Asymptotic Results for Self-Similar Markov Processes *

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Dedicated to Professor Miklós Csörgő, on the occasion of his 65th birthday

Let $X(t)$ $(t \in \mathbf{R}_+)$ be an α-self-similar Markov process on $\mathbf{R}^d$ or $\mathbf{R}^d_+$. For two types of such processes, the lower functions for $X(t)$ are studied. As a consequence, the results of Khoshnevisan (1996) and Knight (1973) on the lower class of the maximum of Brownian motion normalized by its local time at 0 are recovered.

1. Introduction

The class of α-self-similar (α-*s.s.*) Markov processes on $(0, \infty)$ and on $[0, \infty)$ were introduced and studied by Lamperti [14], who used the name "semi-stable". A very important and useful result in Lamperti [14] is Theorem 4.1, which relates, through random time change, a $[0, \infty)$-valued self-similar Markov process with a real-valued Lévy process and hence makes it possible to study sample path properties of α-*s.s.* Markov processes $[0, \infty)$ by using known results for Lévy processes. See Lamperti [14], Vuolle-Apiala [24], Liu [16], Li et al [15], Xiao and Liu [28], and the references therein.

Graversen and Vuolle-Apiala [6] generalized some of Lamperti's results, including the above mentioned Theorem 4.1, to $\mathbf{R}^d$-valued isotropic α-*s.s.* Markov processes. See also Kiu [11]. Vuolle-Apiala and Graversen [26], based on the results in [6], studied the duality of isotropic α-*s.s.* Markov processes on $\mathbf{R}^d \backslash \{0\}$. However, it seems very difficult to apply the results of Graversen and Vuolle-Apiala [6] to study sample path properties such as lower functions, the exact Hausdorff measure and the local times of (isotropic) $\mathbf{R}^d$-valued α-*s.s.* Markov processes via corresponding results for Lévy processes, because one has to deal with the angular process (see [6]). This consideration motivates us to study the sample path properties of α-*s.s.* Markov processes in $\mathbf{R}^d$ through their Markov property and transition functions, and it turns out that we can generalize many results about Brownian motion and stable Lévy processes to more general α-*s.s.* Markov processes.

In this paper, we study lower functions (escape rates) for certain α-*s.s.* Markov processes on $\mathbf{R}^d$ or $\mathbf{R}^d_+$. There has been a lot of work on the lower functions for Brownian motion and stable Lévy processes. See Dvoretsky and Erdös [4] and Spitzer [18] for the Brownian motion case; Takeuchi [19] for symmetric stable processes; Takeuchi [20] and Taylor [23] for transient stable processes; Takeuchi and Watanabe [21] for the Cauchy process; Fristedt [5] and Breiman [2] for stable subordinators. The methods in all these papers depend

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heavily on the special properties of Brownian motion or stable Lévy processes and hence cannot be applied directly to α-*s.s.* Markov processes. Recently, Khoshnevisan [9] gave a different method for estimating hitting probabilities of Lévy processes and studied the escape rates for Lévy processes with stable components.

The rest of the paper is organized as follows. In Section 2 we recall briefly the definition of α-*s.s.* Markov processes and prove, by modifying an argument of Khoshnevisan [9], a basic estimate about the hitting probability of strong Markov processes on $\mathbf{R}^d$. In Section 3, we study the escape rates for certain transient α-*s.s.* Markov processes including stable Lévy processes and their relatives. Theorem 3.1 generalizes the previous results about Brownian motion and stable Lévy processes mentioned above. In Section 4, we study the lower functions of the α-*s.s.* Markov process in $\mathbf{R}^d_+$ defined by

$$X(t) = (X_1(t), \cdots, X_d(t)) \ ,$$

where $X_1, \cdots .X_d$ are independent copies of the extremal α-*s.s.* Markov process considered by Lamperti [13] [14]. As a consequence of Theorem 4.1, we recover the results of Khoshnevisan [10] and Knight [12] on the lower class of the maximum of Brownian motion normalized by its local time at 0.

We will use $K, K_1, \cdots, K_8$ to denote positive and finite constants whose precise values are not important and may be different in each appearance.

2. Definitions and a Basic Estimate

Throughout this paper, $(E, \mathcal{B})$ denotes $\mathbf{R}^d$, $\mathbf{R}^d\backslash\{0\}$ or $\mathbf{R}^d_+$ with the usual Borel σ-algebra, Δ a point attached to E as an isolated point. Ω denotes the space of all functions ω from $[0, \infty)$ to $E \cup \{\Delta\}$ having the following properties:

(i) $\omega(t) = \Delta$ for $t \geq \tau$, where $\tau = \inf\{t \geq 0;\ \omega(t) = \Delta\}$;

(ii) ω is right continuous and has a left limit at every $t \in [0, \infty)$.

Let $\alpha > 0$ be a given constant. A stochastic process $X = (X(t), P^x)$ with state space $E \cup \{\Delta\}$ is called an α-self-similar Markov process if there exists a transition function $\mathbf{P}(t, x, A)$ satisfying

$$\mathbf{P}(0, x, A) = I_A(x) \quad \text{for all} \quad x \in E,\ A \in \mathcal{B} \tag{2.1}$$

where $I_A(\cdot)$ denotes the indicator function of A, and

$$\mathbf{P}(t, x, A) = \mathbf{P}(at, a^\alpha x, a^\alpha A) \ \text{ for all } \ t > 0,\ a > 0,\ x \in E,\ A \in \mathcal{B} \tag{2.2}$$

such that $(X(t), P^x)$ is a time homogeneous Markov process with a transition function $\mathbf{P}(t, x, A)$ and for every $x \in E$, $X(t) \in \Omega$ P^x-almost surely.

For $d > 1$, X is called an isotropic α-*s.s.* Markov process if its transition function further satisfies the following condition

$$\mathbf{P}(t, x, A) = \mathbf{P}(t, \phi(x), \phi(A)) \ \text{ for all } \ t \geq 0,\ x \in E,\ A \in \mathcal{B},\ \phi \in O(d) \tag{2.3}$$

where $O(d)$ denotes the group of orthogonal transformations on $\mathbf{R}^d$.

REMARK Condition (2.2) is equivalent to the statement that for every $a > 0$, the P^x-distribution of $X(t)$ $(t \geq 0)$ is equal to the $P^{a^\alpha x}$-distribution of $a^{-\alpha}X(at)$ $(t \geq 0)$. We write this self-similar property as

$$(X(\cdot), P^x) \stackrel{\mathrm{d}}{=} (a^{-\alpha}X(a\cdot), P^{a^\alpha x}) \quad \text{for every} \quad a > 0. \tag{2.4}$$

It is easy to see that all $1/\alpha$-strictly stable Lévy processes on $\mathbf{R}^d$ are α-$s.s.$ Markov processes and $1/\alpha$ symmetric stable Lévy processes are isotropic a-$s.s.$ Markov processes. It is proved by Graversen and Vuolle-Apiala [6] that if $X(t)$ is an isotropic α-$s.s.$ Markov process on E, then $(|X(t)|, P^{|x|})$ is an α-$s.s.$ Markov process on $|E|$; and if $X(t)$ is an α-$s.s.$ Markov process on $\mathbf{R}^d$, then for every $\gamma > 0$

$$(X(t)^{<\gamma>}, P^{x^{<1/\gamma>}})$$

is also an $\alpha\gamma$-$s.s.$ Markov process on $\mathbf{R}^d$, where $0^{<\gamma>} = 0$ and $x^{<\gamma>} = x|x|^{\gamma-1}$ for $x \neq 0$.

The Bessel processes form exactly the class of $1/2$-$s.s.$ diffusions on $(0, \infty)$. We refer to Revuz and Yor [17] for the definition and properties of Bessel processes.

More examples of α-$s.s.$ Markov processes can be found in Lamperti [14], Graversen and Vuolle-Apiala [6], Vuolle-Apiala and Graversen [26] and Vuolle-Apiala [25].

From now on, we will only consider α-$s.s.$ Markov processes with the strong Markov property. It was shown by Lamperti [14] and Graversen and Vuolle-Apiala [6] that every self-similar Markov process on $(0, \infty)$ and every isotropic self-similar Markov process on $\mathbf{R}^d \backslash \{0\}$ is automatically a strong Markov process with respect to a right-continuous filter of σ-algebras.

The following estimates on hitting probabilities for strong Markov processes generalize Theorem 1.1 of Khoshnevisan [9] and will prove to be useful later.

Proposition 2.1 *Let $X(t)$ $(t \geq 0)$ be a time homogeneous strong Markov process in $\mathbf{R}^d$ with transition function $\mathbf{P}(t, x, A)$. Then for every $x \in \mathbf{R}^d$, $c > b > 0$ and $r > 0$ we have*

$$\begin{aligned} \frac{1}{2}\frac{\int_b^c \mathbf{P}(t, x, B(0, r))dt}{\sup_{|y|\leq r} \int_0^c \mathbf{P}(t, y, B(0, r))dt} &\leq P^x\Big(|X(t)| \leq r \ \text{ for some } \ b \leq t \leq c\Big) \\ &\leq \frac{\int_b^{2c-b} \mathbf{P}(t, x, B(0, r))dt}{\inf_{|y|\leq r} \int_0^{c-b} \mathbf{P}(t, y, B(0, r))dt} \end{aligned} \tag{2.5}$$

where $B(0, r) = \{u \in \mathbf{R}^d : |u| \leq r\}$.

Proof . Define $T = \inf\{s \geq b : \ |X(s)| \leq r\}$. Then by the strong Markov property, we have

$$\begin{aligned} E^x \int_b^{2c-b} 1_{\{|X(t)|\leq r\}} dt &\geq E^x\Big[E^{X(T)}\Big(\int_T^{2c-b} 1_{\{|X(t-T)|\leq r\}} dt\Big); \ T \leq c\Big] \\ &\geq \inf_{|y|\leq r}\Big\{\int_0^{c-b} \mathbf{P}(t, y, B(0, r))dt\Big\} \cdot P^x(T \leq c) \end{aligned}$$

This proves the upper bound in (2.5).

On the other hand, the Cauchy-Schwartz inequality gives

$$\begin{aligned} E^x \int_b^c 1_{\{|X(t)|\le r\}} dt &= E^x\Big[\int_b^c 1_{\{|X(t)|\le r\}} dt;\ T\le c\Big] \\ &\le \Big[E^x\Big(\int_b^c 1_{\{|X(t)|\le r\}} dt\Big)^2\Big]^{1/2}\cdot\Big[P^x(T\le c)\Big]^{1/2}. \end{aligned}$$

Thus

$$P^x(T\le c)\ge \frac{\Big(E^x\int_b^c 1_{\{|X(t)|\le r\}}dt\Big)^2}{E^x\Big(\int_b^c 1_{\{|X(t)|\le r\}}dt\Big)^2}. \tag{2.6}$$

The expectation in the denominator of (2.6) is

$$\begin{aligned} &2\int_b^c ds\int_s^c P^x(|X(s)|\le r,\ |X(t)|\le r)dt \\ &= 2\int_b^c ds\int_s^c\int_{|y|\le r}\mathbf{P}(t-s,y,B(0,r))\mathbf{P}(s,x,dy)dt \\ &= 2\int_b^c ds\int_{|y|\le r}\Big(\int_s^c \mathbf{P}(t-s,y,B(0,r))dt\Big)\mathbf{P}(s,x,dy) \\ &\le 2\sup_{|y|\le r}\Big\{\int_0^c \mathbf{P}(t,y,B(0,r))dt\Big\}\int_b^c \mathbf{P}(s,x,B(0,r))ds. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) yields the lower bound in (2.5).

We will consider two classes of α-*s.s.* Markov processes, for which we can get useful estimates for the hitting probabilities, and study their asymptotic properties.

3. Type I α-*s.s.* Markov Processes

In this section, we consider α-*s.s.* Markov processes in $\mathbf{R}^d$ satisfying the strong Markov property and the following condition: there exist positive constants η_0, β, K_1 and K_2 such that for every $r>0$ and $x\in\mathbf{R}^d$ with $|x|\le\eta_0$ we have

$$K_1\min\{1,r^\beta\}\le \mathbf{P}(1,x,B(0,r))\le K_2\min\{1,r^\beta\}. \tag{3.1}$$

Clearly, Condition (3.1) is satisfied by strictly stable Lévy processes $X(t)$ with $\beta=d$, because its transition function is translation invariant and $X(1)$ has a bounded density function; by $X(t)^{<\gamma>}$ with $\beta=d/\gamma$ and by a Bessel process of dimension δ (not necessarily an integer) with $\beta=\delta$. It should be noticed that the transition functions of the last two processes are not translation invariant.

The following lemma is a corollary of Proposition 2.1.

Lemma 3.1 *Let $X(t)$ $(t\ge 0)$ be an α-s.s. Markov process in $\mathbf{R}^d$ verifying (3.1). For any given $c>b>0$, there exist positive constants η_1, K_3 and K_4 such that for every $0<r\le\eta_1$ and $x\in\mathbf{R}^d$ with $|x|\le\eta_1$ we have*

$$K_3 r^{\beta-1/\alpha}\le P^x\Big(|X(t)|\le r\ \ \textit{for some}\ \ b\le t\le c\Big)\le K_4 r^{\beta-1/\alpha} \tag{3.2}$$

if $\alpha\beta>1$; and

$$\frac{K_3}{\log 1/r}\le P^x\Big(|X(t)|\le r\ \ \textit{for some}\ \ b\le t\le c\Big)\le \frac{K_4}{\log 1/r} \tag{3.3}$$

if $\alpha\beta=1$.

Proof . For any $y \in \mathbf{R}^d$ with $|y| \le r$, by (2.2) and (3.1) we have

$$\begin{aligned}\int_0^{c-b} \mathbf{P}(t, y, B(0,r))dt &= \int_0^{c-b} \mathbf{P}(1, y/t^\alpha, B(0, r/t^\alpha))dt \\ &\ge K_1 \int_{r^{1/\alpha}}^{c-b} \min\Big(1, \frac{r^\beta}{t^{\alpha\beta}}\Big)dt \\ &\ge \begin{cases} Kr^{1/\alpha} & \text{if } \alpha\beta > 1 \\ Kr^\beta \log 1/r & \text{if } \alpha\beta = 1 \end{cases}\end{aligned} \tag{3.4}$$

On the other hand, by (2.2) and (3.1) we have for $\alpha\beta \ge 1$

$$\int_b^{2c-b} \mathbf{P}(t, x, B(0,r))dt \le Kr^\beta \ . \tag{3.5}$$

Now the right inequalities in (3.2) and (3.3) follow from Proposition 2.1, (3.4) and (3.5). The left inequalities can be proved similarly.

With the help of Lemma 3.1, we can prove the following result in a standard way.

Proposition 3.1 *Let $X(t)$ $(t \ge 0)$ be an α-s.s. Markov process in $\mathbf{R}^d$ verifying (3.1). If $\alpha\beta = 1$, then singletons are polar, but $(X(t), P^0)$ is neighborhood recurrent. If $\alpha\beta > 1$, then $X(t)$ is transient in the sense that for every $x \in \mathbf{R}^d$*

$$P^x(|X(t)| \to \infty \quad as \quad t \to \infty) = 1 \ .$$

It is easy to prove that if $\alpha\beta < 1$, then $X(t)$ $(t \ge 0)$ hits points with positive probability and has a square-integrable local time. These results, together with the joint continuity and Hölder conditions of the local times, will be proved in a subsequent paper.

We will make use of the following extension of the Borel-Cantelli lemma due to Chung and Erdös [3].

Lemma 3.2 *Let $\{A_n\}$ be an infinite sequence of events satisfying the following conditions:*

(i). $\sum_{n=1}^\infty P(A_n) = \infty$;

(ii). For every pair $h, m \in \mathbf{N}, h \le m$, there exist positive constants $K(h)$ and $H(h,m) > m$ such that for every $n \ge H(h,m)$,

$$P(A_n | A_h^c \cap A_{h+1}^c \cap \cdots \cap A_m^c) > K(h)P(A_n),$$

where $P(A|B)$ is the conditional probability of A given B and A^c denotes the complement of A;

(iii). There exist two absolute positive constants K_5, K_6 with the following property: to each A_m there corresponds a set (may be infinite) of events $A_{m_1}, \cdots A_{m_s}$ in $\{A_n\}$ such that

$$\sum_{k=1}^s P(A_m \cap A_{m_k}) < K_5 P(A_m)$$

and that for any other A_n than $A_{m_k} (1 \le k \le s)$ and $n > m$,

$$P(A_m \cap A_n) \le K_6 P(A_m)P(A_n);$$

Then $P(A_n\ i.\ o.) = 1$.

Now we prove the main theorem of this section, which extends the results of Dvoretsky and Erdös [4], Spitzer [18], Takeuchi [19], Taylor [23] and Takeuchi and Watanabe [21].

Theorem 3.1 *Let $X(t)$ $(t \geq 0)$ be an α-s.s. Markov process in $\mathbf{R}^d$ verifying (3.1). For any positive non-increasing function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$, set*

$$\mathcal{I}(\phi) = \begin{cases} \int_1^\infty t^{-1}\phi(t)^{\beta-1/\alpha}dt & \text{if} \quad \alpha\beta > 1 \\ \int_1^\infty (t|\log\phi(t)|)^{-1}dt & \text{if} \quad \alpha\beta = 1. \end{cases}$$

Then P^0-almost surely

$$\liminf_{t\to\infty} \frac{|X(t)|}{t^\alpha\phi(t)} = \begin{cases} \infty & \text{if} \quad \mathcal{I}(\phi) < \infty \\ 0 & \text{if} \quad \mathcal{I}(\phi) = \infty. \end{cases}$$

Proof . For fixed $\gamma > 0$ and $c > 1$, consider the sequence of events

$$A_n = \left\{|X(t)| \leq \gamma c^{n\alpha}\phi(c^n) \ \text{ for some } \ c^n \leq t \leq c^{n+1}\right\} .$$

It follows from (2.4) and Lemma 3.1 that

$$\begin{aligned} P^0(A_n) &= P^0\left(|X(t)| \leq \gamma\phi(c^n) \ \text{ for some } \ 1 \leq t \leq c\right) \\ &\leq \begin{cases} K_\gamma\phi(c^n)^{\beta-1/\alpha} & \text{if } \alpha\beta > 1 \\ K_\gamma|\log\phi(c^n)|^{-1} & \text{if } \alpha\beta = 1 \end{cases} \end{aligned}$$

When $\mathcal{I}(\phi) < \infty$, by the monotonicity of $\phi(t)$ we have $\sum_{n=1}^\infty P^0(A_n) < \infty$. By the easy part of the Borel-Cantelli lemma and the arbitrariness of γ, we obtain

$$\liminf_{t\to\infty} \frac{|X(t)|}{t^\alpha\phi(t)} = \infty \quad P^0\text{-almost surely.}$$

Now suppose that $\mathcal{I}(\phi) = \infty$. Then (2.4) and Lemma 3.1 imply $\sum_{n=1}^\infty P^0(A_n) = \infty$. It remains to verify that Conditions (ii) and (iii) of Lemma 3.2 are satisfied. We only consider the case of $\alpha\beta > 1$ and the proof for the case $\alpha d = 1$ is similar. For any fixed pair (h, m) of integers with $h < m$, we choose a positive constant $a_{h,m}$ such that

$$P^0\left(A_h^c \cap A_{h+1}^c \cap \cdots \cap A_m^c \cap B_m\right) \geq \frac{1}{2}P^0\left(A_h^c \cap A_{h+1}^c \cap \cdots \cap A_m^c\right),$$

where $B_m = \{|X(c^{m+1})| \leq a_{h,m}\}$. Then for every $n > m$,

$$P^0\left(A_n|A_h^c \cap A_{h+1}^c \cap \cdots \cap A_m^c\right) \geq \frac{1}{2}P^0\left(A_n|A_h^c \cap \cdots \cap A_m^c \cap B_m\right) \tag{3.6}$$

By the Markov property, we have

$$\begin{aligned} &P^0\left(A_n \cap A_h^c \cap \cdots \cap A_m^c \cap B_m\right) \\ &= E^0\Big[P^{X(c^{m+1})}\left(|X(t - c^{m+1})| \leq \gamma c^{n\alpha}\phi(c^n) \ \text{ for some } c^n \leq t \leq c^{n+1}\right); \\ &\qquad\qquad A_h^c \cap \cdots \cap A_m^c \cap B_m\Big] \end{aligned} \tag{3.7}$$

On event B_m, we have

$$
\begin{aligned}
& P^{X(c^{m+1})}\left(|X(t-c^{m+1})| \le \gamma c^{n\alpha}\phi(c^n) \ \text{ for some } \ c^n \le t \le c^{n+1}\right) \\
\ge & \inf_{|y|\le a_{h,m}} P^y\left(|X(t-c^{m+1})| \le \gamma c^{n\alpha}\phi(c^n) \text{ for some } c^n \le t \le c^{n+1}\right) \\
= & \inf_{|y|\le a_{h,m}} P^{yc^{-n\alpha}}\left(|X(t-c^{m-n+1})| \le \gamma\phi(c^n) \ \text{ for some } 1 \le t \le c\right)
\end{aligned} \tag{3.8}
$$

where the last equality follows from the self-similarity (2.4). We choose $H(h,m)$ large enough such that for every $n \ge H(h,m)$, we have $a_{h,m}c^{-n\alpha} \le \eta_1$. It follows from Lemma 3.1 that (3.8) is at least

$$
KP^0\left(|X(t)| \le \gamma\phi(c^n) \ \text{ for some } 1 \le t \le c\right) = KP^0(A_n). \tag{3.9}
$$

Combining (3.6) - (3.9), we have for every $n \ge H(h,m)$

$$
P^0\left(A_n | A_h^c \cap A_{h+1}^c \cdots \cap A_m^c\right) \ge K_7 P^0(A_n)\ .
$$

Hence condition (ii) of Lemma 3.2 is verified. To verify condition (iii), we define, as in Takeuchi [19]

$$
\sigma_n = \begin{cases} \inf\{t \in [c^n, c^{n+1}] : \ |X(t)| \le \gamma c^{n\alpha}\phi(c^n)\} & \text{if there is such a } \ t \\ c^{n+1}+1 & \text{otherwise} \end{cases}
$$

Then for any $m < n$, it follows from the strong Markov property that

$$
\begin{aligned}
P^0(A_m \cap A_n) \ &= \ P^0\left(\sigma_m \le c^{m+1}, \ \sigma_n \le c^{n+1}\right) \\
&= \int_{c^m}^{c^{m+1}} P^0\left(\sigma_n \le c^{n+1} | \sigma_m = s\right) P^0(\sigma_m \in ds) \\
&\le \int_{c^m}^{c^{m+1}} P^{X(\sigma_m)}\Big(|X(t)| \le \gamma c^{n\alpha}\phi(c^n) \ \text{ for some } \\
&\qquad\qquad c^n - c^{m+1} \le t \le c^{n+1} - c^m\Big) P^0(\sigma_m \in ds) \\
&\le \sup_{|y|\le \gamma c^{m\alpha}\phi(c^m)} P^y\Big(|X(t)| \le \gamma c^{n\alpha}\phi(c^n) \ \text{ for some } \\
&\qquad\qquad c^n - c^{m+1} \le t \le c^{n+1} - c^m\Big) P^0(A_m)\ .
\end{aligned} \tag{3.10}
$$

Similar to (3.8) and (3.9), by (2.4) and Lemma 3.1 we see that when $m-n$ is large enough, say $n - m \ge n_0$, we have

$$
P^0(A_m \cap A_n) \le KP^0(A_m)P^0(A_n)\ . \tag{3.11}
$$

On the other hand, for $m < n < n_0$, $P^0(A_m \cap A_n) \le P^0(A_m)$. Thus condition (iii) of Lemma 3.2 is satisfied. Finally since $\gamma > 0$ is arbitrary, we have proved that P^0-almost surely

$$
\liminf_{t\to\infty} \frac{|X(t)|}{t^\alpha \phi(t)} = 0\ .
$$

This completes the proof of Theorem 3.1.

Remark Theorem 3.1 also holds for $t \to 0$. The proof is easier, we only need to verify (i) and (iii) in Lemma 3.2 and then use the Blumenthal zero-one law ([1]).

4. Type II α-$s.s.$ Markov Processes

It is known that stable subordinators on $\mathbf{R}_+$ of index β are $1/\beta$-$s.s.$ Markov processes which do not satisfy (3.1) and their lower functions are different from those obtained in Section 3. See Fristedt [5] and Breiman [2]. In this section we consider another class of α-$s.s.$ Markov processes in $\mathbf{R}_+^d$ whose transition functions do not satisfy (3.1) either and study their lower functions. We remark that the argument of this section can be applied to stable subordinators and give a different proof of Theorem 1 of Breiman [2].

Now fix a positive constant ξ. For every $t \geq 0$, $x \in \mathbf{R}_+^d$ and $[0,b] \hat{=} \prod_{i=1}^d [0, b_i]$, we set

$$\mathbf{P}(t, x, [0,b]) = \begin{cases} 0 & \text{if } x \notin [0,b] \\ \prod_{i=1}^d \exp\left(-\frac{\xi t}{b_i^{1/\alpha}}\right) & \text{if } x \in [0,b] \end{cases} \tag{4.1}$$

Then it is easy to verify that $\mathbf{P}(t,x,A)$ is a transition function satisfying (2.1) and (2.2). It follows from the general theory that there exists an α-$s.s.$ Markov process $X(t)$ $(t \geq 0)$ on $\mathbf{R}_+^d$ with the above $\mathbf{P}(t,x,A)$ as its transition function. Also the form of the transition function implies that each component of $X(t)$ is increasing almost surely.

For $d = 1$, these α-$s.s.$ Markov processes arise in the study of the statistics of extremes (see Lamperti [13]). In the case of $d = 1$, $\alpha = 1$ and $\xi = 1$ the inverse of $X(t)$ plays important roles in some limit theorems for the occupation times of two-dimensional Brownian motion and a class of one-dimensional diffusion processes (see Kasahara and Kotani [8]), and fractional Brownian motion (Kasahara and Kosugi [7]). Watanabe [27] proved that $X(t)$ is the limit process for a certain class of sums of i.i.d. random variables.

Even though $X(t)$ is not isotropic, the strong Markov property is easy to prove.

Lemma 4.1 *For each $x \in \mathbf{R}_+^d$, $X = (X(t), \mathcal{F}_{t+}, P^x)$ is a strong Markov process.*

Proof . By Theorem 8.11 in Chapter 1 of Blumenthal and Getoor [1], it is sufficient to prove that

$$x \to \int_0^\infty e^{-\lambda t} \mathbf{P}_t f(x) dt \tag{4.2}$$

is continuous for $\lambda > 0$ and $f \in \mathcal{C}_c(\mathbf{R}_+^d)$, the space of real continuous functions on $\mathbf{R}_+^d$ with compact support.

Let $x \in \mathbf{R}_+^d$ and $f \in \mathcal{C}_c(\mathbf{R}_+^d)$ be fixed. We first prove that for each $t > 0$

$$\lim_{y \to x} \mathbf{P}_t f(y) = \mathbf{P}_t f(x) \ . \tag{4.3}$$

By (4.1) and a change of variables, we have for any $t > 0$

$$\begin{aligned} \mathbf{P}_t f(y) &= \int_{\mathbf{R}_+^d} f(u) \mathbf{P}(t, y, du) \\ &= \int_{\mathbf{R}_+^d \cap [y,\infty)} f(u) \left(\frac{\xi t}{\alpha}\right)^d \prod_{i=1}^d \left[\exp\left(-\frac{\xi t}{u_i^{1/\alpha}}\right) u_i^{-1-1/\alpha}\right] du \\ &= \int_{\mathbf{R}_+^d \cap [x,\infty)} f(u + y - x) \left(\frac{\xi t}{\alpha}\right)^d \prod_{i=1}^d \left[\exp\left(-\frac{\xi t}{(u_i + y_i - x_i)^{1/\alpha}}\right)\right. \\ &\qquad \left. \cdot (u_i + y_i - x_i)^{-1-1/\alpha}\right] du \ . \end{aligned} \tag{4.4}$$

Since $f(u)$ is continuous with compact support, we have

$$\lim_{y \to x} f(u+y-x) \prod_{i=1}^{d} \exp\Big(-\frac{\xi t}{(u_i+y_i-x_i)^{1/\alpha}}\Big) \cdot (u_i+y_i-x_i)^{-1-1/\alpha}$$
$$= f(u) \prod_{i=1}^{d} \exp\Big(-\frac{\xi t}{u_i^{1/\alpha}}\Big) \cdot u_i^{-1-1/\alpha}$$

for each $u \in \mathbf{R}_+^d \cap [x, \infty)$. Hence (4.3) follows from (4.4) and the dominated convergence theorem. Using the dominated convergence theorem again proves the continuity of the function in (4.2).

From now on, we take $\xi = 1$ and we will not distinguish $[0,r]^d$ from $B(0,r) \cap \mathbf{R}_+^d$. The following lemma is also a corollary of Proposition 2.1 and Lemma 4.1, which we state without proof.

Lemma 4.2 *Let $X(t)$ $(t \geq 0)$ be an α-s.s. Markov process in $\mathbf{R}_+^d$ with transition function (4.1). For given constants $c > b > 0$, $r > 0$ and $x \in \mathbf{R}_+^d$ with $|x| \leq r$ we have*

$$\begin{aligned} \exp\Big(-\frac{bd}{r^{1/\alpha}}\Big) &\leq P^x\Big(|X(t)| \leq r \ \textit{ for some } \ b \leq t \leq c\Big) \\ &\leq \frac{1-\exp(-2(c-b)d/r^{1/\alpha})}{1-\exp(-(c-b)d/r^{1/\alpha})} \cdot \exp\Big(-\frac{bd}{r^{1/\alpha}}\Big) \ . \end{aligned} \tag{4.5}$$

Now we state the main result of this section. An argument of Talagrand [22] makes it possible for us to drop the "classical" monotonicity assumption on ϕ for this kind of results.

Theorem 4.1 *Let $X(t)$ $(t \geq 0)$ be an α-s.s. Markov Process in $\mathbf{R}_+^d$ with transition function (4.1). Consider a positive bounded function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ such that $t^\alpha \phi(t)$ is nondecreasing. Then P^0-almost surely $|X(t)| \geq t^\alpha \phi(t)$ for all t large enough if and only if*

$$\mathcal{J}(\phi) = \int_1^\infty \frac{1}{\phi(t)^{1/\alpha}} \exp\Big(-\frac{d}{\phi(t)^{1/\alpha}}\Big)\frac{dt}{t} < \infty \ .$$

Proof of Sufficiency. Suppose that $\mathcal{J}(\phi) < \infty$ and, without loss of generality, $t^\alpha \phi(t)$ is right continuous. Then similar to Lemma 3.1 of Talagrand [22], we have

$$\lim_{t \to \infty} \phi(t) = 0 \ . \tag{4.6}$$

We define an increasing sequence $\{t_n\}$ inductively as follows. Let $t_0 \geq 1$. Having defined t_n, we set

$$u_{n+1} = t_n\Big(1 + \phi(t_n)^{1/\alpha}\Big) \tag{4.7}$$

$$v_{n+1} = \inf\Big\{v > t_n : \ v^\alpha \phi(v) \geq t_n^\alpha \phi(t_n)(1 + L\phi(t_n)^{1/\alpha})\Big\} \ , \tag{4.8}$$

where $L > 1$ is a constant to be determined later, and define $t_{n+1} = \min\{u_{n+1}, \ v_{n+1}\}$. Then by Lemma 3.2 of Talagrand [22],

$$\lim_{n \to \infty} t_n = \infty \ . \tag{4.9}$$

For $n \geq 1$, consider the event

$$A_n = \Big\{ |X(t)| \leq t_n^\alpha \phi(t_n)(1 + L\phi(t_n)^{1/\alpha}) \text{ for some } t_n \leq t \leq t_{n+1} \Big\} .$$

Since, by (4.7), (4.6) and (4.9), for n large enough, $t_{n+1}/t_n \leq 2$, it follows from (2.4) and (4.5) that

$$\begin{aligned} P^0(A_n) &\leq P^0\Big(|X(t)| \leq \phi(t_n)(1 + L\phi(t_n)^{1/\alpha}) \text{ for some } 1 \leq t \leq 2 \Big) \\ &\leq K \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}(1 + L\phi(t_n)^{1/\alpha})^{1/\alpha}} \Big) \\ &\leq K \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}} \Big) \qquad (4.10) \end{aligned}$$

for n large enough. Suppose that we have proved

$$\sum_{n=1}^{\infty} \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}} \Big) < \infty , \qquad (4.11)$$

it follows from the Borel-Cantelli lemma that P^0-almost surely for n large enough, say, $n \geq n_0$

$$\inf_{t_n \leq t \leq t_{n+1}} |X(t)| \geq t_n^\alpha \phi(t_n)(1 + L\phi(t_n)^{1/\alpha}) . \qquad (4.12)$$

Thus for every $t \geq t_{n_0}$ we can find $n \geq n_0$ such that $t_n \leq t < t_{n+1}$. Since $t < v_{n+1}$, by (4.8) and (4.12) we have

$$t^\alpha \phi(t) < t_n^\alpha \phi(t_n)(1 + L\phi(t_n)^{1/\alpha}) \leq |X(t)| .$$

It remains to establish (4.11). We start with the following lemma.

Lemma 4.3 *(i) If n is large enough and $t_{n+1} = u_{n+1} < v_{n+1}$, then*

$$\exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}} \Big) \leq K \int_{t_n}^{t_{n+1}} \frac{1}{\phi(t)^{1/\alpha}} \exp\Big(-\frac{d}{\phi(t)^{1/\alpha}} \Big) \frac{dt}{t} , \qquad (4.13)$$

where $K > 0$ is a constant depending on α only.

(ii) We can choose L large such that if $t_{n+1} = v_{n+1}$ and n large enough, we have

$$\exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}} \Big) \leq \frac{1}{2} \exp\Big(-\frac{d}{\phi(t_{n+1})^{1/\alpha}} \Big) . \qquad (4.14)$$

Proof . (i) If $t_{n+1} = u_{n+1} < v_{n+1}$, then by (4.7) and (4.8) we have

$$t_{n+1} - t_n = t_n \phi(t_n)^{1/\alpha} \qquad (4.15)$$

$$t_{n+1}^\alpha \phi(t_{n+1}) \leq t_n^\alpha \phi(t_n)(1 + L\phi(t_n)^{1/\alpha}) . \qquad (4.16)$$

Since $t^\alpha \phi(t)$ is nondecreasing, we have

$$I_n \hat{=} \int_{t_n}^{t_{n+1}} \frac{1}{\phi(t)^{1/\alpha}} \exp\Big(-\frac{d}{\phi(t)^{1/\alpha}} \Big) \frac{dt}{t} \geq \frac{t_{n+1} - t_n}{t_{n+1}\phi(t_{n+1})^{1/\alpha}} \exp\Big(-\frac{t_{n+1} d}{t_n \phi(t_n)^{1/\alpha}} \Big) .$$

It follows from (4.6), (4.15) and (4.16) that for n large enough

$$I_n \ge \frac{t_n \phi(t_n)^{1/\alpha}}{t_{n+1}\phi(t_{n+1})^{1/\alpha}} \exp\Big(-d - \frac{d}{\phi(t_n)^{1/\alpha}}\Big) \ge K \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) \ .$$

This proves (4.13).

(ii) If $t_{n+1} = v_{n+1}$, then

$$t_{n+1}^\alpha \phi(t_{n+1}) \ge t_n^\alpha \phi(t_n)(1 + L\phi(t_n)^{1/\alpha}) \ . \tag{4.17}$$

Since $t_{n+1} \le u_{n+1}$, we have

$$\frac{t_{n+1}}{t_n} \le 1 + \phi(t_n)^{1/\alpha} \ . \tag{4.18}$$

It follows from (4.17) and (4.18) that for n large enough

$$\begin{aligned}\exp\Big(-\frac{d}{\phi(t_{n+1})^{1/\alpha}}\Big) &\ge \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) \exp\Big(\frac{d}{\phi(t_n)^{1/\alpha}} - \frac{(1+\phi(t_n)^{1/\alpha})d}{\phi(t_n)^{1/\alpha}(1+L\phi(t_n)^{1/\alpha})^{1/\alpha}}\Big) \\ &\ge \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) \exp\Big(\frac{(L/\alpha - 1)d}{(1+L\phi(t_n)^{1/\alpha})^{1/\alpha}}\Big) \\ &\ge 2\exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big)\end{aligned}$$

for suitable chosen L(e. g. $L = 2\alpha$). This proves (4.14).

To prove (4.11), we set $J = \{n : t_{n+1} = u_{n+1} < v_{n+1}\}$. Then by (4.13) we have

$$\sum_{n\in J} \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) < \infty \ . \tag{4.19}$$

We denote the elements of J by $n(j)$ $(j \ge 1)$. Then for every k with $n(j-1) < k < n(j)$, by (4.14) we have

$$\exp\Big(-\frac{d}{\phi(t_k)^{1/\alpha}}\Big) \le 2^{-k} \exp\Big(-\frac{d}{\phi(t_{n(j)})^{1/\alpha}}\Big) \ .$$

So

$$\sum_{n(j-1)<k<n(j)} \exp\Big(-\frac{d}{\phi(t_k)^{1/\alpha}}\Big) \le K \exp\Big(-\frac{d}{\phi(t_{n(j)})^{1/\alpha}}\Big) \ . \tag{4.20}$$

Combining (4.19) and (4.20) proves (4.11).

Now we proceed to prove the necessity of $\mathcal{J}(\phi) < \infty$.

Lemma 4.4 *We can assume that* $\lim_{t\to\infty} \phi(t) = 0$.

Proof . If $\limsup_{t\to\infty} \phi(t) > 0$, then there exists a sequence of positive numbers $\{t_n\}$ such that

$$\frac{t_{n+1}}{t_n} \ge 2 \quad \text{and} \quad \phi(t_n) \ge \delta$$

for some $\delta > 0$. Then by Lemma 4.2 we have

$$\sum_{n=1}^{\infty} P^0\Big(|X(t)| \le t_n^\alpha \ \text{ for some } \ t_n \le t \le t_{n+1}\Big) = \infty \ .$$

Now the same proof as that of Theorem 3.1 yields P^0-almost surely $X(t_n) \le t_n^\alpha \phi(t_n)$ infinitely often.

Lemma 4.5 *Without loss of generality, we can further assume that $t^\alpha \phi(t)$ is continuous and $\phi(t) \geq (2 \log\log t)^{-\alpha}$.*

Proof . Since $t^\alpha \phi(t)$ is non-decreasing and hence it only has countably many discontinuities, we can define a continuous, non-decreasing function $t^\alpha \psi(t)$ such that $t^\alpha\psi(t) \leq t^\alpha \phi(t)$ and $\mathcal{J}(\psi) = \infty$. The proof of the second part is very similar to the proof of Lemma 7 of Takeuchi [19].

Lemma 4.6 *Suppose that $\mathcal{J}(\phi) = \infty$ and the conditions in Lemmas 4.4 and 4.5 are satisfied. Then there exists a sequence $\{t_n\}$ with the following properties:*

$$\sum_{n=1}^{\infty} \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) = \infty \tag{4.21}$$

$$t_{n+1} \geq t_n(1 + \phi(t_n)^{1/\alpha}) \tag{4.22}$$

and for n large enough

$$m \leq n \;\Rightarrow\; \frac{\phi(t_n)}{t_n^\alpha} \leq 2\frac{\phi(t_m)}{t_m^\alpha} \ . \tag{4.23}$$

Proof . We construct a sequence $\{t_n\}$ inductively. Let $t_0 = 1$. Having defined t_n, we choose $s_n \geq t_n$ such that

$$\sup\Big\{\frac{\phi(t)}{t^\alpha} : t \geq t_n\Big\} = \frac{\phi(s_n)}{s_n^\alpha} \ , \tag{4.24}$$

which is possible because of Lemma 4.4 and the continuity of ϕ, and define

$$t_{n+1} = s_n(1 + \phi(s_n)^{1/\alpha}) \ . \tag{4.25}$$

Clearly (4.22) holds. To derive (4.23), we notice that by (4.24), for $m \leq n$

$$\frac{\phi(t_n)}{t_n^\alpha} \leq \frac{\phi(s_{m-1})}{s_{m-1}^\alpha} \ . \tag{4.26}$$

Also $s_{m-1}^\alpha \phi(s_{m-1}) \leq t_m^\alpha \phi(t_m)$ and $t_m^{2\alpha} \leq 2 s_{m-1}^{2\alpha}$ for m large enough. Hence we have

$$\frac{\phi(s_{m-1})}{s_{m-1}^\alpha} \leq \frac{t_m^\alpha \phi(t_m)}{s_{m-1}^{2\alpha}} \leq 2\frac{\phi(t_m)}{t_m^\alpha} \ . \tag{4.27}$$

Combining (4.26) and (4.27) yields (4.23). In order to prove (4.21), it is enough to show

$$I_n = \int_{t_n}^{t_{n+1}} \frac{1}{\phi(t)^{1/\alpha}} \exp\Big(-\frac{d}{\phi(t)^{1/\alpha}}\Big)\frac{dt}{t} \leq K \exp\Big(-\frac{d}{\phi(t_{n+1})^{1/\alpha}}\Big) \ . \tag{4.28}$$

We write

$$I_n = \int_{t_n}^{s_n} + \int_{s_n}^{t_n} = I_n^1 + I_n^2 \ .$$

Then by the monotonicity of $t^\alpha\phi(t)$ and (4.25) we have

$$\begin{aligned} I_n^2 &\leq \frac{t_{n+1} - s_n}{s_n \phi(s_n)^{1/\alpha}} \exp\Big(-\frac{s_n d}{t_{n+1}\phi(t_{n+1})^{1/\alpha}}\Big) \\ &\leq K \exp\Big(-\frac{d}{\phi(t_{n+1})^{1/\alpha}}\Big) \ . \end{aligned} \tag{4.29}$$

To estimate I_n^1, we notice that by (4.24)

$$t_n \le t \le s_n \ \Rightarrow\ \phi(t) \le \frac{t^\alpha \phi(s_n)}{s_n^\alpha}\ . \tag{4.30}$$

It follows from (4.30), the fact that $x \to xe^{-x}$ is decreasing for x large and a change of variable that

$$\begin{aligned} I_n^1 &\le \int_{t_n}^{s_n} \frac{s_n}{t^2\phi(s_n)^{1/\alpha}} \exp\Big(-\frac{s_n d}{t\phi(s_n)^{1/\alpha}}\Big)dt \\ &\le \int_0^1 \frac{1}{t^2\phi(s_n)^{1/\alpha}} \exp\Big(-\frac{d}{t\phi(s_n)^{1/\alpha}}\Big)dt \\ &= \frac{1}{d}\exp\Big(-\frac{d}{\phi(s_n)^{1/\alpha}}\Big) \\ &\le K\exp\Big(-\frac{d}{\phi(t_{n+1})^{1/\alpha}}\Big)\ , \end{aligned} \tag{4.31}$$

where the last inequality follows from the monotonicity of $t^\alpha\phi(t)$ and (4.25). By (4.29) and (4.31) we obtain (4.28), and hence (4.21).

Even though the sequence $\{t_n\}$ constructed above increases to infinity, it may not increase fast enough. So we need an extra procedure to obtain the appropriate sequence. For each n, let $k(n)$ be defined by

$$2^{k(n)} \le \frac{1}{\phi(t_n)^{1/\alpha}} < 2^{k(n)+1}$$

and let $I_k = \{n:\ k(n) = k\}$. Then by Lemma 4.4 each I_k is finite. For each $k \ge 1$, we denote $N_k = \exp(d2^{k-2})$.

Lemma 4.7 *There exists a set $J \subseteq \mathbf{N}$ with the following properties:*

$$\sum_{n\in J} \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) = \infty \tag{4.32}$$

and for every pair $m, n \in J$, $m < n$, such that

$$\#(I_{k(n)} \cap [m,n]) > N_{k(n)} \tag{4.33}$$

where $\#A$ denotes the cardinality of A, we have

$$\frac{t_n}{t_m} \ge \exp\left(\exp(d2^{k(n)-3})\right)\ . \tag{4.34}$$

Proof . By the definition of $k(n)$ we have

$$\exp(-d2^{k(n)+1}) \le \exp\Big(-\frac{d}{\phi(t_n)^{1/\alpha}}\Big) \le \exp(-d2^{k(n)})\ . \tag{4.35}$$

Given $m, k \in \mathbf{N}$, we define

$$U_{m,k} = \{i < m:\ i \in I_k,\ \#(I_k \cap [i,m)) \le N_k\}\ .$$

Then by (4.35)

$$\sum_{i\in U_{m,k}} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \le N_k \exp(-d2^k) \le \exp(-d2^{-(k-1)}).$$

Thus

$$\begin{aligned}
&\sum_{k\ge k(m)+3} \sum_{i\in U_{m,k}} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \\
&\le \sum_{k\ge k(m)+3} \exp(-d2^{-(k-1)}) \\
&\le \exp\Big(-\frac{d}{\phi(t_m)^{1/\alpha}}\Big) \sum_{k\ge k(m)+3} \exp\left(d2^{k(m)+1} - d2^{-(k-1)}\right) \\
&\le \exp\Big(-\frac{d}{\phi(t_m)^{1/\alpha}}\Big) \sum_{l\ge 1} \exp\left(d2^{k(m)+1}(2^l-1)\right) \\
&\le \frac{1}{2}\exp\Big(-\frac{d}{\phi(t_m)^{1/\alpha}}\Big) \ . \qquad (4.36)
\end{aligned}$$

We set

$$V_m = \bigcup_{k\ge k(m)+3} U_{m,k} \ .$$

Then (4.36) implies

$$\sum_{l\le p}\sum_{m\in I_l}\sum_{i\in V_m} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \le \frac{1}{2}\sum_{l\le p}\sum_{m\in I_l} \exp\Big(-\frac{d}{\phi(t_m)^{1/\alpha}}\Big) \ . \qquad (4.37)$$

Now let

$$J = \mathbf{N}\setminus \bigcup_{m=1}^{\infty} V_m \ .$$

We observe that if $i \in V_m$, then $k(i) \ge k(m) + 3$ and hence

$$\Big(\bigcup_m V_m\Big) \bigcap \Big(\bigcup_{l\le p} I_l\Big) \subseteq \bigcup_{k(m)\le p} V_m \ .$$

Thus by (4.37) we have

$$\sum_{i\notin J, k(i)\le p} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \le \frac{1}{2}\sum_{k(i)\le p} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \ .$$

This implies

$$\sum_{i\in J, k(i)\le p} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \ge \frac{1}{2}\sum_{k(i)\le p} \exp\Big(-\frac{d}{\phi(t_i)^{1/\alpha}}\Big) \ .$$

Letting $p \to \infty$, we get (4.32).

Now we prove (4.34). Consider $m, n \in J$, $m < n$ and denote $k = k(n)$. By (4.25) for each $i \in I_k$ we have

$$t_{i+1} \ge t_i(1 + \phi(t_i)^{1/\alpha}) \ge t_i(1 + 2^{-k-1}) \ .$$

Thus, when (4.33) holds, we see that for m, hence k, large enough

$$\frac{t_n}{t_m} \geq \left(1 + 2^{-k-1}\right)^{N_k} \geq \exp\left(\exp(d2^{k-3})\right) .$$

Proof of Necessity of Theorem 4.1. For each $n \in J$, let

$$A_n = \{|X(t)| \leq t_n^\alpha \phi(t_n) \ \text{ for some } \ t_n \leq t \leq t_{n+1}\} \ .$$

Then by Lemma 4.2 we have

$$P^0(A_n) \geq \exp\left(-\frac{d}{\phi(t_n)^{1/\alpha}}\right) ,$$

and by (4.32)

$$\sum_{n \in J} P^0(A_n) = \infty \ . \tag{4.38}$$

It remains to verify that conditions (ii) and (iii) of Lemma 3.2 are satisfied. Since the verification of condition (ii) is the same as in the proof of Theorem 3.1, we will omit it. To verify condition (iii), we set

$$J' = \{n \in J : \ t_m < t_n \leq 2t_{m+1}\}$$

and for each $k \geq 1$, let

$$J_k = \{n \in J \cap I_k : \ t_n > 2t_{m+1} \ \text{ and } \ \#(I_k \cap [m, n]) \leq N_k\} \ .$$

Finally we set

$$J'' = J \backslash \left(J' \cup \bigcup_{k=1}^\infty J_k\right) \ .$$

It is sufficient to prove the following: for every $m \in J$

$$\sum_{n \in J'} P^0(A_m \cap A_n) \leq KP(A_m) \tag{4.39}$$

$$\sum_{k=1}^\infty \sum_{n \in J_k} P^0(A_m \cap A_n) \leq KP(A_m) \tag{4.40}$$

and

$$\sum_{n \in J''} P^0(A_m \cap A_n) \leq KP(A_m)P(A_n) \ . \tag{4.41}$$

To prove (4.39), we notice, by (4.23), that for $n \in J'$ with $t_m < t_n \leq 2t_{m+1}$ we have

$$t_n^\alpha \phi(t_n) \leq 2^{2\alpha+1} t_{m+1}^\alpha \phi(t_{m+1}) \ . \tag{4.42}$$

It follows from (4.22) that

$$t_n - t_{m+1} \geq (n - m - 1) t_{m+1} \phi(t_{m+1})^{1/\alpha} \ . \tag{4.43}$$

Hence by (3.11), (4.5), (4.42) and (4.43), we have

$$\begin{aligned} P^0(A_m \cap A_n) &\le K_8 P^0(A_m) \exp\Big(-\frac{(t_n - t_{m+1})d}{t_n \phi(t_n)^{1/\alpha}}\Big) \\ &\le K_8 P^0(A_m) \exp(-(n-m-1)/K) \ . \end{aligned}$$

This implies (4.39) immediately.

Suppose now $t_n \ge 2t_{m+1}$, then

$$\begin{aligned} P^0(A_m \cap A_n) &\le K_8 P^0(A_m) \exp\Big(-\frac{(t_n - t_{m+1})d}{t_n \phi(t_n)^{1/\alpha}}\Big) \\ &\le K_8 P^0(A_m) \exp\Big(-\frac{d}{2\phi(t_n)^{1/\alpha}}\Big) \ . \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n \in J_k} P^0(A_m \cap A_n) &\le K N_k P^0(A_m) \exp\Big(-\frac{2^k d}{2}\Big) \\ &\le K P^0(A_m) \exp(-d 2^{k-1}) \end{aligned}$$

and (4.40) follows.

For every $n \in J''$, we denote $k = k(n)$ and $k' = k(m)$. Since $n \notin J_k$ we have

$$\#(I_k \cap [m,n]) > N_k$$

and hence (4.34) holds. Thus

$$\begin{aligned} P^0(A_m \cap A_n) &\le K_8 P^0(A_m) \exp\Big(-\frac{(t_n - t_{m+1})d}{t_n \phi(t_n)^{1/\alpha}}\Big) \\ &\le K P^0(A_m) P^0(A_n) \exp\Big(\frac{t_{m+1} d}{t_n \phi(t_n)^{1/\alpha}}\Big) \\ &\le K P^0(A_m) P^0(A_n) \exp\Big(\exp(-\exp(2^k)) 2^{k+1} d\Big) \\ &\le K P^0(A_m) P^0(A_n) \ . \end{aligned}$$

This proves (4.41) and hence Theorem 4.1.

As an immediate corollary, we have the following Chung type law of iterated logarithm.

Theorem 4.2 *Let $X(t)$ $(t \ge 0)$ be an α-s.s. Markov process in $\mathbf{R}^d$ with transition function (4.1). Then P^0-almost surely*

$$\liminf_{t\to\infty} \frac{|X(t)|}{t^\alpha / (\log\log t)^\alpha} = d \ .$$

An application of Theorem 4.1 is to recover and improve the following result of Khoshnevisan [10] and Knight [12] for the lower class of the maximum of Brownian motion normalized by its local time at 0.

Let $B(t)$ $(t \ge 0)$ be a real-valued Brownian motion starting from 0. We denote

$$M(t) = \max_{0 \le s \le t} B(s)$$

and $l(t)$ the local time of $B(t)$ at the level 0.

Corollary 4.1 *Consider a positive bounded function* $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $t\phi(t)$ *is nondecreasing. Then* P^0*-almost surely* $M(t) \geq l(t)\phi(l(t))$ *for all* t *large enough if and only if*

$$\int_1^\infty \frac{1}{\phi(t)} \exp\Big(-\frac{1}{\phi(t)}\Big)\frac{dt}{t} < \infty \ .$$

In particular,

$$\liminf_{t\to\infty} \frac{\log\log l(t)}{l(t)} M(t) = 1 \ .$$

Proof . Let $d = 1$ and $\alpha = 1$ in Theorem 4.1, then $X(t)$ is the canonical extremal process. It follows from Theorem 2.1 in Watanabe [27] that

$$(M(l^{-1}(\cdot), P^x) \stackrel{\mathrm{d}}{=} (X(\cdot), P^x)$$

where $l^{-1}(t)$ is the right continuous inverse function of $l(t)$. Hence Corollary 4.1 follows immediately from Theorem 4.1.

Remark With a little more effort, similar results can be proved for a symmetric $1/\alpha$-stable Lévy process with $\alpha \in (1, 2]$. See Khoshnevisan [11] for more general results using a different approach.

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