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Triangular array limits for continuous time random walks

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Abstract

A continuous time random walk (CTRW) is a random walk subordinated to a renewal process, used in physics to model anomalous diffusion. Transition densities of CTRW scaling limits solve fractional diffusion equations. This paper develops more general limit theorems, based on triangular arrays, for sequences of CTRW processes. The array elements consist of random vectors that incorporate both the random walk jump variable and the waiting time preceding that jump. The CTRW limit process consists of a vector-valued Lévy process whose time parameter is replaced by the hitting time process of a real-valued nondecreasing Lévy process (subordinator). We provide a formula for the distribution of the CTRW limit process and show that their densities solve abstract space–time diffusion equations. Applications to finance are discussed, and a density formula for the hitting time of any strictly increasing subordinator is developed.

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1. Introduction

A continuous time random walk (CTRW) is a random walk subordinated to a renewal process. It is specified in terms of a sequence of independent, identically distributed random vectors

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(Y_i, J_i) where Y_i represents a particle jump and $J_i > 0$ is the waiting time preceding that jump. Let $S(n) = X_1 + \cdots + X_n$ denote the particle location after n jumps and let $T(n) = J_1 + \cdots + J_n$ be the time of the nth jump. Then $N_t = \max\{n \geq 0 : T(n) \leq t\}$ is the number of jumps by time $t > 0$ and the CTRW $X(t) = S(N_t)$ represents the particle location at time $t > 0$. The CTRW is useful in physics for modeling anomalous diffusion. Heavy tailed particle jumps lead to superdiffusion, where a cloud of particles spreads faster than the classical Brownian motion, and heavy tailed waiting times lead to subdiffusion. If Y_i belongs to the strict domain of normal attraction of a stable law with index α then as $c \to \infty$ we get $c^{-1/\alpha} S([ct]) \Rightarrow A(t)$, a stable Lévy motion with superdiffusive scaling $A(ct) \stackrel{d}{=} c^{1/\alpha} A(t)$. Densities of $A(t)$ solve a diffusion equation that involves a fractional derivative in space of order α. If the waiting times J_i belong to the domain of normal attraction of a stable law with index $\beta < 1$ then $c^{-1/\beta} T([ct]) \Rightarrow D(t)$, a stable subordinator with $D(ct) \stackrel{d}{=} c^{1/\beta} D(t)$. The renewal process N_t is a kind of inverse to the process $T(n)$, and it follows that $c^{-\beta} N_{ct} \Rightarrow E(t)$, the inverse or hitting time process of the stable subordinator, with inverse scaling $E(ct) \stackrel{d}{=} c^{\beta} E(t)$. In this case $c^{-\beta/\alpha} X([ct]) \Rightarrow M(t) = A(E(t))$, a non-Markovian limit with scaling $M(ct) \stackrel{d}{=} c^{\beta/\alpha} M(t)$. Densities of the CTRW scaling limit $M(t)$ solve a space–time fractional diffusion equation that also involves a fractional time derivative of order β; see [9,39] for complete details.

Continuous time random walks and the associated fractional diffusion equations are useful in physics [42,43], finance [23,33,41,47,48,52], and hydrology [10,12,55]. In applications to hydrology, the heavy tailed particle jumps capture the velocity irregularities caused by a heterogeneous porous media, and the waiting times model particle sticking or trapping. In applications to finance, the particle jumps are price changes or log-returns, separated by a random waiting time between trades. One principal motivation for this work comes from the application to finance, where price jumps typically exhibit power law tails but finite variance. In this situation, the scaling limits in the preceding paragraph lose the power law tails, since the limit $A(t)$ is Gaussian. A more delicate limiting procedure based on triangular arrays yields Lévy process limits that combine heavy tails with finite variance. A similar triangular array approach was already used in [40] to develop models for ultraslow diffusion.

Consider a sequence of continuous time random walks indexed by a scale parameter $c > 0$. Take $\{J_i^{(c)} : j = 1, 2, \ldots\}$ nonnegative i.i.d. random variables representing the waiting times between particle jumps and $T^{(c)}(n) = \sum_{i=1}^n J_i^{(c)}$ the time of the n-th jump. Let $\{Y_i^{(c)} : i = 1, 2, \ldots\}$ be i.i.d. random vectors on $\mathbb{R}^d$ representing the particle jumps and $S^{(c)}(n) = \sum_{i=1}^n Y_i^{(c)}$ the location after n jumps. Define $N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$, the number of jumps by time $t \geq 0$ and

$$X^{(c)}(t) = S^{(c)}(N_t^{(c)}) = \sum_{i=1}^{N_t^{(c)}} Y_i^{(c)} \tag{1.1}$$

the position of the particle at time $t \geq 0$ and scale $c > 0$. Observe that we do not necessarily assume that the waiting times $\{J_i^{(c)} : i \geq 1\}$ and the particle jumps $\{Y_i^{(c)} : i \geq 1\}$ are independent. In fact we allow dependence between the waiting time before the particle jump and the particle jump. More precisely we assume that for each $c > 0$ the sequence $(Y_i^{(c)}, J_i^{(c)})$, $i = 1, 2, \ldots,$ of $\mathbb{R}^d \times \mathbb{R}_+$-valued random vectors are i.i.d., allowing arbitrary dependence between the waiting time $J_i^{(c)}$ and the following jump $Y_i^{(c)}$. In this paper, we develop limit theorems for these CTRW sequences using a triangular array approach. Then we prove a density formula for hitting times of strictly increasing subordinators, which may be of independent interest. The main result of this paper is a formula for the distribution of the CTRW limit process under a weak technical condition, see Theorem 3.6. Finally we derive governing equations for the density of the CTRW limit process. As a special case, governing equations for the uncoupled case (where the waiting times $J_i^{(c)}$ and the jumps $Y_i^{(c)}$ are independent) are discussed in detail. The governing equations are generalized Cauchy problems involving pseudo-differential operators in space and time, related to the generators of semigroups associated with the Lévy processes that emerge as the triangular array limits in space and time.

2. Limit theorems

In order to obtain a triangular array limit for the CTRW sequence $\{X^{(c)}(t)\}_{t\geq 0}$ as $c \to \infty$ we need to impose certain assumptions on the triangular array $\Delta = \{(Y_i^{(c)}, J_i^{(c)}) : i \geq 1, c > 0\}$. For each fixed $c > 0$ the random vectors $(Y_i^{(c)}, J_i^{(c)})$, $i = 1, 2, \ldots$, are assumed to be i.i.d. on $\mathbb{R}^d \times \mathbb{R}_+$. Let

$$S^{(c)}(t) = \sum_{i=1}^{[t]} Y_i^{(c)} \quad \text{and} \quad T^{(c)}(t) = \sum_{i=1}^{[t]} J_i^{(c)}$$

denote the row sums. We assume that Δ is given so that

$$\{(S^{(c)}(cu), T^{(c)}(cu))\}_{u\geq 0} \Rightarrow \{(A(u), D(u))\}_{u\geq 0} \quad \text{as } c \to \infty \tag{2.1}$$

in the J_1 topology on $D([0,\infty), \mathbb{R}^d \times \mathbb{R}_+)$, where $\{(A(u), D(u))\}_{u\geq 0}$ is a Lévy process on $\mathbb{R}^d \times \mathbb{R}_+$. Observe that $\{D(u)\}_{u\geq 0}$ is necessarily a subordinator.

Recall that $N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$ and let

$$E(t) = \inf\{u \geq 0 : D(u) > t\} \tag{2.2}$$

denote the hitting time of the subordinator $\{D(u)\}_{u\geq 0}$ obtained in (2.1). It follows from Theorem 21.3 of [51] that if assumption (3.7) stated below holds, then the subordinator has strictly increasing sample path almost surely and hence the hitting time process $\{E(t)\}_{t\geq 0}$ has continuous nondecreasing sample path almost surely. Moreover it is easy to see that $\{E(t)\}_{t\geq 0}$ is strictly increasing at some $t_0 > 0$ if and only if $\{D(u)\}_{u\geq 0}$ is continuous at $E(t_0)$. For any element $x \in D([0,\infty), S)$ for some complete separable metric space S let $\mathrm{Disc}(x) = \{t \geq 0 : x(t-) \neq x(t)\}$ denote the set of discontinuities of x.

Theorem 2.1. *Assume that (2.1) and (3.7) holds. If*

$$\mathrm{Disc}(\{A(t)\}_{t\geq 0}) \cap \mathrm{Disc}(\{D(t)\}_{t\geq 0}) = \emptyset \quad a.s. \tag{2.3}$$

then

$$\{X^{(c)}(t)\}_{t\geq 0} \Rightarrow \{M(t)\}_{t\geq 0} \quad as\ c \to \infty \tag{2.4}$$

in the M_1-topology on $D([0,\infty), \mathbb{R}^d)$, where $M(t) = A(E(t))$ is a random time change of the first component $\{A(t)\}_{t\geq 0}$ in (2.1) caused by the hitting time process $\{E(t)\}_{t\geq 0}$ of the second component $\{D(t)\}_{t\geq 0}$ in (2.1).

Proof. Since the argument is similar to [9, Theorem 3.1] we only sketch the proof. A continuous mapping argument on $D(\mathbb{R}_+, \mathbb{R}^d \times \mathbb{R}_+)$ using $(x, y) \mapsto (x, y^{-1})$ shows that $(S^{(c)}(ct), c^{-1}N_t^{(c)}) \to (A(t), E(t))$ as $c \to \infty$ in the M_1-topology. Then another continuous mapping argument using $(x, y) \mapsto x \circ y$ (composition) yields (2.4). The technical condition is needed to satisfy condition (i) in [59, Theorem 13.2.4]. □

Remark 2.2. Since $E(t)$ is almost surely continuous and as a Lévy process $\{A(t)\}_{t\geq 0}$ almost surely does not have any fixed points of discontinuity, it follows from Theorem 11.6.6 of [59] that (2.4) also holds in the sense of convergence of all finite dimensional marginal distributions, and hence under the conditions of Theorem 2.1 we have

$$X^{(c)}(t) \Rightarrow A(E(t)) \quad \text{as } c \to \infty$$

in distribution for any fixed $t > 0$.

Remark 2.3. Observe that condition (2.3) is rather strong. In fact it is close to independence of $\{A(t)\}_{t\geq 0}$ and $\{D(t)\}_{t\geq 0}$; see Lemma 15.6 in [28]. It is a challenging open problem to find weaker conditions such that $X^{(c)}(t) \Rightarrow A(E(t))$ as $c \to \infty$ at least for any fixed point in time or for all finite dimensional marginals.

Since in general the processes $\{A(t)\}_{t\geq 0}$ and $\{D(t)\}_{t\geq 0}$ are dependent, the distribution of $M(t)$ can have a complicated structure; see [9]. In the next result, we consider the important special case in which $Y_i^{(c)}$ and $J_i^{(c)}$ are independent. Then the processes $\{A(t)\}_{t\geq 0}$ and $\{E(t)\}_{t\geq 0}$ are independent, and the distributional properties of $M(t) = A(E(t))$ can be obtained via a conditioning argument.

Corollary 2.4. *Assume that (2.1) and (3.7) hold. If the triangular array elements $Y_i^{(c)}$ and $J_i^{(c)}$ are independent for each i and c then (2.4) holds in the M_1-topology on $D([0, \infty), \mathbb{R}^d)$, where $M(t) = A(E(t))$.*

Proof. In this case the components of the limit in (2.1) are independent stochastic processes. Then it is easy to check that the independent Lévy processes $\{A(x)\}$ and $\{D(x)\}$ have (almost surely) no simultaneous jumps, so that (2.3) holds. Then the result follows from Theorem 2.1. □

We conclude this section with some examples to illustrate the practical application of the triangular array convergence for continuous time random walks.

Example 2.5. If J_i are nonnegative independent and identically distributed random variables in the strict domain of attraction of a stable law with index $\beta < 1$ then there exists a regularly varying sequence of positive reals (b_n) with index $-1/\beta$ such that

$$b_n(J_1 + \cdots + J_n) \Rightarrow D, \tag{2.5}$$

where D is stable with index β and $D > 0$ almost surely. Write $b(t) = b_{[t]}$ and let $J_i^{(c)} = b(c)J_i$. Then $T^{(c)}(ct) = b(c)\sum_{i=1}^{[ct]} J_i \Rightarrow D(t)$ for any fixed $t > 0$ and furthermore it follows from Theorem 4.1 in our paper [39] that

$$\{T^{(c)}(ct)\}_{t\geq 0} \Rightarrow \{D(t)\}_{t\geq 0} \quad \text{as } c \to \infty, \tag{2.6}$$

in the J_1 topology, where $\{D(t)\}$ is a stable subordinator. If (Y_i) are i.i.d. $\mathbb{R}^d$-valued random variables that belong to the strict generalized domain of attraction of some operator stable law with exponent E, then there exists a regularly varying sequence of linear operators (B_n) with index $-E$ such that

$$B_n \sum_{i=1}^{n} Y_i \Rightarrow A \quad \text{as } n \to \infty, \tag{2.7}$$

where A is strictly operator stable with exponent E. Write $B(t) = B_{[t]}$ and let $Y_i^{(c)} = B(c)Y_i$. Another application of Theorem 4.1 in [39] shows that

$$\{S^{(c)}(ct)\}_{t\geq 0} \Rightarrow \{A(t)\}_{t\geq 0} \quad \text{as } c \to \infty, \tag{2.8}$$

in the J_1 topology, where $\{A(t)\}$ is an operator Lévy motion on $\mathbb{R}^d$. Finally, in the case where (J_i, Y_i) are i.i.d. random vectors of dimension $d+1$, and allowing dependence between the waiting times J_i and the jumps Y_i, if we assume that

$$\sum_{i=1}^{n} (B_n Y_i, b_n J_i) \Rightarrow (A, D) \tag{2.9}$$

and we define $Y_i^{(c)}$ and $J_i^{(c)}$ as before, then we obtain the joint convergence (2.1) by another application of Theorem 4.1 in [39], since (J_i, Y_i) belong to the strict generalized domain of attraction of the operator stable random vector (A, D). Then the CTRW limit theorem [9, Theorem 3.1] is a special case of Theorem 2.1. For the situation where the waiting times and jumps are independent, see [39, Theorem 4.2],

Example 2.6. The classical example that requires the triangular array construction is a sequence of random walks that converges to a Brownian motion with drift. Given a sequence of independent and identically distributed random vectors (Y_i) on $\mathbb{R}^d$ with mean μ and finite second moments, we define for each scale $c > 0$ the array elements $Y_i^{(c)} = c^{-1}\mu + c^{-1/2}(Y_i - \mu)$. Then a classical computation yields

$$S^{(c)}(ct) = \frac{[ct]}{c}\mu + c^{-1/2} \sum_{i=1}^{[ct]} (Y_i - \mu) \Rightarrow A(t)$$

which is a Brownian motion with drift $\mathbb{E}A(t) = t\mu$, with convergence in the J_1 topology on $D([0, \infty), \mathbb{R}^d)$. The two spatial scales are necessary to retain both the Gaussian and the drift components in the limit, since each has a different scaling. For heavy tailed random vectors with finite mean, a similar approach leads to an operator Lévy motion with drift. Then the constructions of the previous example can be applied to obtain joint convergence (2.1) in this case. The drift is important in finance, for example, where it represents the average rate of growth for the log-price $A(t)$ of an asset. In a similar way, we can also add a drift to the subordinator $D(t)$ by replacing J_i by $J_i + \mu$ in Example 2.5 (note that μ is not the mean waiting time) and using two time scales; see [4,8].

Example 2.7. Given (B_i) i.i.d. with density $p(\beta)$ supported on $(0, 1)$, for any scale $c \geq 1$ let $J_i^{(c)}$ be nonnegative i.i.d. random variables with $P\{J_i^{(c)} > u | B_i = \beta\} = c^{-1}u^{-\beta}$ for $u \geq c^{-1/\beta}$. This amounts to letting $J_i^{(c)} = c^{-1/\beta} J_i$ conditionally on $B_i = \beta$ where J_i are i.i.d. random variables with slowly varying probability tails. The triangular array construction leads to a richer asymptotic theory than the usual methods for very heavy tails. In particular, Corollary 3.5 in [40] shows that under certain regular variation assumptions on the mixing density $p(\beta)$ we have

$$\{T^{(c)}(ct)\}_{t\geq 0} \Rightarrow \{D(t)\}_{t\geq 0} \quad \text{as } c \to \infty$$

in the J_1 topology on $D([0,\infty),[0,\infty))$, where $\{D(t)\}$ is a subordinator with Lévy measure $t\phi$ and $\phi(r,\infty)$ is slowly varying at infinity. Combining this triangular array for waiting times with an independent array of jumps yields another example that satisfies (2.1). In applications to physics $D(t)$ is called an ultrafast subordinator, and its inverse process $\{E(t)\}$ is used to build models of ultraslow diffusion.

Example 2.8. In applications to finance, the waiting times $J_i^{(c)}$ represent the times between transactions and the jumps $Y_i^{(c)}$ are the price jumps (or log-returns). There is considerable evidence in finance for heavy tailed price jumps, where the probability of a jump larger in magnitude than $r > 0$ falls off like $r^{-\alpha}$, or more generally, regularly varying probability tails. A multivariable theory of regular variation is developed in [37], and applications to finance are suggested in [7,38]. Mandelbrot [34] and Fama [21] pioneered the use of heavy tail distributions in finance. Mandelbrot [34] presents graphical evidence that historical daily price changes in cotton have heavy tails with $\alpha \approx 1.7$, so that the mean exists but the variance is infinite. Jansen and de Vries [26] argue that daily returns for many stocks and stock indices have heavy tails with $3 < \alpha < 5$, and discuss the possibility that the October 1987 stock market plunge might be just a heavy tailed random fluctuation. Loretan and Phillips [32] use similar methods to estimate heavy tails with $2 < \alpha < 4$ for returns from numerous stock market indices and exchange rates. In this case the limiting process $A(u)$ is Gaussian if a classical scaling is used, but the underlying CTRW has power law jumps, and this important feature would be lost in the asymptotic analysis. Triangular array asymptotics allow the power law probability tail to persist in the limit. In the simplest model of this type is one takes

$$(Y_i^{(c)}, J_i^{(c)}) = (A(c^{-1}i) - A(c^{-1}(i-1)), D(c^{-1}i) - D(c^{-1}(i-1)))$$

i.i.d. where $\{(A(u), D(u))\}_{u\geq 0}$ is a Lévy process with $D(u)$ a subordinator and $A(u)$ has a Lévy measure ϕ_A with power law tails. For example, if $\phi_A\{x : |x| > r\} \sim Cr^{-\alpha}$ when $r > r_0$ for some (any) $r_0 > 0$, $C > 0$ and $\alpha > 0$, then it follows from [51, Theorem 25.3] that $\mathbb{E}\|A(u)\|^\rho$ exists for $0 < \rho < \alpha$ and diverges for $\rho \geq \alpha$. This includes the case where $A(u)$ is compound Poisson with Pareto jump distribution. Then we have $\{(S^{(c)}(cu), T^{(c)}(cu))\} = \{(A(c^{-1}[cu]), D(c^{-1}[cu]))\}$ and then (2.1) holds in the J_1 topology on $D([0,\infty), \mathbb{R}^d \times \mathbb{R}_+)$ in view of [28, Theorem 16.14].

The Lévy measure or jump intensity describes the constituent price jumps in this model, and thus allows the coding of dependence between various stocks or other financial issues; see [38] for an illustration. It also allows the modeling of dependence between waiting times and price jumps; see [41,53]. If $D(u)$ is a stable subordinator then the hitting time process $E(t)$ has Mittag–Leffler distributions, see [16,17,39]. A coupled model can be obtained by taking $(Y_i^{(c)}, J_i^{(c)}) \stackrel{d}{=} (X(D(c^{-1})), D(c^{-1}))$ where $\{(X(u), D(u))\}_{u\geq 0}$ is a Lévy process with $D(u)$ a subordinator and $\{X(u)\}$ another Lévy process independent of $\{D(u)\}$. An example in [41] illustrates a reasonable fit to a set of high-resolution (tick-by-tick) data for bond futures with a stable subordinator in time and $\{X(u)\}$ a Brownian motion, so that $A(u) = X(D(u))$ is symmetric stable with index $2\beta < 2$. Extending to triangular array CTRW limits allows the consideration of similar models with $\alpha > 2$, which seems to be the most common case in finance.

3. The limit process

In this section we analyze the distribution of the triangular array CTRW limit $M(t) = A(E(t))$ under weak technical conditions on the underlying space–time Lévy process $\{(A(u), D(u))\}_{u\geq 0}$. Before we formulate our main result let us state the general assumptions needed in the proof.

3.1. General assumptions

In this paper we will denote the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{R}$ by $\hat{f}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} f(x)\mathrm{d}x$, the Laplace transform of a function $g : \mathbb{R}_+ \to \mathbb{R}$ by $\tilde{g}(t) = \int_0^\infty \mathrm{e}^{-st} g(s)\mathrm{d}s$, and the Fourier–Laplace transform of a function $h : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ by $\bar{h}(k,s) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-st-\mathrm{i}\langle k,x\rangle} h(x,t)\mathrm{d}x\mathrm{d}t$. For probability measures μ we adopt a similar notation, $\tilde{\mu}(s) = \int \mathrm{e}^{-st}\mu(\mathrm{d}t)$ and so forth. Let $\{(A(u), D(u))\}_{u\geq 0}$ be a Lévy process on $\mathbb{R}^d \times \mathbb{R}_+$ with Lévy representation $[(a,0), Q, \phi]$. That is, the Fourier–Laplace transform (FLT) of the probability measure $P_{(A(u),D(u))}$ is given by

$$\bar{P}_{(A(u),D(u))}(k,s) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-st-\mathrm{i}\langle k,x\rangle}\, P_{(A(u),D(u))}(\mathrm{d}x,\mathrm{d}t) = \mathrm{e}^{-u\psi(k,s)} \tag{3.1}$$

for $k \in \mathbb{R}^d$ and $s > 0$, with

$$\psi(k,s) = \mathrm{i}\langle a,k\rangle + Q(k) + \int_{\mathbb{R}^d\times\mathbb{R}_+\setminus\{(0,0)\}} \left(1 - \mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\mathrm{e}^{-st} - \frac{\mathrm{i}\langle k,x\rangle}{1+\|x\|^2}\right)\phi(\mathrm{d}x,\mathrm{d}t), \tag{3.2}$$

where $a \in \mathbb{R}^d$ is some shift, $Q(k) = \langle k, Ak\rangle$ is a nonnegative definite quadratic form on $\mathbb{R}^d$ and $\phi(\mathrm{d}x,\mathrm{d}t)$ is a Lévy measure on $\mathbb{R}^d \times \mathbb{R}_+ \setminus \{(0,0)\}$; see for example [13, Theorem 4.3.19]. That is, $\phi(\mathrm{d}x,\mathrm{d}t)$ assigns finite measure to sets bounded away from the origin and

$$\int_{0<\|x\|^2+t\leq 1} (\|x\|^2+t)\phi(\mathrm{d}x,\mathrm{d}t) < \infty.$$

Note that by Lemma 2.1 of [9] the function $\psi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}$ with $\psi(0,0) = 0$ and $\operatorname{Re}\psi \geq 0$ is uniquely determined and continuous. We will call ψ the Fourier–Laplace symbol of this Lévy process.

We denote by $\phi_A(\mathrm{d}x) = \phi(\mathrm{d}x, \mathbb{R}_+)$ the Lévy measure of the Lévy process $\{A(u)\}_{u\geq 0}$. By setting $s = 0$ in the representation (3.1) we see that

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\, P_{A(u)}(\mathrm{d}x) = \mathrm{e}^{-u\psi_A(k)}, \tag{3.3}$$

where

$$\psi_A(k) = \mathrm{i}\langle a,k\rangle + Q(k) + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} - \frac{\mathrm{i}\langle k,x\rangle}{1+\|x\|^2}\right)\phi_A(\mathrm{d}x) \tag{3.4}$$

is the Fourier symbol of the Lévy process $\{A(u)\}$. Similarly, we let $\phi_D(\mathrm{d}t) = \phi(\mathbb{R}^d, dt)$ denote the Lévy measure of $\{D(u)\}$. By setting $k = 0$ in the representation (3.1) we see that

$$\int_0^\infty \mathrm{e}^{-st}\, P_{D(u)}(\mathrm{d}t) = \mathrm{e}^{-u\psi_D(s)}, \tag{3.5}$$

where

$$\psi_D(s) = \int_0^\infty \left(1 - \mathrm{e}^{-sv}\right)\phi_D(\mathrm{d}v) \tag{3.6}$$

is the Laplace symbol of the Lévy process $\{D(u)\}$. Note that $\{D(u)\}$ is a subordinator, i.e., a Lévy process with nondecreasing sample paths. Note also that we assume that the drift term of the subordinator is zero. The main results in this paper require that

$$\phi_D(0,\infty) = \infty \tag{3.7}$$

and

$$\int_0^1 y|\ln y|\,\phi_D(\mathrm{d}y) < \infty. \tag{3.8}$$

Assumption (3.7) implies that the process $\{D(u)\}$ is strictly increasing, since in this case the set of jumps of $D(u)$ is almost surely dense in $(0,\infty)$; see for example Theorem 21.3 in [51]. Observe that (3.8) is a rather weak technical condition on the subordinator, since the integral must converge if the $|\ln y|$ term is omitted. Finally we note that if $A(u)$ and $D(u)$ are independent, we have

$$\bar{P}_{(A(u),D(u))}(k,s) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-st-\mathrm{i}\langle k,x\rangle}\, P_{A(u)}(\mathrm{d}x) P_{D(u)}(\mathrm{d}t) = \mathrm{e}^{-u\psi_A(k)}\mathrm{e}^{-u\psi_D(s)} \tag{3.9}$$

so that $\psi(k,s) = \psi_A(k) + \psi_D(s)$.

Recall the definition $E(t) = \inf\{u \ge 0 : D(u) > t\}$ of the inverse or hitting time process. Then

$$\{E(t) \le x\} = \{D(x) \ge t\}. \tag{3.10}$$

Since $D(t)$ is strictly increasing, $E(t)$ is almost surely continuous. Before we state the main result of this section, we first provide a preliminary result on the distribution of $E(t)$ which may be of independent interest. In the sequel, measurability of functions $g : \mathbb{R}^d \to \mathbb{R}$ is always understood to mean measurable with respect to the σ-field of Lebesgue-measurable sets in $\mathbb{R}^d$. Furthermore, let λ^d denote the Lebesgue measure on $\mathbb{R}^d$.

Theorem 3.1. *Under assumption (3.7), for all $t > 0$, the random variable $E(t)$ has the Lebesgue density*

$$f(x,t) = \int_0^t \phi_D(t-y,\infty) P_{D(x)}(\mathrm{d}y). \tag{3.11}$$

Moreover, the mapping $(x,t) \mapsto f(x,t)$ is measurable.

Proof. For fixed $z > 0$ let

$$L(z,t) = \int_0^z f(u,t)\mathrm{d}u \quad \text{and} \quad R(z,t) = P\{E(t) \le z\}.$$

It is enough to show that $L(z,t) = R(z,t)$ for all $z, t > 0$. Observe that $R(z,t) = P\{D(z) \ge t\}$ and define the occupation measure

$$W(\mathrm{d}y) = \int_0^\infty P_{D(u)}(\mathrm{d}y)\mathrm{d}u.$$

In view of [29], Corollary 6.2 on p. 119, we get

$$\begin{aligned}\int_0^\infty f(z,t)\mathrm{d}z &= \int_0^\infty \int_0^t \phi_D(t-y,\infty) P_{D(z)}(\mathrm{d}y)\mathrm{d}z\\ &= \int_0^t \phi_D(t-y,\infty) W(\mathrm{d}y) = 1\end{aligned}$$

for all $t > 0$ and hence $z \mapsto L(z,t)$ is a distribution function for any fixed $t > 0$.

Let us now compute the double Laplace transforms of the functions $R(z,t)$ and $L(z,t)$. For $s,\xi>0$ let

$$\tilde{L}(\xi,s)=\int_0^\infty e^{-st}\left(\int_0^\infty e^{-\xi z}d_zL(z,t)\right)dt=\int_0^\infty e^{-\xi z}\left(\int_0^\infty e^{-st}f(z,t)dt\right)dz,$$

where $d_zL(z,t)$ denotes integration with respect to the z-variable. For fixed $s>0$ we compute using Tonelli's theorem and a simple change of variables

$$\begin{aligned}\int_0^\infty e^{-st}f(z,t)dt &= \int_0^\infty e^{-st}\left(\int_0^t \phi_D(t-y,\infty)P_{D(z)}(dy)\right)dt\\ &= \left(\int_0^\infty e^{-su}\phi_D(u,\infty)du\right)\left(\int_0^\infty e^{-sy}\,P_{D(z)}(dy)\right).\end{aligned}$$

In view of (3.5) a change of variables yields

$$\int_0^\infty e^{-su}\phi_D(u,\infty)du=\int_0^\infty e^{-su}\int_u^\infty \phi_D(dz)du=\frac{1}{s}\psi_D(s). \tag{3.12}$$

Hence

$$\int_0^\infty e^{-st}f(z,t)dt=\frac{1}{s}\psi_D(s)e^{-z\psi_D(s)}. \tag{3.13}$$

Therefore we get

$$\tilde{L}(\xi,s)=\frac{\psi_D(s)}{s}\int_0^\infty e^{-\xi z}e^{-z\psi_D(s)}dz=\frac{1}{s}\frac{\psi_D(s)}{\xi+\psi_D(s)}.$$

On the other hand we need to compute

$$\tilde{R}(\xi,s)=\int_0^\infty e^{-st}\left(\int_0^\infty e^{-\xi z}d_zR(z,t)\right)dt.$$

Integration by parts together with $R(0,t)=P\{D(0)\geq t\}=0$ for $t>0$ yields

$$\int_0^\infty e^{-\xi z}d_zR(z,t)=\xi\int_0^\infty e^{-\xi z}R(z,t)dz.$$

Moreover, an application of Fubini together with (3.5) yields

$$\int_0^\infty e^{-st}R(z,t)dt=\int_0^\infty e^{-st}P\{D(z)\geq t\}dt=\frac{1}{s}\left(1-e^{-z\psi_D(s)}\right)$$

and hence

$$\tilde{R}(\xi,s)=\frac{\xi}{s}\int_0^\infty\left(1-e^{-z\psi_D(s)}\right)e^{-\xi z}dz=\frac{\xi}{s}\left(\frac{1}{\xi}-\frac{1}{\xi+\psi_D(s)}\right)=\frac{1}{s}\frac{\psi_D(s)}{\xi+\psi_D(s)}. \tag{3.14}$$

Hence we have shown that for all $\xi,s>0$ we have $\tilde{L}(\xi,s)=\tilde{R}(\xi,s)$. The uniqueness theorem of the Laplace transform applied to the t-variable implies that for any $\xi>0$ we have

$$\int_0^\infty e^{-\xi z} d_z L(z,t) = \int_0^\infty e^{-\xi z} d_z R(z,t) \tag{3.15}$$

for Lebesgue almost all $t > 0$. However, we wish to establish (3.11) for every $t > 0$. If we can show that for any fixed $\xi > 0$ both

$$t \mapsto \int_0^\infty e^{-\xi z} d_z L(z,t) \tag{3.16}$$

and

$$t \mapsto \int_0^\infty e^{-\xi z} d_z R(z,t) \tag{3.17}$$

are right-continuous, then it would follow that (3.15) holds for all $t > 0$ and all $\xi > 0$. Applying the uniqueness theorem for the Laplace transform (see e.g. [22], Theorem 1 on p. 430) again to (3.15) it would follow that $L(z,t) = R(z,t)$ for all $t > 0$ and all $z > 0$ and the proof would be complete.

Using Theorem 21.3 in [51] we know by assumption (3.7) that the sample paths of the subordinator $\{D(u)\}_{u\geq 0}$ are strictly increasing almost surely and hence $t \mapsto E(t)$ is continuous almost surely and hence in distribution. Therefore the mapping in (3.17) is continuous for any $\xi > 0$, by the continuity theorem for Laplace transforms; e.g. see [22, Theorem 4, p. 431].

It remains to show that for any fixed $\xi > 0$ the function

$$t \mapsto \int_0^\infty e^{-\xi z} d_z L(z,t) = \int_0^\infty e^{-\xi z} f(z,t) dz$$

is right-continuous. For $t, h > 0$ write

$$\begin{aligned}
&\int_0^\infty e^{-\xi z} f(z,t) dz - \int_0^\infty e^{-\xi z} f(z,t+h) dz \\
&\quad = \int_0^\infty e^{-\xi z} \int_0^t [\phi_D(t-y,\infty) - \phi_D(t+h-y,\infty)]\, P_{D(z)}(dy) dz \\
&\qquad - \int_0^\infty e^{-\xi z} \int_t^{t+h} \phi_D(t+h-y,\infty) P_{D(z)}(dy) dz \\
&\quad = I_h - J_h.
\end{aligned}$$

Let

$$g_h(z,y) = e^{-\xi z} [\phi_D(t-y,\infty) - \phi_D(t+h-y,\infty)].$$

Since $v \mapsto \phi_D(v,\infty)$ is right-continuous it follows that $g_h(z,y) \to 0$ as $h \downarrow 0$ for all $z \geq 0$ and $0 \leq y < t$. Moreover $g_h(z,y) \leq \phi_D(t-y,\infty)$ where by (3.11)

$$\int_0^\infty \int_0^t \phi_D(t-y,\infty) P_{D(z)}(dy) dz = 1.$$

In view of dominated convergence we have $I_h \to 0$ as $h \downarrow 0$.

Next we have, using Corollary 6.2 in [29] again, that

$$\begin{aligned}
J_h &\leq \int_0^\infty \int_t^{t+h} \phi_D(t+h-y,\infty) P_{D(x)}(dy) dx \\
&= \int_t^{t+h} \phi_D(t+h-y,\infty) W(dy)
\end{aligned}$$

$$= \int_0^{t+h} \phi_D(t+h-y,\infty)W(\mathrm{d}y) - \int_0^t \phi_D(t+h-y,\infty)W(\mathrm{d}y)$$
$$= 1 - \int_0^t \phi_D(t+h-y,\infty)W(\mathrm{d}y).$$

Rewriting Eqs. (6.1) and (6.7) on pp. 116–117 of [29] in our notation, we have for any $r > 0$ and $x \geq 0$

$$P\{D(E(r)) > r + x\} = \int_0^r \phi_D(r+x-y,\infty)W(\mathrm{d}y).$$

Hence

$$\int_0^t \phi_D(t+h-y,\infty)W(\mathrm{d}y) = P\{D(E(t)) > t+h\}$$

and therefore

$$1 - \int_0^t \phi_D(t+h-y,\infty)W(\mathrm{d}y) = P\{D(E(t)) \leq t+h\} = P\{D(E(t)) - t \leq h\}$$
$$= G_t(h).$$

By Proposition 5 on p. 119 of [29] we know that $G_t(h)$ is a continuous function in $h \geq 0$ with $\lim_{h\downarrow 0} G_t(h) = G_t(0)$. Since

$$1 - G_t(0) = P\{D(E(t)) > t\} = \int_0^t \phi_D(t-y,\infty)W(\mathrm{d}y) = 1$$

using [29], Corollary 6.2 again, we have $G_t(0) = 0$. Hence we have shown that

$$J_h \leq G_t(h) \to G_t(0) = 0 \quad \text{as } h \downarrow 0$$

which proves that $t \mapsto L_z(t)$ is right-continuous in any $t > 0$. The measurability of $(x,t) \mapsto f(x,t)$ follows by approximating the integrand from below by simple functions and using the continuity in distribution of $x \mapsto D(x)$. Now the proof is complete. $\square$

Remark 3.2. In the special case of Example 2.7, the hitting time density $f(x,t)$ in (3.11) was computed in [40] under a technical condition on the continuity of the Laplace transform. That result was strengthened in [31, Theorem 3.1] using a deep result from analysis, the Carasso–Kato theorem [19], along with some multivariable regular variation arguments. Theorem 3.1 gives a more elementary proof, and extends the result to an arbitrary strictly increasing subordinator, under weaker assumptions.

Example 3.3. Here we relate the density formula (3.11) to the formula in [39] for the hitting time density of a stable subordinator. Suppose that $D > 0$ is a β-stable random variable with the bounded C^∞-density g_β normalized so that $\tilde{g}_\beta(s) = \exp(-s^\beta)$. Note that this normalization corresponds to the Lévy measure $\phi_D(t,\infty) = t^{-\beta}/\Gamma(1-\beta)$ for $t > 0$. Next observe that

$$z g_\beta(z) = \frac{\beta}{\Gamma(1-\beta)} \int_0^z (z-y)^{-\beta} g_\beta(y)\mathrm{d}y \quad \text{for } z > 0. \tag{3.18}$$

To see that (3.18) holds, compute the Laplace transform of both sides, and use uniqueness of the Laplace transform for continuous functions. The Laplace transform of the left-hand side is $(-\frac{\mathrm{d}}{\mathrm{d}s})\tilde{g}_\beta(s)$; for the right-hand side use (3.13) with $\psi_D(s) = s^\beta$. Now let $\{D(x)\}_{x\geq 0}$

be the β-stable subordinator with $D(1) = D$. Then every $D(x)$ has density $g(x, y) = x^{-1/\beta} g_\beta (x^{-1/\beta} y)$. Hence, by Theorem 3.1 the density of the hitting time $E(t)$ is given by

$$f(x,t) = \frac{x^{-1/\beta}}{\Gamma(1-\beta)} \int_0^t (t-y)^{-\beta} g_\beta(x^{-1/\beta} y)\mathrm{d}y. \tag{3.19}$$

If we let $z = tx^{-1/\beta}$ in (3.18), a simple change of variable together with (3.19) yields

$$\begin{aligned} tx^{-1/\beta} g_\beta(tx^{-1/\beta}) &= \frac{\beta}{\Gamma(1-\beta)} \int_0^{tx^{-1/\beta}} \left(tx^{-1/\beta} - y\right)^{-\beta} g_\beta(y)\mathrm{d}y \\ &= \frac{\beta x}{\Gamma(1-\beta)} \left(x^{-1/\beta} \int_0^t (t-y)^{-\beta} g_\beta(x^{-1/\beta} y)\mathrm{d}y\right) \\ &= \beta x f(x,t). \end{aligned}$$

Hence

$$f(x,t) = \frac{t}{\beta} x^{-1-1/\beta} g_\beta(tx^{-1/\beta}) \tag{3.20}$$

which agrees with [39], Corollary 3.1(c).

Example 3.4. In this simple example, we extend the formula (3.20), for the hitting time density of a stable subordinator, to the case of a stable subordinator with drift. As in Example 3.3, let g_β denote the density of a standard β-stable random variable $D > 0$ with $\tilde{g}_\beta(s) = \exp(-s^\beta)$. Let $D_0(t)$ be a stable subordinator with $D_0(1) = D$ and let $D(t) = at + D_0(t)$ for some $a > 0$. Then the inverse or hitting time process $E(t)$ defined by (2.2) has a density, which can be calculated as follows. Note that (3.10) still holds, and hence we can write $P\{E(t) \le x\} = P\{D(x) \ge t\} = P\{D_0(x) \ge t - ax\}$. Recall that $D_0(x)$ is identically distributed with $x^{1/\beta} D$, and let $G_\beta(y) = P\{D \le y\}$, so that $\frac{\mathrm{d}}{\mathrm{d}y} G_\beta(y) = g_\beta(y)$. Then the random variable $E(t)$ has density

$$\begin{aligned} f(x,t) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[1 - G_\beta(x^{-1/\beta}(t-ax))\right] \\ &= \left(\frac{t-ax}{\beta x} + a\right) x^{-1/\beta} g_\beta(x^{-1/\beta}(t-ax)) \quad \text{for } 0 < x < t/a. \end{aligned} \tag{3.21}$$

Since $D(x) > ax$, the density of the inverse process $E(t)$ is zero on $x \notin (0, t/a)$. Note that (3.21) reduces to (3.20) when $a = 0$.

Example 3.4 illustrates that the density formula for the hitting time of a subordinator with drift is considerably different. A complete analysis of this case is beyond the scope of this paper. However, we can determine the double Laplace transform of the distribution of $E(t)$, using arguments from the proof of Theorem 3.1. Suppose, then, that $\{D(t)\}$ is a subordinator with $\mathbb{E}(\mathrm{e}^{-sD(t)}) = \mathrm{e}^{-u\psi_D(s)}$ where the Laplace symbol

$$\psi_D(s) = as + \int_0^\infty \left(1 - \mathrm{e}^{-sv}\right) \phi_D(\mathrm{d}v) \tag{3.22}$$

for some $a \ge 0$. We emphasize that assumption (3.7) is not needed here, as the next result holds for any subordinator.

Corollary 3.5. *For all $t > 0$, the distribution $R(z,t) = P\{E(t) \le z\}$ of the hitting time (2.2) satisfies*

$$\tilde{R}(\xi, s) = \int_0^\infty \mathrm{e}^{-st} \left(\int_0^\infty \mathrm{e}^{-\xi z} \mathrm{d}_z R(z,t) \right) \mathrm{d}t = \frac{1}{s} \frac{\psi_D(s)}{\xi + \psi_D(s)}, \tag{3.23}$$

where the Laplace symbol $\psi_D(s)$ of the subordinator $D(t)$ is given by (3.22).

Proof. The argument is exactly the same as (3.14) in Theorem 3.1. □

Recall from above that the limiting process of a triangular array CTRW sequence is of the form $M(t) = A(E(t))$, where $E(t)$ is the hitting time process of the subordinator $\{D(u)\}_{u\ge 0}$. Note that in the space–time process $\{(A(u), D(u))\}_{u\ge 0}$ the processes $\{A(u)\}_{u\ge 0}$ and $\{D(u)\}_{u\ge 0}$ are usually dependent, so $\{A(u)\}_{u\ge 0}$ and $E(t)$ are dependent and hence the distribution of $M(t) = A(E(t))$ can have quite a complicated structure; see [9]. However, the following theorem provides a formula in the general case under weak technical conditions.

Theorem 3.6. *Assume that conditions (3.7) and (3.8) hold. Then for any fixed $t > 0$ we have*

$$P_{M(t)}(\mathrm{d}x) = \int_0^\infty \int_0^t \phi_D(t-u, \infty) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}u)\mathrm{d}s. \tag{3.24}$$

Moreover we have for any $\xi > 0$ and $k \in \mathbb{R}^d$ that

$$\int_0^\infty \mathrm{e}^{-\xi t} \hat{P}_{M(t)}(k)\mathrm{d}t = \frac{1}{\xi} \cdot \frac{\psi_D(\xi)}{\psi(k,\xi)}, \tag{3.25}$$

where $\psi(k,\xi)$ is given by (3.2).

Proof. First note that (3.24) means for any Borel set $S \subset \mathbb{R}^d$

$$P\{M(t) \in S\} = \int_0^\infty \int_0^t \phi_D(t-u, \infty) P_{(A(s),D(s))}(S, \mathrm{d}u)\mathrm{d}s, \tag{3.26}$$

or equivalently for any bounded continuous function f on $\mathbb{R}^d$

$$\int_{\mathbb{R}^d} f(x) P_{M(t)}(\mathrm{d}x) = \int_0^\infty \int_{\mathbb{R}^d} \int_0^t f(x)\phi_D(t-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}u)\mathrm{d}s.$$

Before we go into the details of the proof let us describe its main idea. We first show that the FLT of the right-hand side of (3.24) is equal to the right-hand side of (3.25). Then we show that (3.25) holds true. This implies by uniqueness of the FLT that (3.26) holds true for Lebesgue almost all $t > 0$. Using (right-)continuity together with results from [29] we then show as in the proof of Theorem 3.1 that (3.24) holds for all $t > 0$.

Define a family $(\rho_t(\mathrm{d}x) : t > 0)$ of measures on $\mathbb{R}^d$ by the right-hand side of (3.24), that is

$$\rho_t(\mathrm{d}x) = \int_0^\infty \int_0^t \phi_D(t-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}u)\mathrm{d}s. \tag{3.27}$$

First note that by (3.12) we have for $u, \xi > 0$ that

$$\int_u^\infty \mathrm{e}^{-\xi t} \phi_D(t-u,\infty)\mathrm{d}t = \frac{\psi_D(\xi)}{\xi}\mathrm{e}^{-\xi u}.$$

Moreover, observe that by Theorem 3.1 we have

$$\begin{aligned}\rho_t(\mathbb{R}^d) &= \int_0^\infty \int_0^t \phi_D(t-u,\infty) P_{(A(s),D(s))}(\mathbb{R}^d, \mathrm{d}u)\mathrm{d}s \\ &= \int_0^\infty \int_0^t \phi_D(t-u,\infty) P_{D(s)}(\mathrm{d}u)\mathrm{d}s = \int_0^\infty f(s,t)\mathrm{d}s = 1\end{aligned}$$

showing that $\rho_t(\mathrm{d}x)$ is a probability measure for any $t > 0$. Hence, by Fubini's theorem together with (3.1), we get for $\xi > 0$ and $k \in \mathbb{R}^d$ that

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\xi t}\hat{\rho}_t(k)\mathrm{d}t = \int_{t=0}^\infty \mathrm{e}^{-\xi t}\int_{s=0}^\infty \int_0^t \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\phi_D(t-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s\mathrm{d}t \\ &= \int_{t=0}^\infty \mathrm{e}^{-\xi t}\int_{s=0}^\infty \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{[0,t]}(u)\phi_D(t-u,\infty)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle} P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s\mathrm{d}t \\ &= \int_{s=0}^\infty \int_0^\infty \int_{\mathbb{R}^d}\left(\int_{t=0}^\infty \mathrm{e}^{-\xi t}\mathbf{1}_{[0,t]}(u)\phi_D(t-u,\infty)\mathrm{d}t\right)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle} P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\ &= \int_{s=0}^\infty \int_0^\infty \int_{\mathbb{R}^d}\left(\int_{t=u}^\infty \mathrm{e}^{-\xi t}\phi_D(t-u,\infty)\mathrm{d}t\right)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle} P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\ &= \frac{\psi_D(\xi)}{\xi}\int_{s=0}^\infty\left(\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-\xi u}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle} P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\right)\mathrm{d}s \\ &= \frac{\psi_D(\xi)}{\xi}\int_0^\infty \mathrm{e}^{-s\psi(k,\xi)}\mathrm{d}s \\ &= \frac{1}{\xi}\frac{\psi_D(\xi)}{\psi(k,\xi)}.\end{aligned}$$

Note that the last equality in the chain of equations above holds true since for any $\xi > 0$

$$\begin{aligned}\mathrm{Re}\,\psi(k,\xi) &\geq \int_{\mathbb{R}^d\times\mathbb{R}_+}\left(1+\cos(\langle k,x\rangle)\mathrm{e}^{-\xi t}\right)\phi(\mathrm{d}x,\mathrm{d}t) \\ &\geq \int_{\mathbb{R}^d\times\mathbb{R}_+}\left(1-\mathrm{e}^{-\xi t}\right)\phi(\mathrm{d}x,\mathrm{d}t) = \int_0^\infty\left(1-\mathrm{e}^{-\xi t}\right)\phi_D(\mathrm{d}t) = \psi_D(\xi) > 0\end{aligned}$$

using $\phi_D \neq 0$. This shows that the FLT of the right-hand side of (3.24) equals the right-hand side of (3.25).

We now show that (3.25) holds using $M(t) = A(E(t))$. For Borel sets $S \subset \mathbb{R}^d$ and $t, s > 0$ let

$$H_s(t) = P\{A(s) \in S, D(s) < t\} \tag{3.28}$$

and note that by the inversion formula for the Fourier transform (see, e.g., Proposition 2.5(xi) of Sato [51]) and the Lévy–Khinchin formula the mapping

$$(s,t) \mapsto P\{A(s) \in S, D(s) < t\}$$

is measurable. Observe further that by Theorem 3.1

$$P\{M(t) \in S\} = \int_0^\infty P\{A(s) \in S | E(t) = s\} f(s,t)\mathrm{d}s, \tag{3.29}$$

where $f(s,t)$ is the density of $E(t)$.

The proof of (3.25) is based on the following lemma.

Lemma 3.7. *For $h, t, s > 0$ let*

$$q_h(s,t) = P\{A(s) \in S | s < E(t) \le s+h\}.$$

Then we have

(a) *For all $h > 0$ the mapping $(s,t) \mapsto q_h(s,t)$ is measurable.*

(b)

$$\lim_{h \downarrow 0} q_h(s,t) = P\{A(s) \in S | E(t) = s\}$$

for λ^2-almost every (s,t). Hence there exists a version of $P\{A(s) \in S | E(t) = s\}$ such that

$$(s,t) \mapsto P\{A(s) \in S | E(t) = s\}$$

is measurable.

(c) *For any $\xi > 0$ we have*

$$\int_0^\infty \mathrm{e}^{-\xi t} P\{A(s) \in S | E(t) = s\} f(s,t) \mathrm{d}t = \psi_D(\xi) \tilde{H}_s(\xi) \tag{3.30}$$

for λ^1-almost every $s \ge 0$, where $\tilde{H}_s(\xi) = \int_0^\infty \mathrm{e}^{-\xi t} H_s(t) \mathrm{d}t$ denotes the Laplace transform of $H_s(t)$ in t.

Proof. (a) Observe that

$$q_h(s,t) = \frac{P\{A(s) \in S, s < E(t) \le s+h\}}{P\{s < E(t) \le s+h\}}.$$

Using (3.10) we have

$$P\{s < E(t) \le s+h\} = P\{D(s+h) \ge t\} - P\{D(s) \ge t\}$$

and hence $(s,t) \mapsto P\{s < E(t) \le s+h\}$ is measurable. Moreover, we can write

$$\begin{aligned} &P\{A(s) \in S, s < E(t) \le s+h\} \\ &\quad = P\{A(s) \in S, E(t) > s\} - P\{A(s) \in S, E(t) > s+h\} \\ &\quad = P\{A(s) \in S, D(s) < t\} - P\{A(s) \in S, D(s+h) < t\} \end{aligned} \tag{3.31}$$

which implies that $(s,t) \mapsto q_h(s,t)$ is measurable for any $h > 0$.

(b) Let $F = \{(s,t) : \lim_{h\downarrow 0} q_h(s,t) \text{ exists}\}$. Then F is measurable and hence

$$g(s,t) = \begin{cases} \lim_{h\downarrow 0} q_h(s,t) & (s,t) \in F \\ 0 & (s,t) \in F^c \end{cases}$$

is measurable. Now it follows from a variant of Lebesgue's differentiation theorem (see e.g. [15], exercise 33.16 on p. 444) that for any fixed $t > 0$ we have

$$\lim_{h\downarrow 0} q_h(s,t) = P\{A(s) \in S | E(t) = s\} \quad \text{for } \lambda^1\text{-almost every } s \ge 0.$$

Hence, since F is measurable we have for all $t > 0$ that $\lambda^1(F_t^c) = 0$ where $F_t^c = \{s \ge 0 : (s,t) \in F^c\}$. Then by Tonelli's theorem we know that $\lambda^2(F^c) = 0$ and hence

$$g(s,t) = P\{A(s) \in S | E(t) = s\}$$

for λ^2-almost every (s,t). Since g is measurable, this implies that there exists a version of $P\{A(s) \in S | E(t) = s\}$ which is jointly measurable in (s,t).

For the proof of (c) first note that, since the Lévy process $\{D(u)\}_{u\geq 0}$ has stationary independent increments, a simple conditioning argument yields

$$P\{A(s) \in S, D(s+h) < t\} = \int_0^t H_s(t-\tau) P_{D(h)}(\mathrm{d}\tau). \tag{3.32}$$

Note further that by Lebesgue's differentiation theorem we see, by arguing as in part (a) of the proof, that for each $t > 0$ we have

$$\frac{1}{h} P\{s < E(t) \leq s+h\} = \frac{1}{h}\int_s^{s+h} f(x,t)\mathrm{d}x \to f(s,t) \quad \text{as } h \downarrow 0$$

for λ^1-almost every $s > 0$. Then we can argue as in the proof of part (b) that the same convergence holds for λ^2-almost every (s,t). Hence, by part (b) we obtain

$$\begin{aligned}\frac{1}{h} P\{A(s) \in S, s < E(t) \leq s+h\} &= q_h(s,t)\frac{1}{h} P\{s < E(t) \leq s+h\} \\ &\to P\{A(s) \in S | E(t) = s\} f(s,t)\end{aligned} \tag{3.33}$$

as $h \downarrow 0$ for λ^2-almost every (s,t). Since in view of (3.31) and (3.32) we have

$$P\{A(s) \in S, s < E(t) \leq s+h\} = H_s(t) - \int_0^t H_s(t-\tau) P_{D(h)}(\mathrm{d}\tau)$$

we obtain by (3.33) that

$$\begin{aligned} f_h(s,t) &= \mathrm{e}^{-\xi t}\frac{1}{h}\left(H_s(t) - \int_0^t H_s(t-\tau)P_{D(h)}(\mathrm{d}\tau)\right) \\ &\to \mathrm{e}^{-\xi t} P\{A(s) \in S | E(t) = s\} f(s,t)\end{aligned}$$

as $h \downarrow 0$ for λ^2-almost every (s,t). Therefore

$$\int_0^\infty \mathrm{e}^{-\xi t} P\{A(s) \in S | E(t) = s\} f(s,t)\mathrm{d}t = \int_0^\infty \lim_{h\downarrow 0} f_h(s,t)\mathrm{d}t$$

for λ^1-almost every $s \geq 0$. Observe that by Tonelli's theorem we have

$$\begin{aligned}\int_0^\infty f_h(s,t)\mathrm{d}t &= -\frac{1}{h}\left(\int_0^\infty \mathrm{e}^{-\xi t}\int_0^\infty \mathbf{1}_{[0,t)}(\tau) H_s(t-\tau)P_{D(h)}(\mathrm{d}\tau)\mathrm{d}t - \tilde{H}_s(\xi)\right) \\ &= -\frac{1}{h}\left(\int_0^\infty\left(\int_0^\infty \mathrm{e}^{-\xi t}\mathbf{1}_{[0,t)}(\tau)H_s(t-\tau)\mathrm{d}t\right)P_{D(h)}(\mathrm{d}\tau) - \tilde{H}_s(\xi)\right) \\ &= -\frac{1}{h}\left(\int_0^\infty\left(\int_\tau^\infty \mathrm{e}^{-\xi t}H_s(t-\tau)\mathrm{d}t\right)P_{D(h)}(\mathrm{d}\tau) - \tilde{H}_s(\xi)\right) \\ &= -\tilde{H}_s(\xi)\frac{1}{h}\left(\int_0^\infty \mathrm{e}^{-\xi\tau}P_{D(h)}(\mathrm{d}\tau) - 1\right) \\ &= -\tilde{H}_s(\xi)\frac{1}{h}\left(\mathrm{e}^{-h\psi_D(\xi)} - 1\right) \\ &\to \psi_D(\xi)\tilde{H}_s(\xi)\end{aligned}$$

as $h \downarrow 0$.

Now in view of dominated convergence, in order to prove (3.30) it suffices to show that for any fixed $s > 0$ there exists an integrable function $g : [0, \infty) \to \mathbb{R}_+$ such that $|f_h(s, t)| \le g(t)$ for $0 < h \le 1$. Observe that, using Tonelli's theorem again

$$\begin{aligned}
\int_0^t H_s(t-\tau) P_{D(h)}(\mathrm{d}\tau) &= \int_0^t \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{S\times[0,t-\tau)}(x, v) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) P_{D(h)}(\mathrm{d}\tau) \\
&= \int_0^\infty \int_S \left(\int_0^t \mathbf{1}_{[0,t-\tau)}(v) P_{D(h)}(\mathrm{d}\tau) \right) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) \\
&= \int_0^\infty \int_S P\{D(h) < t - v\}\, P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) \\
&= \int_0^t \int_S P\{D(h) < t - v\}\, P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v).
\end{aligned}$$

Recalling (3.28) we have

$$|f_h(s, t)| = \mathrm{e}^{-\xi t} \int_0^t \int_S \frac{1}{h} P\{D(h) \ge t - v\}\, P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v). \tag{3.34}$$

Since the function $x \mapsto 1 - \mathrm{e}^{-x}$ is strictly increasing, we get from Markov's inequality, for any $x > 0$ that

$$\begin{aligned}
P\{D(h) \ge x\} &= P\{1 - \mathrm{e}^{-x^{-1}D(h)} \ge 1 - \mathrm{e}^{-1}\} \\
&\le \frac{1}{1 - \mathrm{e}^{-1}} \mathbb{E}\left[1 - \mathrm{e}^{-x^{-1}D(h)}\right] = C\left(1 - \mathrm{e}^{-h\psi_D(x^{-1})}\right).
\end{aligned}$$

Using the inequality $1 - \mathrm{e}^{-y} \le y$ for all $y > 0$ we therefore get for some constant $C > 0$

$$\frac{1}{h} P\{D(h) \ge x\} \le C \frac{1 - \mathrm{e}^{-h\psi_D(x^{-1})}}{h} \le C\psi_D(x^{-1})$$

for all $h > 0$.

By (3.34) we therefore get

$$|f_h(s, t)| \le C\mathrm{e}^{-\xi t} \int_0^t \int_S \psi_D\left(\frac{1}{t - v}\right) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) = g(t)$$

for all $t, h > 0$. It remains to show that

$$\int_0^\infty g(t)\mathrm{d}t < \infty. \tag{3.35}$$

Using Tonelli's theorem again we conclude that

$$\begin{aligned}
\int_0^\infty g(t)\mathrm{d}t &= C \int_0^\infty \mathrm{e}^{-\xi t} \int_0^\infty \int_S \mathbf{1}_{[0,t)}(v)\psi_D\left(\frac{1}{t-v}\right) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v)\mathrm{d}t \\
&= C \int_0^\infty \int_S \left(\int_0^\infty \mathrm{e}^{-\xi t} \mathbf{1}_{[0,t)}(v)\psi_D\left(\frac{1}{t-v}\right) \mathrm{d}t \right) P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) \\
&= C \left(\int_0^\infty \mathrm{e}^{-\xi u}\psi_D(u^{-1})\mathrm{d}u \right) \left(\int_0^\infty \int_S \mathrm{e}^{-\xi v}\, P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) \right),
\end{aligned}$$

where

$$\int_0^\infty \int_S \mathrm{e}^{-\xi v}\, P_{(A(s),D(s))}(\mathrm{d}x, \mathrm{d}v) \le \int_0^\infty \mathrm{e}^{-\xi v}\, P_{D(s)}(\mathrm{d}v) = \mathrm{e}^{-s\psi_D(\xi)} < \infty.$$

Moreover, using the fact that $z \mapsto \psi_D(z)$ is monotone, we have for some constant $C > 0$ that

$$\int_0^\infty e^{-\xi u}\psi_D(u^{-1})du = \int_0^1 e^{-\xi u}\psi_D(u^{-1})du + \int_1^\infty e^{-\xi u}\psi_D(u^{-1})du$$
$$\leq \int_0^1 \psi_D(u^{-1})du + C\int_1^\infty e^{-\xi u}du,$$

so it remains to show that

$$\int_0^1 \psi_D(u^{-1})du < \infty. \tag{3.36}$$

In view of Proposition 1 on p. 74 of [14] we know that

$$\frac{\psi_D(z)}{z} \leq C\int_0^{1/z} \phi_D(r,\infty)dr \quad \text{for all } z > 0$$

and hence

$$\psi_D(u^{-1}) \leq Cu^{-1}\int_0^u \phi_D(r,\infty)dr.$$

Therefore

$$\begin{aligned}
\int_0^1 \psi_D(u^{-1})du &\leq C\int_0^1 u^{-1}\int_0^u \phi_D(r,\infty)dt\,du \\
&= C\int_0^1\int_0^1 u^{-1}\mathbf{1}_{[0,u]}(r)du\,\phi_D(r,\infty)dr \\
&= -C\int_0^1 \ln(r)\phi_D(r,\infty)dr \\
&= -C\int_0^1 \ln(r)\int_0^\infty \mathbf{1}_{(r,\infty)}(y)\phi_D(dy)dr \\
&= -C\int_0^1 \ln(r)\int_0^1 \mathbf{1}_{(r,\infty)}(y)\phi_D(dy)dr \\
&\quad - C\int_0^1 \ln(r)\int_1^\infty \mathbf{1}_{(r,\infty)}(y)\phi_D(dy)dr \\
&= A + B.
\end{aligned}$$

Now, by assumption (3.8) we have

$$\begin{aligned}
A &= -C\int_0^1\int_0^1 \ln(r)\mathbf{1}_{(r,\infty)}(y)dr\,\phi_D(dy) \\
&= -C\int_0^1 y\ln(y)\phi_D(dy) + \int_0^1 y\,\phi_D(dy) < \infty.
\end{aligned}$$

Finally

$$\begin{aligned}
B &= -C\int_1^\infty\int_0^1 \mathbf{1}_{(r,\infty)}(y)\ln(r)dr\,\phi_D(dy) \\
&= -C\int_1^\infty \phi_D(dy)\int_0^1 \ln(r)(dr) = C\phi_D(1,\infty) < \infty
\end{aligned}$$

which concludes the proof of Lemma 3.7. □

Proof of Theorem 3.6 (*Continued*). For $\xi > 0$ define a finite measure μ on $\mathbb{R}^d$ by

$$\mu(S) = \int_0^\infty e^{-\xi t} P\{M(t) \in S\} dt$$

for Borel sets $S \subset \mathbb{R}^d$. Then for bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ we have by definition

$$\int_{\mathbb{R}^d} f(x)\mu(dx) = \int_0^\infty e^{-\xi t} \int_{\mathbb{R}^d} f(x) P_{M(t)}(dx) dt. \tag{3.37}$$

In view of (3.29), (3.30) and Tonelli's theorem we compute

$$\mu(S) = \int_0^\infty e^{-\xi t} \int_0^\infty P\{A(s) \in S | E(t) = s\} f(s,t) ds dt = \psi_D(\xi) \int_0^\infty \tilde{H}_s(\xi) ds.$$

Now observe that

$$\begin{aligned}
\tilde{H}_s(\xi) &= \int_0^\infty e^{-\xi t} P\{A(s) \in S, D(s) < t\} dt \\
&= \int_0^\infty e^{-\xi t} \int_{S\times\mathbb{R}_+} \mathbf{1}_{[0,t)}(u) P_{(A(s),D(s))}(dx, du) dt \\
&= \int_{S\times\mathbb{R}_+} \left(\int_u^\infty e^{-\xi t} dt \right) P_{(A(s),D(s))}(dx, du) \\
&= \frac{1}{\xi} \int_{S\times\mathbb{R}_+} e^{-\xi u} P_{(A(s),D(s))}(dx, du).
\end{aligned}$$

Therefore

$$\mu(S) = \frac{\psi_D(\xi)}{\xi} \int_0^\infty \int_{S\times\mathbb{R}_+} e^{-\xi u} P_{(A(s),D(s))}(dx, du) ds$$

and hence we also have

$$\mu(dx) = \frac{\psi_D(\xi)}{\xi} \int_0^\infty \int_0^\infty e^{-\xi u} P_{(A(s),D(s))}(dx, du) ds.$$

Now using (3.1) we get in view of (3.37) that

$$\begin{aligned}
\int_0^\infty e^{-\xi t} \hat{P}_{M(t)}(k) dt &= \hat{\mu}(k) \\
&= \frac{\psi_D(\xi)}{\xi} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} e^{-\xi u} e^{-i\langle k,x\rangle} P_{(A(s),D(s))}(dx, du) ds \\
&= \frac{\psi_D(\xi)}{\xi} \int_0^\infty e^{-s\psi(k,\xi)} ds \\
&= \frac{1}{\xi} \cdot \frac{\psi_D(\xi)}{\psi(k,\xi)}.
\end{aligned}$$

This shows that (3.25) holds.

By uniqueness of the Laplace transform we therefore conclude that for all $k \in \mathbb{R}^d$ and for Lebesgue almost every $t > 0$

$$\hat{P}_{M(t)}(k) = \hat{\rho}_t(k). \tag{3.38}$$

If we can prove that both $t \mapsto \hat{P}_{M(t)}(k)$ and $t \mapsto \hat{\rho}_t(k)$ are right-continuous functions for any $k \in \mathbb{R}^d$ it follows that (3.38) holds true for all $t > 0$ and all $k \in \mathbb{R}^d$. The uniqueness theorem of the Fourier transform then implies that (3.24) holds for any $t > 0$ and the proof is complete.

In view of assumption (3.7) and Theorem 21.3 in [51], we know that the sample paths of $\{D(u)\}_{u\geq 0}$ are almost surely strictly increasing, and hence the sample paths of $E(t)$ are continuous and nondecreasing almost surely. Moreover we can choose a version of $\{A(u)\}_{u\geq 0}$ with right-continuous sample path almost surely. Hence $t \mapsto A(E(t))$ is right-continuous almost surely and also in distribution. The continuity theorem of the Fourier transform implies that $t \mapsto \hat{P}_{M(t)}(k)$ is right-continuous for any $k \in \mathbb{R}^d$.

The proof that $t \mapsto \hat{\rho}_t(k)$ is right-continuous is similar to the second half of the proof of Theorem 3.1. In view of (3.27) we obtain

$$\hat{\rho}_t(k) = \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \phi_D(t-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s$$

and therefore for any $t > 0$ and $h > 0$ we can write

$$\begin{aligned}
\hat{\rho}_t(k) - \hat{\rho}_{t+h}(k) &= \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \phi_D(t-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\
&\quad - \int_0^\infty \int_0^{t+h} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \phi_D(t+h-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\
&= \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \left[\phi_D(t-u,\infty) - \phi_D(t+h-u,\infty)\right] P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\
&\quad - \int_0^\infty \int_t^{t+h} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \phi_D(t+h-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\
&= I_h - J_h.
\end{aligned}$$

As in the proof of Theorem 3.1 we have for $0 < u < t < t+h$ that $0 \leq \phi_D(t-u,\infty) - \phi_D(t+h-u,\infty) \leq \phi_D(t-u,\infty)$. Moreover, using the right-continuity of $v \mapsto \phi_D(v,\infty)$ we have

$$f_h(x,u) = \mathrm{e}^{-\mathrm{i}\langle x,k\rangle} \left[\phi_D(t-u,\infty) - \phi_D(t+h-u,\infty)\right] \to 0 \quad \text{as } h \downarrow 0$$

for all $x \in \mathbb{R}^d$ and all $0 < u < t$. Since $|f_h(x,u)| \leq \phi_D(t-u,\infty)$ and since Theorem 3.1 shows that $f(x,t)$ is a density in x, dominated convergence implies $I_h \to 0$ as $h \downarrow 0$. Finally

$$\begin{aligned}
|J_h| &\leq \int_0^\infty \int_t^{t+h} \int_{\mathbb{R}^d} \phi_D(t+h-u,\infty) P_{(A(s),D(s))}(\mathrm{d}x,\mathrm{d}u)\mathrm{d}s \\
&= \int_0^\infty \int_t^{t+h} \phi_D(t+h-u,\infty) P_{D(s)}(\mathrm{d}u)\mathrm{d}s \to 0
\end{aligned}$$

as $h \downarrow 0$ as in the proof of Theorem 3.1. This concludes the proof of Theorem 3.6. □

Corollary 3.8. *Under the assumptions of Theorem 3.6, assume additionally that for any $s > 0$ the distribution of $(A(s), D(s))$ has a density $p(s,x,u)$ with respect to Lebesgue measure. Then $M(t) = A(E(t))$ has Lebesgue density*

$$m(x,t) = \int_0^\infty \int_0^t \phi_D(t-u,\infty) p(s,x,u)\mathrm{d}u\mathrm{d}s \tag{3.39}$$

with Fourier–Laplace transform

$$\bar{m}(k,\xi) = \frac{1}{\xi}\frac{\psi_D(\xi)}{\psi(k,\xi)}. \tag{3.40}$$

4. Governing equations

Triangular array limits of continuous time random walks (CTRWs) have the distribution specified in Theorem 3.6. In this section, we will show how these CTRW limit distributions are related to certain pseudo-differential equations in space and time. The CTRW limit process is $M(t) = A(E(t))$ where $A(t)$ is a Lévy process and $E(t)$ is the inverse or hitting time process for a subordinator $D(t)$. In the case where $D(t)$ is a stable process with index β, the governing equation involves a fractional derivative ∂_t^β; see for example [39,49,57,60]. If $A(t)$ is a stable process with index α, then the governing equation employs fractional space derivatives of order α; see for example [3,11,20,35,36]. Space–time fractional differential equations are important in physics, finance, and hydrology, where they are used to model anomalous diffusion; see [24,30,42,54,56,58] for an introduction to this diverse literature. The fractional space derivative models superdiffusion, where a cloud of particles spreads at a faster rate than the classical diffusion equation predicts. In terms of stochastic processes, this is the result of replacing a Brownian motion by a stable Lévy motion whose self-similarity (Hurst) index is larger: $1/\alpha > 1/2$. In terms of the random walk model that leads to this limit, superdiffusion arises from particle jumps with regularly varying probability tails with index $-\alpha$, whose variance does not exist. Fractional time derivatives model sticking or trapping, a kind of subdiffusion. When $D(t)$ is a stable subordinator with index β, the inverse process $E(t)$ grows at a sub-linear rate with Hurst index $0 < \beta < 1$, which retards the growth of the plume modeled by the CTRW limit $M(t)$; see [39]. The random waiting times in the underlying CTRW have infinite mean, since their probability tails vary regularly with index $-\beta$.

The Fourier–Laplace symbol of any Lévy process $\{(A(u), D(u))\}_{u\geq 0}$ on $\mathbb{R}^d \times \mathbb{R}_+$ defines a pseudo-differential operator that is also the generator of the corresponding convolution semigroup. Given any $\omega > 0$ let $L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$ denote the collection of real-valued measurable functions on $\mathbb{R}^d \times \mathbb{R}_+$ for which the integral and hence the norm

$$\|f\|_\omega = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-\omega t} |f(x,t)| \mathrm{d}x \mathrm{d}t$$

exists. With this norm, $L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$ is a Banach space, and clearly $L^1(\mathbb{R}^d \times \mathbb{R}_+) \subset L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$. Also, if $f \in L^1(\mathbb{R}^d \times \mathbb{R}_+)$, then $\|f\|_\omega \leq \|f\|_1$. A family of bounded linear operators $\{T(t) : t \geq 0\}$ on a Banach space X such that $T(0)$ is the identity operator and $T(u+v) = T(u)T(v)$ for all $u, v \geq 0$ is called a *semigroup of bounded linear operators on* X. If $\|T(u)f\| \leq M\|f\|$ for all $f \in X$ and all $u \geq 0$ then the semigroup is *uniformly bounded*; if in this case $M \leq 1$ then we have a *contraction semigroup*. If $T(u_n)f \to T(u)f$ in X for all $f \in X$ whenever $u_n \to u$ then the semigroup is *strongly continuous*. It is easy to check that $\{T(u) : u \geq 0\}$ is strongly continuous if $T(u)f \to f$ in X for all $f \in X$ as $u \downarrow 0$. If we write $f \geq g$ if $f(x,t) \geq g(x,t)$ almost everywhere on $\mathbb{R}^d \times \mathbb{R}_+$ then $L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$ is an *ordered Banach space* in the sense of [2], and we say that a semigroup on this space is *positive* if $f \geq 0$ implies that $T(u)f \geq 0$ for all $u \geq 0$. A strongly continuous positive contraction semigroup is also called a *Feller semigroup*. For any strongly continuous semigroup $\{T(u) : u > 0\}$ on a Banach space X we define the *generator*

$$Lf = \lim_{u\to 0^+} \frac{T(u)f - f}{u} \quad \text{in } X \tag{4.1}$$

meaning that $\|u^{-1}(T(u)f - f) - Lf\| \to 0$ in the Banach space norm. The domain $D(L)$ of this linear operator is the set of all $f \in X$ for which the limit in (4.1) exists. Then $D(L)$ is dense in X, and L is closed, meaning that if $f_n \to f$ and $Lf_n \to g$ in X then $f \in D(L)$ and $Lf = g$ (see, for example, [45, Corollary I.2.5]). For any Lévy process $\{(A(u), D(u))\}_{u\geq 0}$ on $\mathbb{R}^d \times \mathbb{R}_+$ we define

$$T(u)f(x,t) = \int_0^t \int_{\mathbb{R}^d} f(x-y, t-r) P_{(A(u),D(u))}(\mathrm{d}y, \mathrm{d}r) \tag{4.2}$$

for all $f \in L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$ and all $u \geq 0$. Proposition 3.1 in [5] (see also Jacob [25]) shows that $\{T(u) : u \geq 0\}$ is a Feller semigroup on $L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$, and Theorem 3.2 in [5] shows that the generator $L = -\psi(-\mathrm{i}D_x, \partial_t)$ of this semigroup is a pseudo-differential operator such that for any $u \in D(L)$ the element $v = \psi(-\mathrm{i}D_x, \partial_t)u$ of the space $L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$ has Fourier–Laplace transform (FLT)

$$\bar{v}(k,s) = \int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-st-\mathrm{i}\langle k,x\rangle} \psi(-\mathrm{i}D_x, \partial_t)u(x,t)\mathrm{d}x\mathrm{d}t = \psi(k,s)\bar{u}(k,s), \tag{4.3}$$

where $\psi(k,s)$ is given by (3.2). Theorem 3.2 in [5] also shows that $D(L)$ contains any $f \in L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$ whose weak first- and second-order spatial derivatives as well as weak first-order time derivatives are in $L^1_\omega(\mathbb{R}^d \times \mathbb{R}_+)$, and that in this case we have

$$\begin{aligned}\psi(-\mathrm{i}D_x, \partial_t)f(x,t) &= \langle a, \nabla f(x,t)\rangle - \langle \nabla, A\nabla f(x,t)\rangle \\ &\quad - \int_{\mathbb{R}^d\times\mathbb{R}_+\setminus\{(0,0)\}} \left(H(t-u)f(x-y,t-u) - f(x,t) + \frac{\langle \nabla f(x,t), y\rangle}{1+\|y\|^2}\right)\phi(\mathrm{d}y, \mathrm{d}u)\end{aligned} \tag{4.4}$$

where $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_d} f)'$ and $H(t) = I\ (t \geq 0)$ is the Heaviside step function.

Suppose that, for any $u > 0$, the distribution of $(A(u), D(u))$ has a density $p(u,x,t)$ with respect to the Lebesgue measure. Then Corollary 3.8 shows that the CTRW scaling limit $M(t)$ has a density $m(x,t)$ given by (3.39). The FLT $\bar{m}(k,s)$ of the density $m(x,t)$ is given by (3.40), and it follows that $\psi(k,s)\bar{m}(k,s) = s^{-1}\psi_D(s)$. We can invert this FLT using (3.12) and (4.3) to obtain

$$\psi(-\mathrm{i}D_x, \partial_t)m(x,t) = \delta(x)\phi_D(t,\infty), \tag{4.5}$$

where $\delta(x)$ is the Dirac delta function. This extends the coupled governing equation (4.7) in [9] to the case of a more general subordinator. Some applications of the coupled space–time equation (4.5) to problems in statistical physics are given in [9]. In these applications, the coupled equation governs the scaling limit of a continuous time random walk where the particle jump length is dependent on the waiting time. Coupled space–time jumps were originally considered in [30,57] to enforce physically meaningful velocity constraints. For example, one model assumes that the jump length is exactly equal to the waiting time, to enforce a constant velocity. In the case where the waiting time scaling limit is a stable subordinator, this leads to a coupled governing equation $(\partial/\partial t + \partial/\partial x)^\beta m(x,t) = \delta(x)t^{-\beta}/\Gamma(1-\beta)$.

Next we consider the uncoupled case, where the limiting jump process $A(t)$ and the waiting time process $D(t)$ are independent. It turns out that the subordination formula $M(t) = A(E(t))$

along with the corresponding density formula leads to a useful decomposition result for certain abstract space–time pseudo-differential equations. The space–time limit density $m(x,t)$, which is the fundamental solution to a generalized Cauchy problem, decomposes into two parts. The first part is the density of the space process $\{A(u)\}$ and it represents the fundamental solution to an abstract Cauchy problem. The second part is the density $f(u,t)$ of the time process $\{E(t)\}$ and it represents the fundamental solution to an inhomogeneous Cauchy problem (4.9). This space–time decomposition illustrates the advantages of the stochastic approach to the study of abstract evolution equations. We say that a function m is a *mild solution* to a space–time pseudo-differential equation, if its (Fourier–Laplace or Laplace–Laplace) transform $\bar{m}$ solves the equivalent algebraic equation in transform space. This is somewhat different from the standard usage for integer-order time derivative equations (e.g., see Pazy [45, Def. 2.3 p. 106]) where a mild solution is defined as a solution to the corresponding integral equation. For abstract evolution equations that involve pseudo-differential operators in time, there is no standard concept of a mild solution, and the usage here is consistent with [4,40]. Some deeper questions regarding strong solutions of these equations are also interesting, but beyond the scope of this paper.

Theorem 4.1. *Assume that conditions (3.6)–(3.8) hold, and that the space limit variable $A(u)$ in (2.1) has a density $p(x,u)$ for any $u>0$. Suppose also that the limiting jump process $A(t)$ and the waiting time process $D(t)$ are independent (uncoupled). Then the uncoupled triangular array CTRW limit process $M(t)=A(E(t))$ from Theorem 3.6 has density $m(x,t)$ given by*

$$m(x,t)=\int_0^\infty p(x,u)f(u,t)\mathrm{d}u, \tag{4.6}$$

where $f(u,t)$ is the density (3.11) of the time variable $E(t)$ defined by (2.2). This density $m(x,t)$ is the fundamental solution to the generalized Cauchy problem

$$\psi_D(\partial_t)m(x,t)=-\psi_A(-\mathrm{i}D_x)m(x,t)+\delta(x)\phi_D(t,\infty), \tag{4.7}$$

in the mild sense. Furthermore, its components $p(x,u)$ and $f(u,t)$ are fundamental solutions in the mild sense of two constituent equations. The space component $p(x,u)$ is the fundamental solution to the Cauchy problem

$$\partial_t p(x,t)=L_A p(x,t); \qquad m(x,0)=\delta(x), \tag{4.8}$$

where $L_A=-\psi_A(-\mathrm{i}D_x)$ is the generator of the semigroup associated with $A(t)$. The time component $f(u,t)$ is the fundamental solution to the inhomogeneous Cauchy problem

$$\partial_x f(x,t)=-\psi_D(\partial_t)f(x,t)+\delta(x)\phi_D(t,\infty) \tag{4.9}$$

corresponding to the inverse or hitting time process $\{E(t)\}$ for the Lévy process $\{D(u)\}$ in (2.1).

Proof. In this uncoupled case where $A(t)$, $D(t)$ are independent, the symbol $\psi(k,s)=\psi_A(k)+\psi_D(s)$ and the pseudo-differential operator $\psi(-\mathrm{i}D_x,\partial_t)=\psi_A(-\mathrm{i}D_x)+\psi_D(\partial_t)$ is uncoupled into space and time components. Then the density $m(x,t)$ has FLT

$$\bar{m}(k,s)=\frac{1}{s}\frac{\psi_D(s)}{\psi_A(k)+\psi_D(s)}, \tag{4.10}$$

and it follows that $\psi_D(s)\bar{m}(k,s)=-\psi_A(k)\bar{m}(k,s)+s^{-1}\psi_D(s)$. Invert this FLT using (3.12) and (4.3) to obtain (4.7).

Suppose that the space limit variable $A(u)$ in (2.1) has a density $p(x,u)$ for any $u>0$. Then it follows immediately from Theorems 3.1 and 3.6 that the uncoupled CTRW scaling limit $M(t)$

has a density (4.6) where $f(u,t)$ is the density of $E(t)$ given by Theorem 3.1. It is well known that $p(x,t)$ solves the Cauchy problem (4.8) where $L_A = -\psi_A(-\mathrm{i}D_x)$ is the generator of the semigroup associated with $A(t)$; see for example [1,45]. In the proof of Theorem 3.1 we showed that the bivariate Laplace transform

$$\tilde{f}(\xi,s) = \int_0^\infty \int_0^\infty \mathrm{e}^{-\xi z - st} f(z,t)\mathrm{d}t\mathrm{d}z = \frac{1}{s}\frac{\psi_D(s)}{\xi + \psi_D(s)}.$$

This rearranges to

$$\xi \tilde{f}(\xi,s) = -\psi_D(s)\tilde{f}(\xi,s) + \frac{\psi_D(s)}{s},$$

and then inverting the double Laplace transform using (3.12) shows that $f(x,t)$ solves an inhomogeneous Cauchy problem (4.9). This completes the proof. □

Remark 4.2. The space–time pseudo-differential equation (4.7) extends the uncoupled governing equation (5.4) in [39] to the case of a more general subordinator.

Remark 4.3. The density formula (4.6) can also be written in the form (3.39) by substituting (3.11) into (4.6).

Example 4.4. Specialize for the moment to the case where $A(t)$ is a Gaussian Lévy process with Fourier symbol $\psi_A(k) = \|k\|^2$, and suppose further that $D(t)$ is a stable subordinator, independent of $A(t)$, with Laplace symbol $\psi_D(s) = s^\beta$. Then we have $\psi_A(-\mathrm{i}D_x) = -\Delta$ where $\Delta = \sum_j \partial^2/\partial x_j^2$ is the Laplacian operator, and $\psi_D(\partial_t) = \partial_t^\beta$, a Riemann–Liouville fractional derivative in time. More information on fractional derivatives can be found in [44,46,50]. Then $m(x,t)$ solves

$$\partial_t^\beta m(x,t) = \Delta m(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)},$$

a form considered in [60] as a model for Hamiltonian chaos. Here we have used the fact, which is easy to check, that $t^{-\beta}/\Gamma(1-\beta)$ has Laplace transform $s^{\beta-1}$ for $0<\beta<1$. Alternatively, one can simply compute the tail of the corresponding Lévy measure $\psi_D(t,\infty)$; compare [9, Theorem 2.2].

Example 4.5. In the situation of Example 2.7, where the CTRW waiting times have slowly varying probability tails, the Laplace symbol $\psi_D(s) = \int_0^1 s^\beta \Gamma(1-\beta)p(\beta)\mathrm{d}\beta$; see [40]. Then it follows from (3.25) that $\psi_D(s)\bar{m} - s^{-1}\psi_D(s) = -\psi_A(k)\bar{m}$, and we get by inverting the FLT that the density $m(x,t)$ of the uncoupled CTRW limit $M(t)$ solves a distributed-order time fractional partial differential equation

$$\int_0^1 \mathcal{D}_t^\beta m(t,x)\Gamma(1-\beta)p(\beta)\mathrm{d}\beta = -\psi_A(-\mathrm{i}D_x)m(t,x). \tag{4.11}$$

Remark 4.6. For initial value problems, it is convenient to introduce the Caputo fractional derivative in time, defined so that $\mathcal{D}_t^\beta g(t)$ has Laplace transform $s^\beta\tilde{g}(s) - s^{\beta-1}g(0)$; see for example [18,46]. Using this definition, and inverting the FLT in Example 4.4 using the initial condition $m(x,0)=\delta(x)$ (or equivalently, $\hat{m}(k,0)=1$) yields

$$\mathcal{D}_t^\beta m(x,t) = \Delta m(x,t). \tag{4.12}$$

When $A(t)$ is an operator Lévy motion, the situation of Example 2.5, the density $m(x,t)$ of the CTRW limit $\{M(t)\}$ satisfies a similar form $\mathcal{D}_t^\beta m(x,t) = L_A m(x,t)$ where L_A is the generator of the semigroup associated with the Lévy process $\{A(u)\}_{u\geq 0}$; see [39, Theorem 5.1]. Note that in this case a smooth density $p(x,t)$ always exists for $t > 0$; see [27, Theorem 4.10.2].

Remark 4.7. In view of the fact that the density $p(x,t)$ of $A(t)$ solves the Cauchy problem (4.8), we call $A(t)$ a *stochastic solution* to this Cauchy problem. Suppose that $D(t)$ is a stable subordinator, independent of $A(t)$, with Laplace symbol $\psi_D(s) = s^\beta$. Then the CTRW limit density $m(x,t)$ solves the *fractional Cauchy problem*

$$\partial_t^\beta m(x,t) = L_A m(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{4.13}$$

a special case of the uncoupled governing equation (4.7). We call (4.7) a *generalized Cauchy problem*. The triangular array CTRW limit $\{M(t)\}$ is the stochastic solution to the generalized Cauchy problem (4.7), and its density $m(x,t)$ is the fundamental (point source) solution. This is the case relevant to Example 2.8 when price jumps and waiting times are independent (uncoupled), or more generally, when they are asymptotically independent in the sense that the two limit processes $\{A(u)\}$ and $\{D(u)\}$ in (2.1) are independent, as in the finance application in [41].

Remark 4.8. In order to avoid distributions, we may define a *generalized Caputo derivative*. For suitable functions $g : \mathbb{R}_+ \to \mathbb{R}$ we specify the generalized Caputo derivative operator $\mathbb{C}_D(\partial t)g(t)$ as the inverse Laplace transform of $\psi_D(s)\tilde{g}(s) - s^{-1}\psi_D(s)g(0)$. Of course this reduces to the usual Caputo derivative when $\{D(u)\}$ is a stable subordinator, and to the distributed-order time fractional derivative operator on the left-hand side of (4.11) in the situation of Example 2.7. With this notation, we see that the uncoupled CTRW limit density $m(x,t)$ solves the abstract equation

$$\mathbb{C}_D(\partial t)m(x,t) = -\psi_A(-\mathrm{i}D_x)m(x,t). \tag{4.14}$$

The Caputo fractional derivative facilitates and clarifies the incorporation of initial values in fractional Cauchy problems; see for example [6]. The extension described here should be similarly useful for generalized Cauchy problems.

Remark 4.9. In the case where the subordinator $D(t)$ has positive drift, the study of CTRW scaling limits seems to require different methods. To facilitate comparison with the case of no drift, suppose $D(t) = at + D_0(t)$ is a subordinator with positive drift, as in Example 3.4. The proof of Theorem 3.6 does not extend, as (3.36) certainly does not hold when $\psi_D(u) = au + \psi_{D_0}(u)$. In the special case of Example 3.4, where $D_0(u)$ is a stable subordinator with index β, we know that the hitting time $E(t)$ has a density $f(x,t)$ given by (3.21). Corollary 3.5 shows that

$$\tilde{f}(\xi,s) = \int_0^\infty \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-z\xi} f(z,t)\mathrm{d}z\mathrm{d}t = \frac{1}{s}\frac{\psi_D(s)}{\xi + \psi_D(s)},$$

where $\psi_D(s) = as + s^\beta$. Rewrite in the form $\xi\,\tilde{f}(\xi,s) = -s^\beta \tilde{f}(\xi,s) - as\,\tilde{f}(\xi,s) + a + s^{\beta-1}$. Invert the double Laplace transform using (3.12) and (4.3) to get

$$\partial_x f(x,t) = -\partial_t^\beta f(x,t) - a\partial_t f(x,t) + \delta(x)\left(a\delta(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\right). \tag{4.15}$$

If $A(t)$ has density $p(x,t)$ for all $u > 0$, then this family of densities solves the Cauchy problem (4.8) where $L_A = -\psi_A(-\mathrm{i}D_x)$ is the generator of the associated semigroup. Then the CTRW scaling limit density $m(x,t)$ given by (4.6) solves

$$\partial_t^\beta m(x,t) + \partial_t m(x,t) = -\psi_A(-\mathrm{i}D_x)m(x,t) + \delta(x)\left(a\delta(t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\right). \tag{4.16}$$

Eq. (4.16) can also be written in the form (4.14), where in this case $\mathbb{C}_D(\partial t)$ is the sum of Caputo derivatives of order 1 and order β.

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References

[1] W. Arendt, C. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, in: Monographs in Mathematics, Birkhäuser-Verlag, Berlin, 2001.
[2] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, U. Schlotterbeck, One-parameter Semigroups of Positive Operators, in: Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin, 1986.
[3] B. Baeumer, M.M. Meerschaert, Stochastic solutions for fractional Cauchy problems, Fractional Calculus Appl. Anal. 4 (2001) 481–500.
[4] B. Baeumer, D.A. Benson, M. Meerschaert, Advection and dispersion in time and space, Physica A 350 (2005) 245–262.
[5] B. Baeumer, M.M. Meerschaert, J. Mortensen, Space–time fractional derivative operators, Proc. Amer. Math. Soc. 133 (8) (2005) 2273–2282.
[6] B. Baeumer, M.M. Meerschaert, S. Kurita, Inhomogeneous fractional diffusion equations, Fractional Calculus Appl. Anal. 8 (4) (2005) 371–386.
[7] G. Balkema, P. Embrechts, High Risk Scenarios and Extremes. A Geometric Approach, in: Zurich Lecture Notes, EMS Publishing House, 2007.
[8] P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorem for continuous time random walks with two time scales, J. Appl. Probab. 41 (2) (2004) 455–466.
[9] P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorems for coupled continuous time random walks, Ann. Probab. 32 (1B) (2004) 730–756.
[10] D. Benson, S. Wheatcraft, M. Meerschaert, Application of a fractional advection–dispersion equation, Water Resour. Res. 36 (2000) 1403–1412.
[11] D. Benson, S. Wheatcraft, M. Meerschaert, The fractional-order governing equation of Lévy motion, Water Resour. Res. 36 (2000) 1413–1424.
[12] D. Benson, R. Schumer, M. Meerschaert, S. Wheatcraft, Fractional dispersion, Lévy motions, and the MADE tracer tests, Transport Porous Media 42 (2001) 211–240.
[13] C. Berg, J.P. Reus Christensen, P. Ressel, Harmonic Analysis on Semigroups, Springer, New York, 1984.
[14] J. Bertoin, Lévy Processes, Cambridge University Press, 1996.
[15] P. Billingsley, Probability and Measure, 3rd ed., Wiley, New York, 1995.
[16] N.H. Bingham, Limit theorems for occupation times of Markov processes, Z. Warsch. Verw. Geb. 17 (1971) 1–22.
[17] L. Bondesson, G. Kristiansen, F. Steutel, Infinite divisibility of random variables and their integer parts, Statist. Probab. Lett. 28 (1996) 271–278.
[18] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, Geophys. J. R. Astr. Soc. 13 (1967) 529–539.
[19] A.S. Carasso, T. Kato, On subordinated holomorphic semigroups, Trans. Amer. Math. Soc. 327 (1991) 867–878.
[20] A. Chaves, A fractional diffusion equation to describe Lévy flights, Phys. Lett. A 239 (1998) 13–16.
[21] E. Fama, The behavior of stock market prices, J. Business 38 (1965) 34–105.
[22] W. Feller, An Introduction to Probability Theory and its Applications, 2nd ed., vol. II, Wiley, New York, 1971.

[23] R. Gorenflo, F. Mainardi, E. Scalas, M. Raberto, Fractional calculus and continuous-time finance. III. The diffusion limit. Mathematical finance (Konstanz, 2000), in: Trends Math., Birkhäuser, Basel, 2001, pp. 171–180.
[24] R. Gorenflo, F. Mainardi, Fractional diffusion processes: Probability distribution and continuous time random walk, Lecture Notes Phys. 621 (2003) 148–166.
[25] N. Jacob, Pseudo-Differential Operators and Markov Processes, Akad. Verl., Berlin, 1996.
[26] D. Jansen, C. de Vries, On the frequency of large stock market returns: Putting booms and busts into perspective, Rev. Econ. Statist. 23 (1991) 18–24.
[27] Z. Jurek, J.D. Mason, Operator-Limit Distributions in Probability Theory, Wiley, New York, 1993.
[28] O. Kallenberg, Foundations of Modern Probability, 2nd ed., Springer, New York, 2002.
[29] H. Kesten, Hitting probabilities of single points for processes with stationary independent increments, Mem. Am. Math. Soc. 93 (1969) 129 p.
[30] J. Klafter, A. Blumen, M.F. Shlesinger, Stochastic pathways to anomalous diffusion, Phys. Rev. A 35 (1987) 3081–3085.
[31] M. Kovács, M.M. Meerschaert, Ultrafast subordinators and their hitting times, Publ. Inst. Math. N.S. 80 (94) (2006) 193–206.
[32] M. Loretan, P. Phillips, Testing the covariance stationarity of heavy tailed time series, J. Empirical Finance 1 (1994) 211–248.
[33] F. Mainardi, M. Raberto, R. Gorenflo, E. Scalas, Fractional calculus and continuous-time finance II: The waiting-time distribution, Physica A 287 (2000) 468–481.
[34] B. Mandelbrot, The variation of certain speculative prices, J. Business 36 (1963) 394–419.
[35] M.M. Meerschaert, D.A. Benson, B. Baeumer, Multidimensional advection and fractional dispersion, Phys. Rev. E 59 (1999) 5026–5028.
[36] M.M. Meerschaert, D.A. Benson, B. Baeumer, Operator Lévy motion and multiscaling anomalous diffusion, Phys. Rev. E 63 (2001) 1112–1117.
[37] M.M. Meerschaert, H.P. Scheffler, Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice, Wiley Interscience, New York, 2001.
[38] M. Meerschaert, H.P. Scheffler, Portfolio modeling with heavy tailed random vectors, in: S.T. Rachev (Ed.), Handbook of Heavy-Tailed Distributions in Finance, Elsevier North-Holland, New York, 2003, pp. 595–640.
[39] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, J. Appl. Probab. 41 (3) (2004) 623–638.
[40] M.M. Meerschaert, H.P. Scheffler, Stochastic model for ultraslow diffusion, Stochastic Process Appl. 116 (9) (2006) 1215–1235.
[41] M.M. Meerschaert, E. Scalas, Coupled continuous time random walks in finance, Physica A 370 (2006) 114–118.
[42] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[43] R. Metzler, J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A 37 (2004) R161–R208.
[44] K. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York, 1993.
[45] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential equations, in: Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
[46] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[47] M. Raberto, E. Scalas, F. Mainardi, Waiting-times and returns in high-frequency financial data: An empirical study, Physica A 314 (2002) 749–755.
[48] L. Sabatelli, S. Keating, J. Dudley, P. Richmond, Waiting time distributions in financial markets, Eur. Phys. J. B 27 (2002) 273–275.
[49] A.I. Saichev, G.M. Zaslavsky, Fractional kinetic equations: Solutions and applications, Chaos 7 (1997) 753–764.
[50] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and derivatives: Theory and Applications, Gordon and Breach, London, 1993.
[51] K.I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.
[52] E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance, Physica A 284 (2000) 376–384.
[53] E. Scalas, R. Gorenflo, F. Mainardi, M.M. Meerschaert, Speculative option valuation and the fractional diffusion equation, in: J. Sabatier, J. Tenreiro Machado (Eds.), Proceedings of the IFAC Workshop on Fractional Differentiation and its Applications, Bordeaux, 2004.

[54] E. Scalas, Five years of continuous-time random walks in econophysics, in: A. Namatame (Ed.), Proceedings of WEHIA 2004, Kyoto, 2004.

[55] R. Schumer, D.A. Benson, M.M. Meerschaert, S.W. Wheatcraft, Eulerian derivation of the fractional advection–dispersion equation, J. Contaminant Hydrol. 48 (2001) 69–88.

[56] R. Schumer, D.A. Benson, M.M. Meerschaert, B. Baeumer, Multiscaling fractional advection–dispersion equations and their solutions, Water Resour. Res. 39 (2003) 1022–1032.

[57] M. Shlesinger, J. Klafter, Y.M. Wong, Random walks with infinite spatial and temporal moments, J. Statist. Phys. 27 (1982) 499–512.

[58] V.V. Uchaikin, V.M. Zolotarev, Chance and Stability. Stable Distributions and their Applications, VSP, Utrecht, 1999.

[59] W. Whitt, Stochastic Process Limits: An Introduction to Stochastic-Process Limits And their Application to Queues, Springer, New York, 2002.

[60] G. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion, Physica D 76 (1994) 110–122.