tag{3.3}$$

where $C_r(i,\ell) = \sum_j \psi_i(j\nu + i - r)\psi_{i+\ell}(j\nu + i + \ell - r)$ and $S_0, S_1, \dots, S_{\nu-1}$ are i.i.d. stable with index $\alpha/2$. If $2 \le \alpha \le 4$ and $E\varepsilon_t = 0$, then for some $a_N \to \infty$ we have

$$\begin{aligned} N a_N^{-2} \sigma_N^2 (\hat{\rho}_i(\ell) - \rho_i(\ell)) \Rightarrow & \frac{\rho_i(\ell)}{\sum_j \psi_i(j)\psi_{i+\ell}(j+\ell)} \sum_{r=0}^{\nu-1} C_r(i,\ell) S_r \\ & - \frac{\rho_i(\ell)}{2\sum_j \psi_i(j)^2} \sum_{r=0}^{\nu-1} C_r(i,0) S_r \\ & - \frac{\rho_i(\ell)}{2\sum_j \psi_{i+\ell}(j)^2} \sum_{r=0}^{\nu-1} C_r(i+\ell,0) S_r, \end{aligned} \tag{3.4}$$

where $\sigma_N^2 = E\varepsilon_t^2 I(|\varepsilon_t| \le a_N)$ and $C_r(i,\ell) = \sum_j \psi_i(j\nu+i-r)\psi_{i+\ell}(j\nu+i+\ell-r)$. If $2 \le \alpha < 4$, then we can choose a_N to satisfy $NP[|\varepsilon_t| > a_N] \to 1$ and $S_0, S_1, \dots, S_{\nu-1}$ i.i.d. stable with index $\alpha/2$. If $\alpha = 4$, then we can choose a_N to satisfy $N a_N^{-4} E\varepsilon_t^4 I(|\varepsilon_t| \le a_N) \to 1$ and $S_0, S_1, \dots, S_{\nu-1}$ i.i.d. normal.

PROOF. Define $V = (\sum_r C_r(i,\ell)S_r, \sum_r C_r(i,0)S_r, \sum_r C_r(i+\ell,0)S_r)$, $g(x,y,z) = x/\sqrt{yz}$ and $V_N = (\hat{\gamma}_i(\ell), \hat{\gamma}_i(0), \hat{\gamma}_{i+\ell}(0))$. If $0 < \alpha < 2$, then Theorem 2.2, the remark following Theorem 2.2 [substitute $\sigma_N^2 = 0$ into (2.4)] and Corollary 2.3 imply that $N a_N^{-2} V_N \Rightarrow V$. Apply the continuous mapping theorem (recall that the components of V are almost surely positive in this case) to obtain $g(N a_N^{-2} V_N) = g(V_N) \Rightarrow g(V)$ which is equivalent to (3.3). If $\alpha \ge 2$, define $c_N = N a_N^{-2} \sigma_N^2$ and $V_0 = \sum_j (\psi_i(j)\psi_{i+\ell}(j+\ell), \psi_i(j)^2, \psi_{i+\ell}(j+\ell)^2)$. Theorem 2.2 and Corollary 2.3 imply that $c_N(\sigma_N^{-2} V_N - V_0) \Rightarrow V$ and we will invoke the delta method [e.g., see Billingsley (1979), page 340] to obtain (3.4). Write $g(\sigma_N^{-2} V_N) - g(V_0) = Dg(V_0)(\sigma_N^{-2} V_N - V_0) + O(\sigma_N^{-2} V_N - V_0)^2$ and note that $c_N \to \infty$ so that we must have $\sigma_N^{-2} V_N - V_0 \to 0$ in probability. Then $c_N(g(\sigma_N^{-2} V_N) - g(V_0)) = Dg(V_0) c_N(\sigma_N^{-2} V_N - V_0) + o_P(1) \Rightarrow Dg(V_0)V$, which is equivalent to (3.4). □

REMARKS. The parameters of the stable laws appearing in the limit in (3.3) were specified in the remarks following the proof of Theorem 2.2. If $2 < \alpha < 4$, the limit in (3.4) is stable with index $\alpha/2$, mean zero, skewness 1 and dispersion $\sum_r |D_r|^{\alpha/2}$, where $\sum_r D_r S_r$ represents the right-hand side of (3.4). If $\alpha = 4$ the limit in (3.4) is normal with mean zero and variance $\sum_r D_r^2$. Since $\sum_r C_r(t,\ell) = \sum_j \psi_t(j)\psi_{t+\ell}(j+\ell)$, when $\nu = 1$, (3.3) reduces to $\hat{\rho}_i(\ell) \to \rho_i(\ell)$ in probability, which agrees with the result of Davis and Resnick (1985a) for the classical moving average model. Similarly (3.4) reduces to $Na_N^{-2}\sigma_N^2(\hat{\rho}_i(\ell) - \rho_i(\ell)) \to 0$ in probability. It is well known that in this case $N^{1/2}(\hat{\rho}_i(\ell) - \rho_i(\ell))$ is asymptotically normal; see, for example, Brockwell and Davis [(1991), Proposition 7.3.8]. Since $Na_N^{-2}\sigma_N^2$ varies regularly with index $1-2/\alpha < 1/2$, these norming constants tend to infinity slower than $N^{1/2}$, so there is no contradiction. If we view the classical moving average model as a special case of the periodic model (1.1) in which $\nu = 1$, then mathematically it is a degenerate special case. If we assume that $\nu > 1$ but that the coefficients ψ_t in (1.1) do not depend on t, then the classical sample autocovariance is given by $\hat{\gamma}(\ell) = \nu^{-1}(\hat{\gamma}_0(\ell) + \cdots + \hat{\gamma}_{\nu-1}(\ell))$ and the classical sample autocorrelation $\hat{\rho}(\ell) = \hat{\gamma}(\ell)/\hat{\gamma}(0)$ is very different from $\hat{\rho}_i(\ell)$. The latter can also be viewed as an entry in the autocorrelation matrix of a vector moving average; see Section 4.

COROLLARY 3.2. *The convergence in* (3.2) *and* (3.3) *holds jointly for all seasons* $i = 0, 1, \dots, \nu - 1$ *and lags* $\ell = 0, \dots, h-1$.

To prove the corollary, the continuous mapping arguments extend immediately.

COROLLARY 3.3. *Theorem* 3.1 *still holds if we define* $\hat{\rho}_i(\ell)$ *by* (3.1), *where* $\hat{\gamma}_i(\ell)$ *is given by* (2.6). *Therefore, we need not assume* $E\varepsilon_t = 0$ *if* $\alpha \geq 2$.

For the proof, apply Corollary 2.4 together with Theorem 2.2 in the foregoing proof of Theorem 3.1.

4. Vector moving averages.

The periodic moving average model (1.1) is mathematically equivalent to a vector moving average. If we let $Z_t = (\varepsilon_{t\nu}, \dots, \varepsilon_{(t+1)\nu-1})'$ and $Y_t = (X_{t\nu}, \dots, X_{(t+1)\nu-1})'$, then we can rewrite (1.1) in the form

$$Y_t = \sum_{j=-\infty}^{\infty} \Psi_j Z_{t-j}, \tag{4.1}$$

where Ψ_t is the $\nu \times \nu$ matrix with ij entry $\psi_i(t\nu + i - j)$ and we number the rows and columns $0, 1, \dots, \nu - 1$ for ease of notation. In this section we apply our results on periodic moving averages to vector moving averages, using this mathematical equivalence. If ε_t is RV(α), then Z_t has i.i.d. components with regularly varying tails and we will also say that Z_t is RV(α). If $\alpha \in (0, 2)$, then Z_t belongs to the domain of attraction of a multivariable stable law with index α and the components of this limit law are in fact i.i.d. stable laws with the same index α. If $\alpha \geq 2$, then Z_t belongs to the domain of attraction of a multivariate normal law whose components are i.i.d. univariate normal. Our first result shows that vector moving averages have essentially the same asymptotic behavior as the underlying RV(α) sequence. When $E\|Z_t\|^2 < \infty$ this follows directly from the central limit theorem, so we only consider the remaining case.

THEOREM 4.1. *Suppose that Y_t is a vector moving average of some RV(α) sequence Z_t with $E\|Z_t\|^2 = \infty$. Then for some $a_N \to \infty$ and nonrandom vectors b_N we have*

$$Na_N^{-1}\left(N^{-1}\sum_{t=0}^{N-1} Y_t - \sum_{j=-\infty}^{\infty} \Psi_j b_N\right) \Rightarrow \sum_{j=-\infty}^{\infty} \Psi_j S, \tag{4.2}$$

where S is multivariable stable with index α if $\alpha < 2$ and multivariate normal if $\alpha \geq 2$.

Define the sample autocovariance $\hat{\Gamma}(h) = N^{-1}\sum_{t=0}^{N-1}(Y_t - \bar{Y}_N)(Y_{t+h} - \bar{Y}_N)'$, where $\bar{Y}_N = N^{-1}\sum_{t=0}^{N} Y_t$, and define the autocovariance matrix by $\Gamma(h) = E(Y_t - \bar{Y})(Y_{t+h} - \bar{Y})'$ if it exists (it always exists if $\alpha > 2$), where $\bar{Y} = EY_t$. Note that the ij entry of $\Gamma(h)$ is $\gamma_i(h\nu + j - i)$ and likewise for $\hat{\Gamma}(h)$. Let $\operatorname{diag}(d_0, \ldots, d_{\nu-1})$ denote the $\nu \times \nu$ matrix with diagonal elements $d_0, \ldots, d_{\nu-1}$ and zeroes elsewhere.

THEOREM 4.2. *Suppose that Y_t is a vector moving average of some RV(α) sequence Z_t with $E\|Z_t\|^4 = \infty$. Then for some $a_N \to \infty$ and nonrandom vectors b_N we have*

$$Na_N^{-2}\left(\hat{\Gamma}(h) - b_N \sum_{j=-\infty}^{\infty} \Psi_j \Psi'_{j+h}\right) \Rightarrow \sum_{j=-\infty}^{\infty} \Psi_j \operatorname{diag}(S_0, \ldots, S_{\nu-1})\Psi'_{j+h}, \tag{4.3}$$

where $S_0, \ldots, S_{\nu-1}$ are i.i.d. stable with index $\alpha/2$ if $\alpha < 4$ and i.i.d. normal if $\alpha = 4$.

The limit in (4.3) is a random matrix whose entries are dependent. If $0 < \alpha < 4$, then the ij entry is stable with index $\alpha/2$ and dispersion $\sum_r |C_r(i, \ell)|^{\alpha/2}$, where $\ell = h\nu + j - i$. If $\alpha = 4$, then the ij entry is normal with variance $\sum_r |C_r(i, \ell)|^2$. The autocorrelation matrix $R(h)$ has ij entry equal to $\rho_i(h\nu + j - i)$ and likewise for the sample autocorrelation matrix $\hat{R}(h)$. The asymptotic behavior can be obtained by reference to (3.3) and (3.4). For example, if $0 < \alpha < 2$, then $\hat{R}(h) \Rightarrow M(h)$, where the ij entry of $M(h)$ is given by the right-hand side of (3.3) with $\ell = h\nu + j - i$. Then $\hat{R}(h)$ converges to a random matrix whose entries are dependent and each entry is a function of dependent stable laws.

The results of this section are intended to guide further research. Periodic moving averages correspond to vector moving averages for a very special class of i.i.d. random vectors. It is natural to consider more general sequences Z_t, since we do not expect these vectors to have i.i.d. components in most real applications. The treatment of arbitrary sequences of i.i.d. random vectors with regularly varying tails requires a fundamentally multivariable approach. Meerschaert (1988) introduced regular variation on $\mathbb{R}^d$. Meerschaert (1993) solved the domains of attraction problem on $\mathbb{R}^d$ using regular variation, extending the approach of Feller (1971). We are currently investigating the application of these methods to moving averages of i.i.d. random vectors with regularly varying tails.

5. Applications. In this section we discuss the reformulation of a periodic ARMA model as a periodic moving average. Then we apply our results on periodic moving averages. We restict our attention to the case where the innovations have finite variance but infinite fourth moment ($2 < \alpha < 4$), which is relevant to applications in economics. We will say that $\tilde{X}_t$ follows a PARMA$_\nu(p, q)$ model [a periodic ARMA(p, q) model with period ν] if there exists a RV(α) sequence $\{\varepsilon_t\}$ such that

$$X_t - \sum_{j=1}^{p} \phi_t(j) X_{t-j} = \sigma_t \varepsilon_t - \sum_{j=1}^{q} \theta_t(j) \sigma_{t-j} \varepsilon_{t-j} \tag{5.1}$$

holds almost surely for all t, where $X_t = \tilde{X}_t - \mu_t$ is the mean-standardized process; see, for example, Anderson and Vecchia (1993). The model parameters μ_t, $\phi_t(j)$, $\theta_t(j)$ and σ_t are all assumed periodic with the same period ν. As in Section 4, we can reformulate (5.1) as a vector ARMA model, and then Theorem 11.3.1 in Brockwell and Davis (1991) gives the causality criterion under which (5.1) can be rewritten as a periodic moving average. Then (2.4) and (3.4) give the asymptotic distributions of the sample autocovariances and sample autocorrelations. Brockwell and Davis [(1991), Section 13.3] assume that $x^\alpha P[|\varepsilon_t| > x] \to C$ as $x \to \infty$. In this case we can take $a_N = (CN)^{1/\alpha}$, where C and α can be estimated by the method of Hill (1975). Confidence limits for stable random variables can be obtained by simulation or from unpublished tables; see Samorodnitsky and Taqqu (1994).

EXAMPLE. Consider the PARMA$_\nu(1, 0)$ model $X_t - \phi_t X_{t-1} = \sigma_t \varepsilon_t$, where ε_t is a RV(α) sequence with $2 < \alpha < 4$. If $|\phi_t| < 1$ for all t, then X_t has a causal moving-average representation $X_t = \sum_j \psi_t(j) \varepsilon_{t-j}$, where $\psi_t(\ell) = \phi_t \phi_{t-1} \cdots \phi_{t-\ell+1} \sigma_{t-\ell}$. For example, if $\nu = 2$, $\phi_t = 1/2 + (-1)^t/6$, $\sigma_t = 1/2 - (-1)^t/6$ and $x^\alpha P[|\varepsilon_t| > x] \to 1$ as $x \to \infty$, then the approximate $100(1 - \delta)\%$ confidence interval for $\hat{\rho}_0(1) - \rho_0(1)$ is $\rho_0(1) N^{1/\beta - 1} D_\beta^\beta \sigma I_\beta$, where $\rho_0(1) = 5\sqrt{37}/111$, $\beta = \alpha/2$, $D_\beta = 2(154/925)^\beta$, $\sigma^\beta = \Gamma(2 - \beta)\cos(\pi\beta/2)/(1 - \beta)$ and $P[S \in I_\beta] = 1 - \delta$, where S is centered β stable with skewness 1 and scale factor 1. Standard tables and simulation routines usually assume stable laws with scale factor 1 rather than dispersion 1, which accounts for the appearance of the scale factor σ. Since published tables of quantiles for skewed stable random variables are not available we used S-PLUS to approximate I_β by

TABLE 1
Approximate 95% confidence intervals for $\hat{\rho}_0(1) - \rho_0(1)$

α	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8
Lower	−0.558	−0.188	−0.092	−0.055	−0.036	−0.027	−0.021	−0.019	−0.020
Upper	0.317	0.232	0.133	0.078	0.051	0.036	0.026	0.022	0.021

simulation and produced approximate 95% confidence intervals for $\hat{\rho}_0(1) - \rho_0(1)$ with $N = 1000$, which are given in Table 1 for several values of α. When $\alpha > 4$, the results of Anderson and Vecchia (1993) yield confidence bounds of ± 0.059. Notice that the confidence intervals for the case $2 < \alpha < 4$ may be wider, so that the classical model based on a normal limit may be misleading.

APPENDIX

PROOF OF THEOREM 1.1. Feller [(1971), XVII.5] showed that for the specified d_N, b_N and S_r we have $d_N^{-1}\sum_{t=0}^{N-1}(Z_{t\nu+r} - b_N) \Rightarrow S_r$. If $i \bmod \nu = r$, write $S_i = S_r$. Define $\mathbf{Z}_N = d_N^{-1}(\sum_{t=0}^{N-1}(Z_{t\nu+i-j} - b_N)\colon |j| \le m\nu)$ and $\mathbf{S} = (S_{i-j}\colon |j| \le m\nu)$ and note that $\mathbf{Z}_N \Rightarrow \mathbf{S}$ since any ν consecutive entries of $\mathbf{Z}_N$ are mutually independent and the difference between the j entry and the $j+\nu$ entry tends to zero in probability. Define $c = (c_i(j)\colon |j| \le m\nu)$ and apply the continuous mapping theorem to obtain $c \cdot \mathbf{Z}_N \Rightarrow c \cdot \mathbf{S}$, which can be rewritten in the form

$$Nd_N^{-1}\left(N^{-1}\sum_{t=0}^{N-1} Y_{t\nu+i}(m) - \sum_{|j|\le m\nu} c_i(j) b_N\right) \Rightarrow \sum_{|j|\le m\nu} c_i(j) S_{i-j}, \tag{A.1}$$

where $Y_t(m) = \sum_{|j|\le m\nu} c_i(j) Z_{t-j}$. Notice that $\sum_{j=-\infty}^{\infty} c_i(j) S_{i-j} = \sum_{r=0}^{\nu-1} C_{i,r} S_r$ and so as $m \to \infty$ the limit in (A.1) tends to the limit in (1.3) with probability 1. By a standard result [e.g., see Theorem 6.3.9 in Brockwell and Davis (1991)], the convergence (1.3) will follow once we show that

$$\lim_{m\to\infty} \limsup_{N\to\infty} P[|X_N - X_N(m)| > \varepsilon] = 0 \tag{A.2}$$

for $\varepsilon > 0$ arbitrary, where X_N denotes the left-hand side of (1.3) and $X_N(m)$ is the left-hand side in (A.1). Define $U_a(y) = E|Z_t|^a I(|Z_t| \le y)$ and $V_b(y) = E|Z_t|^b I(|Z_t| > y)$. The regular variation implies that $(a - \beta) y^b U_a(y) \sim (\beta - b) y^a V_b(y)$ as $y \to \infty$ whenever $a > \beta > b$; see Feller (1971). Suppose $0 < \beta < 1$ so that $NV_0(d_N) \to 1$. For an interval U define $xU = xI(|x| \in U)$ and note that $b_N = EZ_t[0, d_N]$ while $X_N - X_N(m) = A + B - C$, where $A = \sum_{|j|>m\nu} c_i(j) d_N^{-1} \sum_{t=0}^{N-1} Z_{t\nu+i-j}[0, d_N]$, $C = Nd_N^{-1} b_N \sum_{|j|>m\nu} c_i(j)$ and $B = \sum_{|j|>m\nu} c_i(j) d_N^{-1} \sum_{t=0}^{N-1} Z_{t\nu+i-j}(d_N, \infty)$. Then $P[|A| > \varepsilon/3] \le 3\varepsilon^{-1} E|A|$, where $E|A| \le Nd_N^{-1} U_1(d_N) \sum_{|j|>m\nu} |c_i(j)|$ and $Nd_N^{-1} U_1(d_N) \sim \beta/(1-\beta)$ as $N \to \infty$. Choose $\delta < \beta$ such that $\sum_j |c_i(j)|^\delta < \infty$ and observe that $P[|B| > \varepsilon/3] \le (3/\varepsilon)^\delta E|B|^\delta$, where $E|B|^\delta \le$

$Nd_N^{-\delta}V_\delta(d_N)\sum_{|j|>m\nu}|c_i(j)|^\delta$ and $Nd_N^{-\delta}V_\delta(d_N) \sim \beta/(\beta-\delta)$. Since $|C| = |Nd_N^{-1}b_N| \le Nd_N^{-1}U_1(d_N) \sim \beta/(1-\beta)$, it follows easily that (A.2) holds in this case. Next suppose $1 \le \beta < 2$ and note that $EA = C$ so that by Chebyshev's inequality we have $P[|A-C| > \varepsilon/2] \le 4\varepsilon^{-2}\sum_{|j|>m\nu}c_i(j)^2 Nd_N^{-2}\operatorname{Var}(Z_t[0,d_N])$, where $Nd_N^{-2}\operatorname{Var}(Z_t[0,d_N]) \le Nd_N^{-2}U_2(d_N) \sim (2-\beta)/\beta$ as $N\to\infty$. If $\beta > 1$, then $P[|B| > \varepsilon/2] \le 2\varepsilon^{-1}E|B|$, where $E|B| \le \sum_{|j|>m\nu}|c_i(j)|d_N^{-1}NV_1(d_N)$ and $Nd_N^{-1}V_1(d_N) \sim \beta/(\beta-1)$ as $N\to\infty$. If $\beta = 1$, choose $\delta < 1$ such that $\sum_j |c_i(j)|^\delta < \infty$ and observe that $P[|B| > \varepsilon/2] \le (2/\varepsilon)^\delta E|B|^\delta$, where $E|B|^\delta \le \sum_{|j|>m\nu}|c_i(j)|^\delta d_N^{-\delta}NV_\delta(d_N)$ and $Nd_N^{-\delta}V_\delta(d_N) \sim 1/(1-\delta)$ as $N\to\infty$. In either case it follows that (A.2) holds when $1 \le \beta < 2$. Finally suppose $\beta = 2$ so that $Nd_N^{-2}U_2(d_N) \to 1$. Then by Chebyshev's inequality $P[|A-C| > \varepsilon/2] \le 4\varepsilon^{-2}\sum_{|j|>m\nu}c_i(j)^2Nd_N^{-2}\operatorname{Var}(Z_t[0,d_N])$, where $Nd_N^{-2}\operatorname{Var}(Z_t[0,d_N]) \le Nd_N^{-2}U_2(d_N) \to 1$. Since $\beta = 2$ and $EZ_t^2 = \infty$ we have $U_2(y)$ slowly varying and $y^{2-b}V_b(y)/U_2(y) \to 0$ as $N\to\infty$ for all $0 \le b < 2$; see Feller (1971). Then $P[|B| > \varepsilon/2] \le 2\varepsilon^{-1}E|B|$, where $E|B| \le \sum_{|j|>m\nu}|c_i(j)|d_N^{-1}NV_1(d_N)$ and $Nd_N^{-1}V_1(d_N) \to 0$. It follows that (A.2) holds in this case as well, which completes the proof. □

PROOF OF COROLLARY 1.2. Apply the argument in the proof of Theorem 1.1 to the moving average $Y_t(k)$ to obtain

$$Nd_N^{-1}\left(N^{-1}\sum_{t=0}^{N-1}Y_{t\nu+i}(k,m) - \sum_{|j|\le m\nu}c_i(j,k)b_N\right) \Rightarrow \sum_{|j|\le m\nu}c_i(j,k)S_{i-j}, \tag{A.3}$$

where $Y_t(k,m) = \sum_{|j|\le m\nu}c_t(j,k)Z_{t-j}$ and

$$\lim_{m\to\infty}\limsup_{N\to\infty}P[|X_N(k) - X_N(k,m)| > \varepsilon] = 0 \tag{A.4}$$

for $\varepsilon > 0$ arbitrary, where $X_N(k)$ denotes the left-hand side of (1.4) and $X_N(k,m)$ is the left-hand side in (A.3). Write (A.3) in the form $V_N(i,k,m) \Rightarrow V(i,k,m)$ and (1.4) in the form $V_N(i,k) \Rightarrow V(i,k)$. Let $\mathbf{V}_N(m) = (V_N(i,k,m)\colon 0 \le i < \nu,\ 0 \le k < h)$ and likewise define $\mathbf{V}(m)$, $\mathbf{V}_N$ and $\mathbf{V}$. The continuous mapping argument in the proof of Theorem 1.1 extends immediately to yield $\mathbf{V}_N(m) \Rightarrow \mathbf{V}(m)$ as $N\to\infty$, and clearly $\mathbf{V}(m)\to\mathbf{V}$ almost surely as $m\to\infty$. To show that $\lim_{m\to\infty}\limsup_{N\to\infty}P[\|\mathbf{V}_N(m)-\mathbf{V}_N\| > \varepsilon] = 0$, use the fact that $P[\|\mathbf{V}_N(m)-\mathbf{V}_N\| > \varepsilon] \le \sum_{i=0}^{\nu-1}\sum_{k=0}^{h-1}P[|V_N(i,k,m) - V_N(i,m) > \varepsilon/(h\nu)]$ for arbitrary $\varepsilon > 0$, along with (A.4). □

PROOF OF LEMMA 2.1. We adapt the proof of Proposition 2.1 in Davis and Resnick (1986), using the same notation as in the proof of Theorem 1.1. Substitute (1.1) into (2.1) and then (2.1) into (2.2) to see that the left-hand side in (2.2) can be written in the form $L = a_N^{-2}\sum_{n=0}^{N-1}\sum_{j\ne k}c_i(j,k)Z_{nj}Z_{nk}$, where $c_i(j,k) = \psi_i(j)\psi_{i+\ell}(k+\ell)$ and $Z_{nj} = \varepsilon_{n\nu+i-j}$. If $0 < \alpha < 2$, then for any $\delta < \alpha$ with $\delta \le 1$ such that (1.2) holds, we have $E|L|^\delta \le a_N^{-2\delta}N\sum_{j\ne k}|c_i(j,k)|^\delta E|Z_{nj}Z_{nk}|^\delta$, where $E|Z_{nj}Z_{nk}|^\delta \le (E|\varepsilon_t|^\delta)^2$ is finite.

Since a_N is regularly varying with index $1/\alpha$ we see that $Na_N^{-2\delta} \to 0$ and so $L \to 0$ in probability in this case. Next suppose $2 \le \alpha < 4$ and write $Z_{nj}Z_{nk}$ in the form $A_{jk} + B_{jk} + C_{jk} + D_{jk}$, where $A_{jk} = (Z_{nj}[0, a_N] - \mu_N)(Z_{nk}[0, a_N] - \mu_N)$, $B_{jk} = \mu_N(Z_{nj}[0, a_N] + Z_{nk}[0, a_N])$, $C_{jk} = Z_{nj}Z_{nk}I(|Z_{nj}| > a_N \text{ or } |Z_{nk}| > a_N)$, and $D_{jk} = -\mu_N^2$, where $\mu_N = EZ_{nj}[0, a_N]$. Substitute back to obtain $L = A+B+C+D$. Now it suffices to show that each of A, B and C tends to zero in probability and $D \to 0$. Write $Z'_{nj} = Z_{nj}[0, a_N] - \mu_N$ so that $A = a_N^{-2}\sum_{n=0}^{N-1}\sum_{j\neq k} c_i(j, k)Z'_{nj}Z'_{nk}$ and then compute $\mathrm{Var}(A) \le a_N^{-4}\sigma_N^4 \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sum_{j\neq k}|c_i(j, k)c_i((n_1 - n_2)\nu + j, (n_1 - n_2)\nu + k)| + |c_i(j, k)c_i((n_1 - n_2)\nu + k, (n_1 - n_2)\nu + j)|$, where $\sigma_N^2 = E|Z'_{nj}|^2$. This triple sum is bounded above by $N\sum_{|n|\le N-1}\sum_{j\neq k}|c_i(j, k)c_i(n\nu + j, n\nu + k)| + |c_i(j, k)c_i(n\nu + k, n\nu + j)|$, where the factor of N is an upper bound for the number of times that n equals d, where $1 - N \le d \le N - 1$. This double sum is in turn bounded above by $(\sum_j |\psi_i(j)|)^2(\sum_j |\psi_{i+\ell}(j)|)^2$, which is finite in view of (1.2). Since a_N is regularly varying with index $1/\alpha$ and σ_N^4 is slowly varying, we have $Na_N^{-4}\sigma_N^4 \to 0$ as $N \to \infty$, hence $\mathrm{Var}(A) \to 0$ and this implies $A \to 0$ in probability. We also have $E|B| \le a_N^{-2}|\mu_N|\sum_{n=0}^{N-1}\sum_{j\neq k}|c_i(j, k)|E(|Z_{nj}[0, a_N]| + |Z_{nk}[0, a_N]|)$, where $\sum_{n=0}^{N-1}\sum_{j\neq k}|c_i(j, k)| \le (\sum_j |\psi_i(j)|)(\sum_j |\psi_{i+\ell}(j)|)$ is finite in view of (1.2), and $E(|Z_{nj}[0, a_N]| + |Z_{nk}[0, a_N]|) \le 2E|\varepsilon_t| < \infty$. Since $E\varepsilon_t = 0$ by assumption in this case, we have $|\mu_N| = |E\varepsilon_t(a_N, \infty)| \le E|\varepsilon_t(a_N, \infty)| = V_1(a_N) \sim \alpha/(\alpha - 1)N^{-1}a_N$ so $Na_N^{-2}|\mu_N| \to 0$ as $N \to \infty$. Then $E|B| \to 0$ and so $B \to 0$ in probability. Similarly $E|C| \le Na_N^{-2}(\sum_j |\psi_i(j)|)(\sum_j |\psi_{i+\ell}(j)|)E|C_{jk}|$, where $E|C_{jk}| \le 2E|Z_{nj}|EZ_{nk}(a_N, \infty)|$ and $Na_N^{-2}E|Z_{nk}(a_N, \infty)| = Na_N^{-2}V_1(a_N) \to 0$. Then $E|C| \to 0$ which implies $C \to 0$ in probability. Since $D = O(Na_N^{-2}\mu_N^2) \to 0$ and since we have already shown that $Na_N^{-2}|\mu_N| \to 0$, we have $D \to 0$ as well, which finishes the proof in the case $2 \le \alpha < 4$. Finally suppose that $\alpha = 4$. The argument is essentially the same, but note that the norming constants a_N are different. We still have $\mathrm{Var}(A) = O(Na_N^{-4}\sigma_N^4)$, where now $\sigma_N^2 \to \sigma^2 = E\varepsilon_1^2 < \infty$ and since $Na_N^{-4}U_4(a_N) \to 1$, we also have $Na_N^{-4} \to 0$, which makes $\mathrm{Var}(A) \to 0$. We also have $E|B| = O(Na_N^{-2}V_1(a_N))$, where $Na_N^{-2}V_1(a_N) \to 0$ and so $E|B| \to 0$. The proof that $E|C| \to 0$ is similar, and $D = o(E|B|)$, so $D \to 0$ as well. □

PROOF OF COROLLARY 2.4. We must show that $Na_N^{-2}\bar{X}_i\bar{X}_{i+\ell} \to 0$ in probability. First suppose that $0 < \alpha < 2$. We will show that in this case $\sqrt{N}a_N^{-1}\bar{X}_i \to 0$ in probability for all i. Theorem 1.1 yields $Na_N^{-1}(\bar{X}_i - b_N\sum_j \psi_i(j)) \Rightarrow Y$, where Y is α stable and $b_N = E\varepsilon_t[0, a_N]$. Then $\sqrt{N}a_N^{-1}(\bar{X}_r - b_N\sum \psi_r(j)) \to 0$ in probability and so it suffices to show that $\sqrt{N}a_N^{-1}b_N \to 0$. If $0 < \alpha < 1$, then $|Na_N^{-1}b_N| \le Na_N^{-1}U_1(a_N) \to \alpha/(1 - \alpha)$ so $\sqrt{N}a_N^{-1}b_N \to 0$. If $\alpha = 1$, then $U_1(a_N)$ is slowly varying and a_N^{-1} is regularly varying with index -1, so again we have $\sqrt{N}a_N^{-1}b_N \to 0$. If $1 < \alpha < 2$, then $b_N \to E\varepsilon_t$ and a_N is regularly varying with index $(1/\alpha) \in (1/2, 1)$ so $\sqrt{N}a_N^{-1}b_N \to 0$. Finally suppose that $\alpha = 2$ and $E\varepsilon_t^2 = \infty$. Without loss of generality we may assume that $E\varepsilon_t = 0$ in (2.6) and then the weak law of large numbers yields $\bar{X}_{i+\ell} \to 0$ in probability, so it suffices to show that $Na_N^{-2}\bar{X}_i \to 0$ in probability. Theorem 1.1 yields $Nd_N^{-1}(\bar{X}_i - b_N \sum_j \psi_i(j)) \Rightarrow Y$ normal, where $Nd_N^{-2}U_2(d_N) \to 1$ and $b_N = E\varepsilon_t[0, d_N]$. Since $b_N = -E\varepsilon_t(d_N, \infty)$, we have $|Nd_N^{-1}b_N| \leq Nd_N^{-1}V_1(d_N) \to 0$ as in the proof of Theorem 1.1, so in fact $Nd_N^{-1}\bar{X}_i \Rightarrow Y$ normal. Both a_N and d_N vary regularly with index 1/2, so $Na_N^{-2}\bar{X}_i = Nd_N^{-1}\bar{X}_i(a_N^{-2}d_N) \to 0$ in probability, which completes the proof. □

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