MTH 133, Lecture 2: Exam 1 Solutions

October, 1999

1. **(10 pts)** Compute the derivative of y with respect to x, where $y = \cos[\ln(x^5)]$.

   Solution 1.

$$\begin{aligned} \frac{d}{dx}\cos[\ln(x^5)] &= \frac{d}{dx}\cos[5\ln(x)] \\ &= -\sin[5\ln(x)]\left(\frac{5}{x}\right) \\ &= \frac{-5\sin[\ln(x^5)]}{x}. \end{aligned}$$

$\square$

   Solution 2.

$$\begin{aligned} \frac{d}{dx}\cos[\ln(x^5)] &= -\sin[\ln(x^5)]\left(\frac{1}{x^5}\,5x^4\right) \\ &= \frac{-5\sin[\ln(x^5)]}{x}. \end{aligned}$$

$\square$

2. **(10 pts)** Compute the derivative of y with respect to x, where $y = xe^{x^2}$.

   Solution.

$$\frac{d}{dx}(xe^{x^2}) = e^{x^2} + xe^{x^2}(2x) = e^{x^2}(1 + 2x^2).$$

$\square$

3. **(15 pts)** Compute

$$\int_0^{\pi/3} \frac{4\sin(\theta)}{1-4\cos(\theta)}\,d\theta.$$

Solution. Use the substitution $u = 1-4\cos(\theta)$, which leads to $du = 4\sin(\theta)\,d\theta$, $u(0) = -3$, and $u(\pi/3) = -1$. This leads to

$$\int_0^{\pi/3} \frac{4\sin(\theta)}{1-4\cos(\theta)}\,d\theta = \int_{-3}^{-1} \frac{1}{u}\,du = \ln(|u|)\Big]_{-3}^{-1} = \ln(1) - \ln(3) = -\ln(3).$$

If you'd rather evaluate the integral using the original (θ) limits, here's how to write your solution:

$$\int \frac{4\sin(\theta)}{1-4\cos(\theta)}\,d\theta = \int \frac{1}{u}\,du = \ln(|u|) + C = \ln(|1-4\cos(\theta)|) + C.$$

So

$$\int_0^{\pi/3} \frac{4\sin(\theta)}{1-4\cos(\theta)}\,d\theta = \ln(|1-4\cos(\theta)|)\Big]_0^{\pi/3} = \ln(1) - \ln(3) = -\ln(3).$$

$\square$

4. **(10 pts)** Compute

$$\int te^{-t^2}\,dt.$$

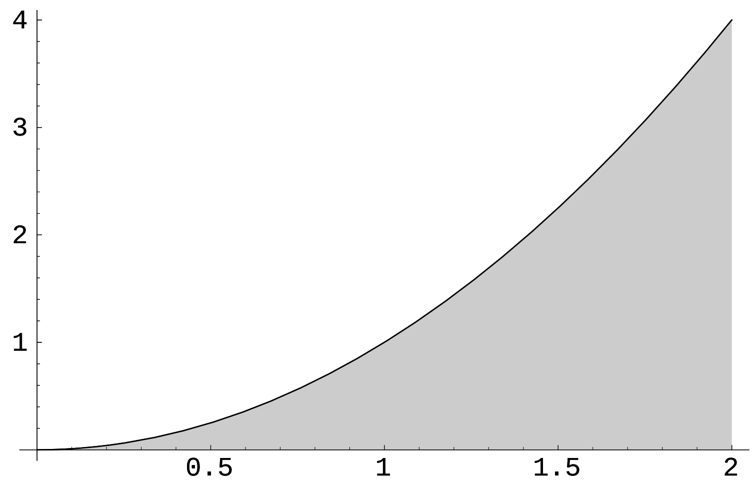
Solution. Use the substitution $u = -t^2$, which leads to $du = -2t\,dt$. This leads to

$$\int te^{-t^2}\,dt = \int e^u(-1/2)\,du = (-1/2)e^u + C = (-1/2)e^{-t^2} + C.$$

$\square$

5. **(15 pts)** Consider the solid obtained by rotating the region bounded by $y = x^2$, $x = 2$, $x = 0$, and $y = 0$ about the y-axis. Set up the integral to compute the volume of this solid. You *do not* need to compute the value of the integral.

Solution. The region is illustrated in the following plot.



A slice of thickness Δy at the point y will be (approximately) a washer with outside radius 2 and inside radius approximately $\sqrt{y}$. This washer will then have volume approximately

$$[\pi(2^2) - \pi((\sqrt{y})^2)]\Delta y,$$

leading to the integral

$$\int_0^4 \pi[4 - y]\, dy.$$

The value of this integral is 8π. (You didn't have to compute this to get credit for this problem.) $\square$

6. **(15 pts)** A rectangular swimming pool is 50 feet long, 20 feet wide, and has a uniform depth of 15 feet. The pool is filled with water, which weighs about 62.4 pounds per cubic foot. Set up the integral to compute the work necessary to pump all the water to the level of the top of the pool. You *do not* need to compute the value of the integral.

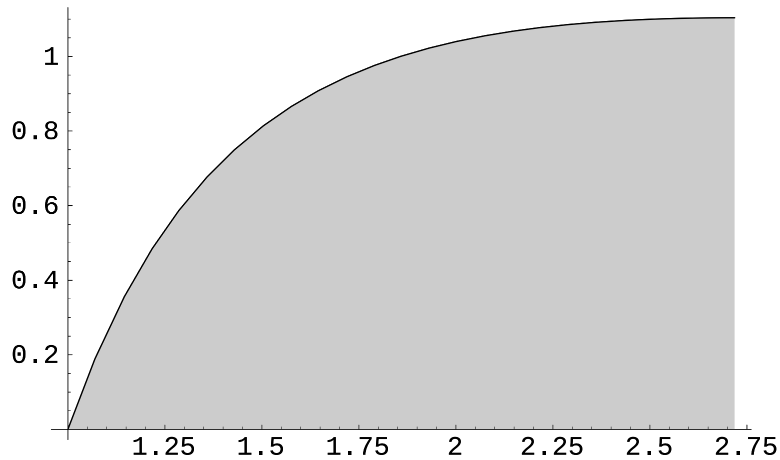
Solution. This problem is *very* similar to the problem on Quiz 3. Let the bottom of the pool be situated at $y = 0$ and the top at $y = 15$. A slice of thickness Δy at y has weight $(62.4)(20)(50)\Delta y = 62400\Delta y$. The distance this slice has to be moved is about $15 - y$ feet. So the work involved in pumping all the water out of the pool is

$$\int_0^{15} 62400(15 - y)\, dy.$$

The value of this integral is 7020000. (You didn't need to compute this value to get credit for this problem.) $\square$

7. **(15 pts)** Compute the area of the region bounded by the curve $y = 3\ln(x)/x$, the x-axis, and the lines $x = 1$ and $x = e$.

Solution. A picture of the region is given in the plot below.



The area of this region is

$$\int_1^e 3\ln(x)/x\,dx.$$

To compute this integral, use the substitution $u = \ln(x)$. Then $du = (1/x)\,dx$. Also $u(1) = 0$ and $u(e) = 1$. So the area becomes

$$\int_0^1 3u\,du = (3u^2/2)\Big]_0^1 = (3/2).$$

□

8. **(10 pts)** Solve the following for y: $8e^{2y} = x$.

Solution. Taking the natural logarithm of both sides, using the fact that the logarithm of a product is the sum of the logarithms, and using the fact that the natural logarithm function is the inverse of the exponential function, we get $\ln(8) + 2y = \ln(x)$. Solve this for y to get $y = [\ln(x) - \ln(8)]/2$. This can be written as

$$y = \frac{1}{2}\ln\left(\frac{x}{8}\right).$$

□