Packing Dimension Profiles and Random Fractals

Yimin Xiao

Department of Statistics and Probability

Michigan State University

URL: www.stt.msu.edu/~xiaoyimi

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1. Hausdorff dimension

For $\beta > 0$, the β-dimensional Hausdorff measure of $E \subset \mathbb{R}^N$ is defined by

$$s^\beta\text{-}m(E) = \quad \lim_{\varepsilon \to 0} \inf \left\{ \sum_i (2r_i)^\beta : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\}.$$

The Hausdorff dimension of E is defined by

$$\dim_{\mathrm{H}} E = \inf\{\beta > 0 : \quad s^\beta\text{-}m(E) = 0\}.$$

2. Packing dimension

Packing measure and packing dimension were introduced by Tricot (1982), Taylor and Tricot (1985). For any $E \subset \mathbb{R}^N$, define

$$s^\beta\text{-}P(E) = \lim_{\varepsilon \to 0} \ \sup \left\{ \sum_i (2r_i)^\beta : \{\overline{B}(x_i, r_i)\} \ \text{ is an } \varepsilon\text{-packing} \right\}$$

The packing measure s^β-p and packing dimension of E are defined as:

$$s^\beta\text{-}p(E) = \inf \left\{ \sum_n s^\beta\text{-}P(E_n) : \quad E \subseteq \bigcup_n E_n \right\}.$$

$$\dim_{\mathrm{P}} E = \inf\{\beta > 0 : \quad s^\beta\text{-}p(E) = 0\}.$$

In general,

$$s^\beta\text{-}m(E) \le s^\beta\text{-}p(E), \quad \dim_{\mathrm{H}} E \le \dim_{\mathrm{P}} E.$$

3. Usefulness of packing dimension

Packing dimension is need for solving the following problems:

- Smallest sets that can be hit by Brownian motion;
- Hausdorff dimension of Cartesian products;
- Hitting probabilities of Limsup random fractals;
- Fractal properties of fractional Brownian motion [see next page for definition].

An (N, d)-fractional Brownian motion **(fBm)** $X = \{X(t),\ t \in \mathbb{R}^N\}$ of index $\alpha \in (0, 1)$ is a Gaussian random field defined by

$$X(t) = (X_1(t), \cdots, X_d(t)),$$

where $X_1, \cdots, X_d$ are independent copies of $Y = \{Y(t),\ t \in \mathbb{R}^N\}$, which is a centered Gaussian random field with $Y(0) = 0$ and

$$\mathbb{E}\Big[(Y(t) - Y(s))^2\Big] = |t - s|^{2\alpha}, \quad (0 < \alpha < 1).$$

Remark When $N = 1$ and $\alpha = 1/2$, then X is Brownian motion in $\mathbb{R}^d$.

3.1 Smallest sets that can be hit by fBm

When B is the ordinary Brownian motion in $\mathbb{R}^d$, Kakutani (1944) proved the following important theorem in probabilistic potential theory:

Theorem 3.1 For any compact set F in $\mathbb{R}^d \backslash \{0\}$,

$$\mathbb{P}\Big\{B([0,\infty)) \cap F \neq \emptyset\Big\} > 0 \iff \mathrm{Cap}_\kappa(F) > 0, \tag{3.1}$$

where

$$\kappa(x,y) = \begin{cases} \frac{1}{|x-y|^{d-2}} & \text{if } d \geq 3 \\ \log \frac{1}{|x-y|} & \text{if } d = 2. \end{cases}$$

Many Extensions of Theorem 3.1:

- Brownian motion: Kaufman (1972), Taylor and Watson (1985).
- Lévy processes: Hawkes (1978), ...
- FBm: Kahane (1983), Testard (1987),
- The Brownian sheet: Khoshnevisan (1997), Khoshnevisan and Shi (1999).
- Additive Lévy processes: Khoshnevisan (2002), Khoshnevisan and Xiao (2002, 2003, 2005, 2007).

In particular, Kaufman (1972) gave conditions on $E \subset \mathbb{R}_+$ and $F \subset \mathbb{R}^d \setminus \{0\}$ such that

$$\mathbb{P}\{B(E) \cap F \neq \emptyset\} > 0.$$

Question 3.1: (Peres (1996)) Given $E \subset \mathbb{R}_+$,

$$\inf\left\{ \dim_{\mathrm{H}} F : F \in \mathcal{B}(\mathbb{R}^d),\ \mathbb{P}\{B(E) \cap F \neq \emptyset\} > 0 \right\} = ?$$

Xiao (1999) solved this problem for fractional Brownian motion $X = \{X(t),\, t \in \mathbb{R}^N\}$ in general.

Theorem 3.2. Let X be an (N, d)-fractional Brownian motion with index $\alpha \in (0, 1)$. Then for any compact set $E \subset \mathbb{R}^N \setminus \{0\}$,

$$\inf\Big\{\dim_{\mathrm{H}} F : F \in \mathcal{B}(\mathbb{R}^d),\ \mathbb{P}\{X(E) \cap F \neq \emptyset\} > 0\Big\}$$
$$= d - \frac{\dim_{\mathrm{P}} E}{\alpha}.$$

3.2 Hausdorff dimension of Cartesian products

Tricot (1982): For any $E,\quad F \subseteq \mathbb{R}^N$,

$$\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F \le \dim_{\mathrm{H}}(E \times F) \le \dim_{\mathrm{H}} E + \dim_{\mathrm{P}} F$$

$$\le \dim_{\mathrm{P}}(E \times F) \le \dim_{\mathrm{P}} E + \dim_{\mathrm{P}} F.$$

Kaufman (1987) defined a dimension adim by

$$\mathrm{adim} E = \sup\{\dim_{\mathrm{H}}(E \times F) - \dim_{\mathrm{H}} F\ ,\quad F \in \mathcal{B}(\mathbb{R}^N)\},$$

Hu and Taylor (1994) defined aDim by

$$\mathrm{aDim} E = \inf\{\dim_{\mathrm{P}}(E \times F) - \mathrm{Dim} F\ ,\quad F \in \mathcal{B}(\mathbb{R}^N)\}.$$

It is easy to see that

$$\dim_{\mathrm{H}} E \le \mathrm{adim} E \le \dim_{\mathrm{P}} E, \quad \dim_{\mathrm{H}} E \le \mathrm{aDim} E \le \dim_{\mathrm{P}} E.$$

QUESTION 3.2: (Hu and Taylor (1994)) For $E \subset \mathbb{R}^N$,

$$\mathrm{adim} E = \dim_{\mathrm{P}} E, \qquad \mathrm{aDim} E = \dim_{\mathrm{H}} E?$$

Xiao (1996), and independently Bishop and Peres (1996), proved that

$$\mathrm{adim} E = \dim_{\mathrm{P}} E, \quad \text{but} \quad \mathrm{aDim} E \neq \dim_{\mathrm{H}} E.$$

3.3 Limsup random fractals

Khintchin's law of iterated logarithm: Let $B = \{B(t), t \geq 0\}$ be a Brownian motion in $\mathbb{R}$. Then for every $t_0 \geq 0$,

$$\limsup_{h \to 0} \frac{\left|B(t_0 + h) - B(t_0)\right|}{\sqrt{2h|\log\log h|}} = 1, \qquad \text{a.s.} \tag{3.2}$$

This describes the local asymptotic behavior of B at a fixed time t_0.

Lévy's modulus of continuity:

$$\lim_{h \to 0} \sup_{t \in [0,1],\ 0 \leq s \leq h} \frac{\left|B(t+s) - B(t)\right|}{\sqrt{2h|\log h|}} = 1., \qquad \text{a.s.}$$

This suggests that there are time-points where $B(t)$ oscillates faster than (3.2) suggests.

For $\lambda \in [0,1]$, Orey and Taylor (1974) considered the set of λ-**fast points** for Brownian motion B:

$$\mathrm{F}(\lambda) = \left\{ t \in [0,1] : \ \limsup_{h \to 0^+} \frac{\big|B(t+h) - B(t)\big|}{\sqrt{2h|\log h|}} \geq \lambda \right\}.$$

and proved that

$$\forall \lambda \in (0,1], \qquad \dim_{\mathrm{H}}\big(\mathrm{F}(\lambda)\big) = 1 - \lambda^2 \, .$$

Kaufman (1974): for $E \subset [0,1]$ with $\dim_{\mathrm{H}} E > \lambda^2$,

$$\mathbb{P}\{\mathrm{F}(\lambda) \cap E \neq \emptyset\} = 1. \tag{3.3}$$

Khoshnevisan, Peres and Xiao (2000) provided a sharp condition for (3.3) to hold in terms of packing dimension.

Theorem 3.3 (Khoshnevisan, Peres and Xiao (2000)) Let B denote linear Brownian motion. For any analytic set $E \subset \mathbb{R}_+$,

$$\sup_{t\in E} \limsup_{h\to 0+} \frac{\big|B(t+h)-B(t)\big|}{\sqrt{2h|\log h|}} = \big(\dim_{\mathrm{P}} E\big)^{1/2}, \quad \text{a.s.}$$

Equivalently,

$$\forall \lambda > 0 \quad \mathbb{P}\Big(\mathrm{F}(\lambda) \cap E \neq \emptyset\Big) = \begin{cases} 1\ , & \text{if } \dim_{\mathrm{P}} E > \lambda^2 \\ 0 & \text{if } \dim_{\mathrm{P}} E < \lambda^2 \end{cases}$$

Moreover, if $\dim_{\mathrm{P}} E > \lambda^2$ then $\dim_{\mathrm{P}}(F(\lambda) \cap E) = \dim_{\mathrm{P}}(E)$ a.s.

3.4 Packing dimension of the images of fBm

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be an (N, d)- fractional Brownian motion with index $\alpha \in (0, 1)$. Kahane (1985) proved that for every Borel set $E \subset \mathbb{R}^N$,

$$\dim_{\rm H} X(E) = \min\left(d, \quad \frac{1}{\alpha}\dim_{\rm H} E\right), \quad \text{a.s.} \tag{3.4}$$

Question 3.3: Does (3.4) holds for $\dim_{\rm P} X(E)$?

The answer is "no". Talagrand and Xiao (1996) proved that there exist compact sets $E \subseteq \mathbb{R}^N$ such that

$$\dim_{\rm P} X(E) < \min\left\{d, \frac{1}{\alpha}\dim_{\rm P} E\right\}, \quad \text{a.s.}$$

Their result suggests the need for a new "packing dimension" for answering Question 3.3.

Similar problems arise in studies of packing dimension of orthogonal projections. See Järvenpää (1994), Falconer, and Howroyd (1996, 1997), ...

4. Packing dimension profiles

Packing dimension profiles were introduced and studied by Falconer and Howroyd (1997) and Howroyd (2001).

4.1 Definition of Falconer and Howroyd

Given a finite Borel measure μ on $\mathbb{R}^N$ and an $s \in (0\,,\infty]$ define

$$F_s^\mu(x\,,r) := \int_{\mathbb{R}^N} \psi_s\left(\frac{x-y}{r}\right)\,\mu(dy),$$

where for $s \in (0\,,\infty)$,

$$\psi_s(x) := \min\left(1\,,|x|^{-s}\right) \qquad \forall\, x \in \mathbb{R}^N,$$

and $\psi_\infty := \mathbf{1}_{\{y\in\mathbb{R}^N:\,|y|\le 1\}}$.

The s-*dimensional packing dimension profile* of μ is defined as

$$\mathrm{Dim}_s\mu = \sup\left\{ t \geq 0 : \liminf_{r\downarrow 0} \frac{F_s^{\mu}(x\,,r)}{r^t} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N \right\}.$$

Packing dimension profiles generalize the packing dimension because for all $s \geq N$,

$$\dim_{\mathrm{P}}\mu = \mathrm{Dim}_s\mu \qquad \text{for all Borel measures } \mu \text{ on } \mathbb{R}^N.$$

Falconer and Howroyd (1997) also defined the s-dimensional packing dimension profile of a Borel set $E \subseteq \mathbb{R}^N$ by

$$\mathrm{Dim}_s E = \sup\left\{ \mathrm{Dim}_s\mu : \ \mu \in \mathcal{M}_c^+(E) \right\}.$$

Using packing dimension profiles, Xiao (1997) solved Question 3.3.

Theorem 4.1 If X is an (N, d)-fBm with index $\alpha \in (0, 1)$. Then for every analytic set $E \subseteq \mathbb{R}^N$, we have

$$\dim_{\mathrm{P}} X(E) = \frac{1}{\alpha}\, \mathrm{Dim}_{\alpha d} E, \quad \text{a.s.}$$

For applications to orthogonal projections, see Falconer and Howroyd (1997).

4.2 Definition of Howroyd

If $E \subseteq \mathbb{R}^N$ and $s > 0$, then a sequence of triples $(w_i\,, x_i\,, r_i)_{i=1}^{\infty}$ is called a $(\psi_s\,, \delta)$-*packing of* E whenever $w_i \geq 0$, $x_i \in E$, $0 < r_i \leq \delta$, and

$$\sup_{i \geq 1} \sum_{j=1}^{\infty} w_j\, \psi_s\left(\frac{x_i - x_j}{r_j}\right) \leq 1.$$

For all $E \subset \mathbb{R}^N$, define

$$\mathcal{P}_0^{\alpha,s}(E)$$
$$= \lim_{\delta \downarrow 0} \sup \left\{ \sum_{i=1}^{\infty} w_i\,(2r_i)^{\alpha} \;:\; (w_i\,, x_i\,, r_i)_{i=1}^{\infty} \text{ is a } (\psi_s\,, \delta)\text{-packing of } E \right\}.$$

Then the α-*dimensional* ψ_s-*packing measure* $\mathcal{P}^{\alpha,s}(E)$ is defined as

$$\mathcal{P}^{\alpha,s}(E) = \inf\left\{\sum_{k=1}^{\infty} \mathcal{P}_0^{\alpha,s}(E_k) :\ E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}.$$

The s-*dimensional packing dimension profile of* E is defined as

$$\text{P-dim}_s E := \inf\left\{\alpha > 0 :\ \mathcal{P}^{\alpha,s}(E) = 0\right\}.$$

Howroyd (2001) proved that if s is an integer, then

$$\text{P-dim}_s E = \text{Dim}_s E \quad \text{all analytic sets } E \subseteq \mathbb{R}^N. \tag{4.1}$$

QUESTION 4.1 (Howroyd (2001)): Does (4.1) hold for all $s > 0$?

Recently, Khoshnevisan and Xiao (2006) gave an affirmative answer to Question 4.1 by proving the following result.

Theorem 4.2 If X is an (N,d)-fractional Brownian motion with index $\alpha \in (0,1)$. Then for every analytic set $E \subseteq \mathbb{R}^N$, we have

$$\dim_{\mathrm{P}} X(E) = \frac{1}{\alpha}\,\mathrm{P}\text{-}\dim_{\alpha d} E, \quad \text{a.s.}$$

By choosing α and d appropriately, Theorems 4.1 and 4.2 together imply that for all real numbers $s > 0$,

$$\mathrm{P}\text{-}\dim_s E = \mathrm{Dim}_s E \quad \text{for all analytic sets } E \subseteq \mathbb{R}^N.$$